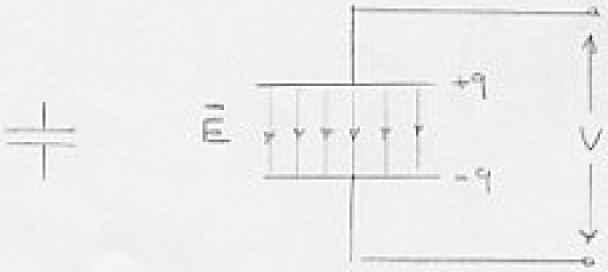
Chapter 33
Inductance

33-1 Capacitors and Inductors:

Simlarity;

Capacitors $\xrightarrow{\text{connection}}$ E



Inductors $\longrightarrow$ $\bar{B}$

33-2 Inductance;

$L = \frac{N\varphi}{i}$ (inductance) (similar to $C = \frac{q}{V}$)

φ: magnetic flux due to i. (windings are said to be linked by this shared flux).

N: number of turns

$N\varphi$: flux linkage

Unit; $1 \text{ henry} = 1\,H = \frac{1\,T \cdot m^2}{A}$

Inductance of a Solenoid:

$N\Phi = (n\ell)(BA)$ number of flux linkages

$B = \mu_0 i n$

$$L = \frac{N\Phi}{i} = \frac{(n\ell)(BA)}{i} = \frac{(n\ell)(\mu_0 i n)(A)}{i}$$

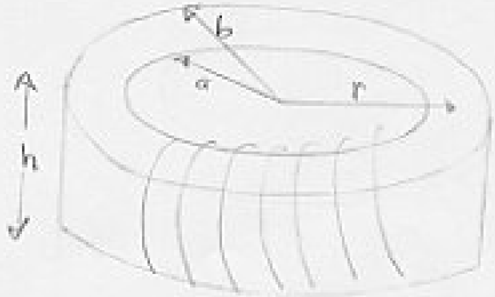
$$= \mu_0 n^2 A \ell$$

$$\frac{L}{\ell} = \mu_0 n^2 A$$ inductance per unit length (solenoid)

Inductance, depends only on geometric factors.

Inductance of a Toroid:

Consider a toroid with rectangular cross-section



We obtained before

$$B = \frac{\mu_0 i N}{2\pi r}$$

$$\Phi = \int B \cdot dA = \int_a^b B h\, dr = \int_a^b \frac{\mu_0 i N}{2\pi r} h\, dr = \frac{\mu_0 i N h}{2\pi}\int_a^b \frac{dr}{r}$$

$$= \frac{\mu_0 i N h}{2\pi} \ln\frac{b}{a}$$

$$L = \frac{N\Phi}{i} = \frac{\mu_0 i N^2 h}{2\pi i} \ln\frac{b}{a}$$

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a} \qquad \text{(toroid)}$$

$$\longrightarrow \mu_0 = 4\pi \times 10^{-7}\ T{\cdot}m/A = 4\pi \times 10^{-7}\ H/m$$

Ex.

A toroid with $N = 1250$, $a = 52$ mm, $b = 95$ mm, $h = 13$ mm

$L = ?$

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a} = \frac{(4\pi \times 10^{-7}\ H/m)(1250)^2(13 \times 10^{-3}\ m)}{2\pi} \ln \frac{95}{52}$$

$$= 2.45 \times 10^{-3}\ H$$

33-3 Self-Induction:

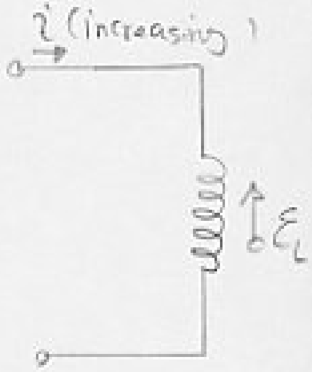
If two coils (inductors) are near each other, a current i in one coil will set up a magnetic flux through the second coil. If we change this flux by changing i, an induced emf will appear in the second coil.

Also,

An induced emf $\mathcal{E}_L$ appears in a coil if we change the current in that same coil.

This process is called self-induction and the emf that appears is called a self-induced emf.

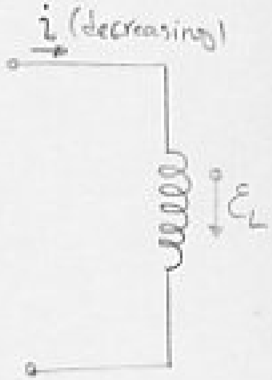
It obeys Faraday's law of induction just as other induced emfs do.



$$L = \frac{N\Phi}{i} \rightarrow N\Phi = Li$$

Faraday's law: $\mathcal{E}_L = -\dfrac{d(N\Phi)}{dt}$

$\rightarrow \mathcal{E}_L = -L\dfrac{di}{dt}$ self-induced emf.



$\mathcal{E}_L$ is in opposition to i

We saw earlier that when $\bar{E}$ and emf are induced by a changing Φ, we cannot define an electric potential.

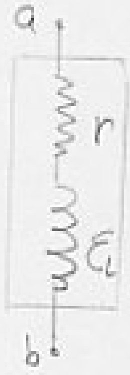
Here also we cannot define a potential within the inductor itself, when the flux is changing.

However, potential can still be defined at points in the circuit outside this region, (where the electric field in the wire and the other circuit elements are due to dist. of charged particles.)

Moreover, we can define a potential difference V_L to be across an inductor.

Ideal inductor: its inner $r = 0$

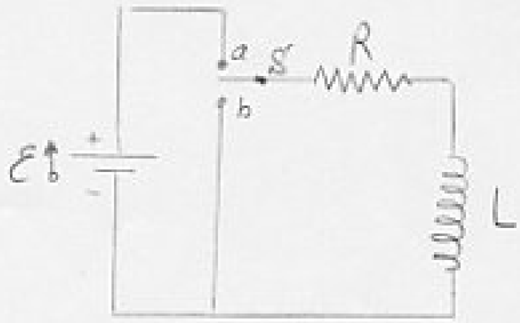
If $r \neq 0$, it will be consider (outside the region of changing flux) and an $\mathcal{E}$ in series.



For ideal inductor; $V_{ab} = \mathcal{E}_L$

" real " ; $V_{ab} \neq \mathcal{E}_L$

33-4 RL Circuits



i) S on a position

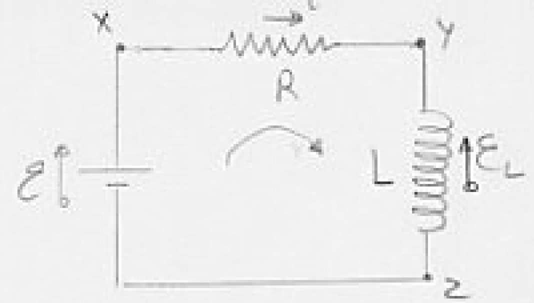
i starts from zero increasing by time.

Induced $\mathcal{E}_L$ opposes the increasing i (Preventing i to reach $\frac{\mathcal{E}}{R}$ its max. value).

i.e. $\mathcal{E}_L$ opposes $\mathcal{E}$.

$V_x > V_Y$ for the dir. shown for i

$V_Y > V_Z$ (look the dir. of $\mathcal{E}_L$ for increasing i)



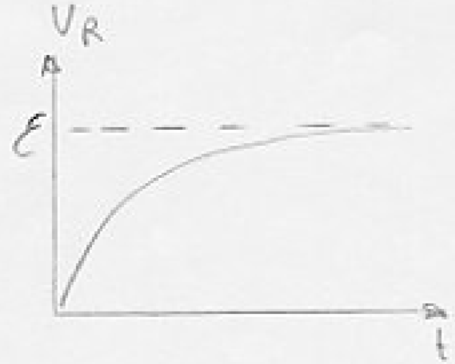
The loop theorem:

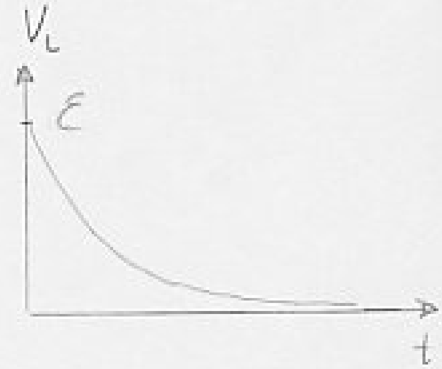
$$-iR \underbrace{- L\frac{di}{dt}}_{\mathcal{E}_L} + \mathcal{E} = 0$$

or $\begin{cases} iR + L\dfrac{di}{dt} = \mathcal{E} \\ i(0) = 0 \end{cases}$ RC circuit

$$\rightarrow \quad i = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right)$$

$\begin{cases} i = \dfrac{\mathcal{E}}{R}\left(1 - e^{t/\tau_L}\right) \\ \tau_L = \dfrac{L}{R} \end{cases}$ rise of current





$$\tau_L = \frac{L}{R} \rightarrow 1\,\frac{H}{\Omega} = 1\,\frac{H}{\Omega}\underbrace{\left(\frac{1V\cdot s}{1H\cdot A}\right)}_{\mathcal{E}_L dt = -L\,di}\underbrace{\left(\frac{1\Omega\cdot A}{1V}\right)}_{V = iR} = 1\,s$$

At $t = \tau_L = \frac{L}{R}$

$$i = \frac{\mathcal{E}}{R}(1 - e^{-1}) = 0.63\,\frac{\mathcal{E}}{R}$$

After a long time that S has been closed on position a,

$\longrightarrow i = \frac{\mathcal{E}}{R}$

we change it to b position

$$\longrightarrow \quad L\frac{di}{dt} + iR = 0$$

$$i = \frac{\mathcal{E}}{R}e^{-t/\tau_L} = i_0 e^{-t/\tau_L}$$ decay of current

i_0 can be different from $\frac{\mathcal{E}}{R}$ in other cases.

Ex.

A solonoid, with $L = 53\,mH$ has $r = 0.37\,\Omega$.
If it is connected to a battery, how long will it take for the current to reach $\frac{1}{2}$ its final equilibrium value?

Sol.

At equilibrium; $t \to \infty \quad i = \frac{\mathcal{E}}{R}$

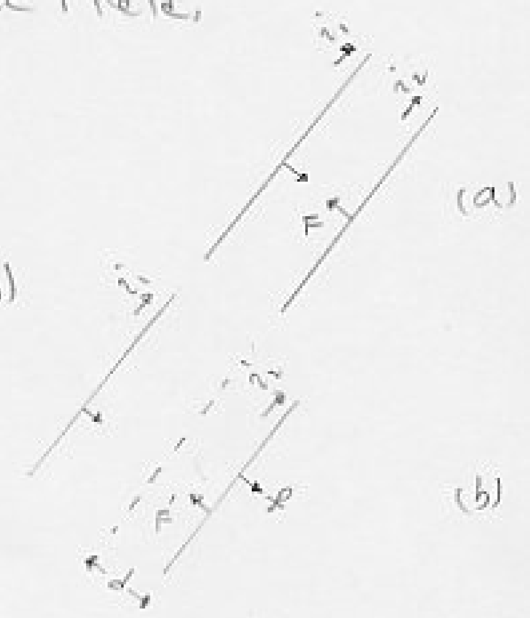
$$\frac{1}{2}\frac{\mathcal{E}}{R} = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}) \qquad t = \tau_L \ln 2 = \frac{L}{R}\ln 2$$

$$t = \frac{53 \times 10^{-3}\,H}{0.37\,\Omega}\ln 2 = 0.10\,s$$

33-5 Energy stored in a Magnetic Field:

$$\mathcal{E} = iR + L\frac{di}{dt}$$ loop theorem (= Conservation of energy)

$$\mathcal{E}i = i^2R + Li\frac{di}{dt}$$



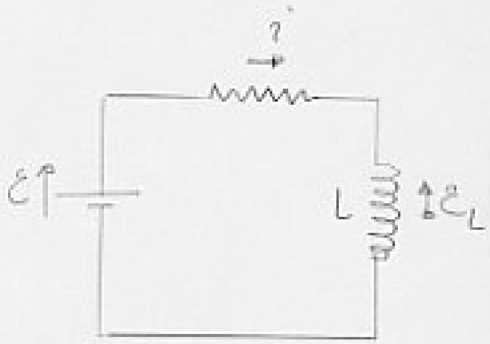
Physical Interpretation:

1) $\mathcal{E}dq$: the work done on dq passing the battery of emf $\mathcal{E}$ by the battery in dt

$$P = \frac{\mathcal{E}dq}{dt} = \mathcal{E}i$$

This is the rate of energy, delivered by the battery to the circuit.

$W = \int_0^d F \cdot dx$ work done by us = stored energy in the mag. field



2) i^2R : the rate of thermal energy in R.

3) $Li\frac{di}{dt}$: the rate at which the energy is stored in the mag. field.

$$\frac{dU_B}{dt} = Li\frac{di}{dt} \longrightarrow dU_B = Li\,di$$

$$\int_0^{U_B} dU_B = \int_0^i Li\,di$$

$$U_B = \frac{1}{2}Li^2$$ magnetic energy (the energy stored by an inductor carrying i)

We may compare this with

$$U_E = \frac{q^2}{2C} \quad \text{for the capacitor}$$

In each case the expression for the stored energy was delivered by setting it equal to the work that must be done to set up the field.

Ex.

A coil has an inductance of $L = 53\,\text{mH}$ and a resistance of $r = 0.35\,\Omega$.

a) If a 12-V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up to its equilibrium value?

Sol.

$$U_B = \frac{1}{2} L i^2$$

$$i_{(t\to\infty)} = \frac{\mathcal{E}}{R} = \frac{12}{0.35} = 34.3\,\text{A}$$

$$U_{B(t\to\infty)} = \frac{1}{2} L i^2_{(t\to\infty)} = \frac{1}{2}(53\times10^{-3}\,\text{H})(34.3\,\text{A})^2 = 31\,\text{J}$$

b) After how many time constants will half of this equilibrium energy be stored in the mag. field?

Sol.

$$U_B = \frac{1}{2} U_{B(t\to\infty)}$$

$$\frac{1}{2} L i^2 = \frac{1}{2}\left(\frac{1}{2} L i^2_{(t\to\infty)}\right)$$

$$i = \frac{1}{\sqrt{2}} i_{(t\to\infty)}$$

But $i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L})$

$$\frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}) = \frac{\mathcal{E}}{\sqrt{2}R} \rightarrow e^{-t/\tau_L} = 1 - \frac{1}{\sqrt{2}} = 0.293$$

$$\rightarrow \frac{t}{\tau_L} = -\ln 0.293 = 1.23 \quad \rightarrow t \simeq 1.2\tau_L$$

Ex.

A 3.56 H inductor is placed in series with a 12.8 Ω resistor, an emf of 3.24 V being suddenly applied across the combination.

a) At 0.278 s (which is one inductive time const.), after the emf is applied, what is the rate P at which energy is being delivered by the battery?

Sol.

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}) \quad \text{at } t = \tau_L, \; i = \frac{\mathcal{E}}{R}(1 - e^{-1})$$

$$i = \frac{3.24}{12.8}(1 - e^{-1}) = 0.1600 \text{ A}$$

$$P = \mathcal{E} i = (3.24)(0.1600) = 0.5184 \text{ W}$$

b) At $t = 0.278\,s$, at what rate P_R is energy appearing as thermal energy in the resistor?

Sol.

$$P = Ri^2 = (12.8)(0.1600)^2 = 0.3277\,W$$

c) At $t = 0.278\,s$ at what rate P_B is energy being stored in the mag. field?

Sol.
$$\frac{di}{dt} = \frac{\mathcal{E}}{R}\frac{R}{L}e^{-t/\tau_L} = \frac{\mathcal{E}}{L}e^{-t/\tau_L}$$

$$\frac{di}{dt} = \frac{3.24\,V}{3.56\,H}e^{-1} = 0.3348\,A/s$$

$$P_B = \frac{dU_B}{dt} = Li\frac{di}{dt} = (3.56\,H)(0.1600\,A)(0.3348\,A/s)$$

$$= 0.1907\,W$$

Note that $P = P_R + P_B$

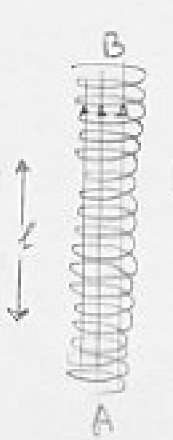
$$P = 0.3277 + 0.1907 = 0.5184\,W$$ (conservation of energy)

33-6 Energy Density of a Magnetic Field:

Consider a special case:

A long solonoid (ideal)

B inside the solonoid is uniform and parallel to the axis of solonoid, and $B = 0$ outside



$$u_B = \frac{U_B}{A\ell}$$

but $U_B = \frac{1}{2} L i^2$ $\longrightarrow$ $u_B = \frac{L i^2}{2A\ell} = \frac{L}{\ell} \frac{i^2}{2A}$

but $\frac{L}{\ell} = \mu_0 n^2 A$ for solonoid

$$\longrightarrow u_B = \frac{1}{2} \mu_0 n^2 i^2$$

but $B = \mu_0 i n$ for solonoid

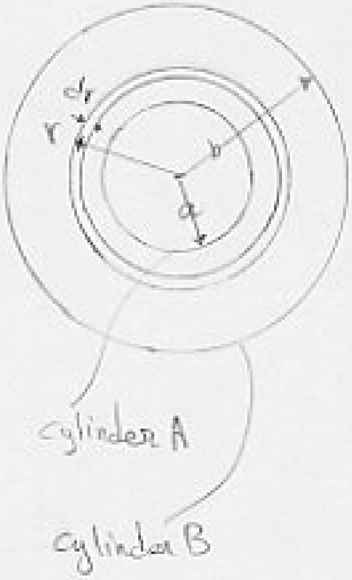
$\rightarrow$ $u_B = \frac{B^2}{2\mu_0}$ magnetic energy density

Even though we derived this result for a special case, but it holds in <u>general</u>.

It is comparable with $u_E = \frac{1}{2} \epsilon_0 E^2$ (in vacuum)

<u>Ex.</u>

A long coaxial cable consists of two thin-walled concentric conducting cylinders with radii a and b. Its central cylinder A carries a steady current i, the outer cylinder B providing the return path.



a) Calculate the energy stored in the mag. field between the cylinders for a length ℓ of such a cable.

Sol.

$$dU = u_B\, dV$$

where $u_B = \dfrac{B^2}{2\mu_0}$

$$\oint B \cdot ds = \mu_0 i \qquad \text{Ampere's law}$$

$$\rightarrow B(2\pi r) = \mu_0 i \quad \rightarrow B = \frac{\mu_0 i}{2\pi r}$$

$$u_B = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2\pi r}\right)^2 = \frac{\mu_0 i^2}{8\pi^2 r^2}$$

$$dV = (2\pi r l)\, dr$$

$$dU = u_B\, dV = \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi r l)\, dr = \frac{\mu_0 i^2 l}{4\pi}\frac{dr}{r}$$

$$U = \int dU = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 i^2 l}{4\pi}\ln\frac{b}{a} \qquad \text{for length } l$$

No energy is stored outside the outer cylinder or inside the inner cylinder (because $B = 0$).

Alternative way:

$$\begin{cases} U_B = \frac{1}{2} L i^2 \\ L = \dfrac{N\Phi}{i} \end{cases}$$

$$\Phi = \int B \cdot dA = \int_a^b \frac{\mu_0 i}{2\pi r}(l\, dr) = \frac{\mu_0 i l}{2\pi}\ln\frac{b}{a}$$

$$N = 1 \quad L = \frac{N\Phi}{i} = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$

$$U_B = \frac{1}{2}L i^2 = \frac{\mu_0 i^2 l}{4\pi}\ln\frac{b}{a}$$

b- What is the stored energy per unit length of the cable if $a = 1.2$ mm, $b = 3.5$ mm, and $i = 2.7$ A?

Sol.

$$\frac{U}{l} = \frac{\mu_0 i^2}{4\pi} \ln\frac{b}{a} = \frac{(4\pi\times10^{-7}\ \mathrm{H/m})(2.7\ \mathrm{A})^2}{4\pi} \ln\frac{3.5}{1.2} = 7.8\times10^{-7}\ \mathrm{J/m}$$

Ex.

Compare the energy required to set up, in a cube 10 cm on edge (a) a uniform electric field of 100 kV/m and (b) a uniform mag. field of 0.1 T. (Both these fields would be judged reasonably large but readily available in a laboratory.)

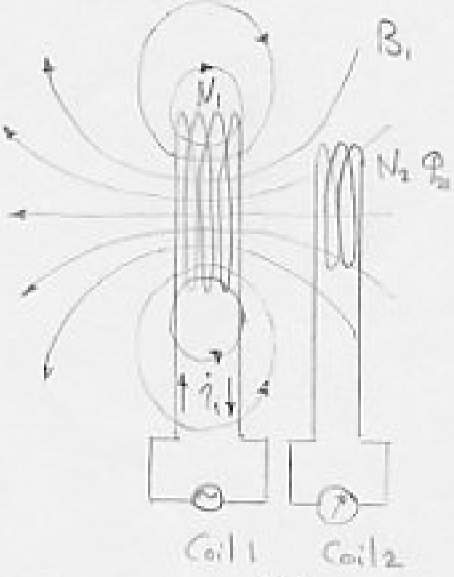
Sol.

$$U_E = u_E V_0 = \frac{1}{2}\epsilon_0 E^2 V_0 = \frac{1}{2}(8.85\times10^{-12}\ \mathrm{F/m})(10^5\ \mathrm{V/m})^2(0.10\ \mathrm{m})^3$$
$$= 4.4\times10^{-5}\ \mathrm{J}$$

$$U_B = u_B V_0 = \frac{B^2}{2\mu_0} V_0 = \frac{(1.0\ \mathrm{T})^2(0.10\ \mathrm{m})^3}{2(4\pi\times10^{-7}\ \mathrm{T\cdot m/A})} = 398\ \mathrm{J}$$

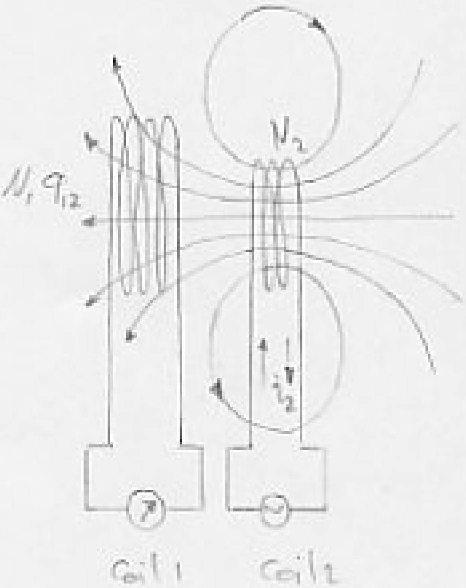
In terms of fields normally available in the lab., <u>much</u> larger amounts of energy can be stored in a mag. field than in an electric field one.

33-7 Mutual Induction:



(a)

If we change i in one coil (with time), an emf given by Faraday's law appears in the second coil. We call this process induction, or more precisely mutual induction, to distinguish it from self-induction.



(b)

Def. - Mutual inductance M_{21} of coil 2 with respect to coil 1 is

$$M_{21} = \frac{N_2 \Phi_{21}}{i_1} \qquad (1)$$

compare this with $L = \frac{N\Phi}{i}$, the def. of self-inductance.

$$(1) \rightarrow \quad M_{21}\, i_1 = N_2 \Phi_{21} \quad \rightarrow \quad M_{21} \frac{di_1}{dt} = N_2 \frac{d\Phi_{21}}{dt}$$

$$\text{Since } \mathcal{E}_2 = -N_2 \frac{d\Phi_{21}}{dt} \quad \rightarrow \quad \mathcal{E}_2 = -M_{21} \frac{di_1}{dt}$$

Compare this with $\mathcal{E} = -L \frac{di}{dt}$

Now we interchange the role of coil 1 and coil 2 (Fig. b):

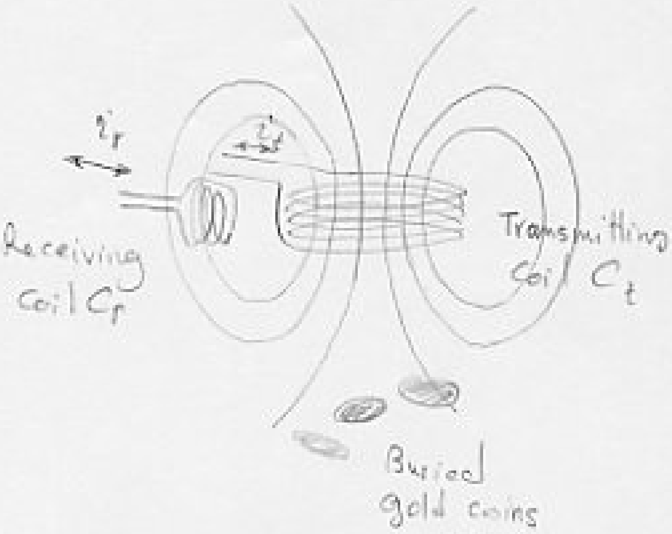
$$\rightarrow \mathcal{E}_1 = -M_{12}\frac{di_2}{dt}$$

But $M_{21} = M_{12} = M$ (without proof)

$$\rightarrow \quad \mathcal{E}_2 = -M\frac{di_1}{dt}$$

$$\mathcal{E}_1 = -M\frac{di_2}{dt}$$

A Metal Detector:



i_t: Sinusoidally current

i_r: Produced by changing φ' which is in turn produced by the currents in metal coins

$C_r \perp C_t$

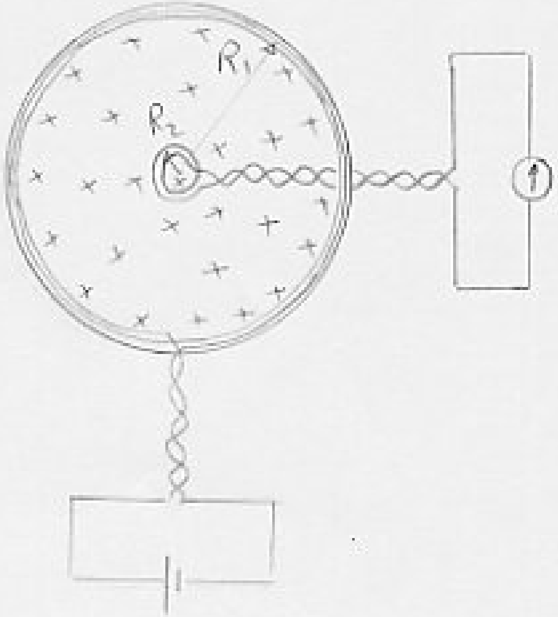
$\rightarrow$ No emf in C_r by φ

changing $i_t \rightarrow$ changing flux φ produced by $C_t \rightarrow$ changing i_c in coins $\rightarrow$ changing flux φ' by the currents of coins

$\rightarrow$ changing i_r in C_r.

Ex.

Consider two circular close-packed coils, the smaller (radius R_2, with N_2 turns) being coaxial with the larger (radius R_1, with N_1 turns) and in the same plane.



a) Derive an expression for the coeff. of mutual inductance M for this arrangement ($R_1 >> R_2$)

sol.

We had for a current loop: $B(z) = \dfrac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}$

$$B(0) = \frac{\mu_0 i}{R} \longrightarrow B_1 = \frac{\mu_0 i_1 N_1}{R_1}$$

Since $R_1 >> R_2$, we may take B_1 to be the mag. field at all points within the boundary of the smaller coil. The number of flux linkages for the smaller coil is then:

$$N_2 \phi_{21} = N_2 (B_1)(\pi R_2^2) = \frac{\pi \mu_0 N_1 N_2 R_2^2 i_1}{2 R_1}$$

Comparing with $M_{21} = M = \dfrac{N_2 \phi_{21}}{i_1}$

$$\rightarrow M = \frac{N_2 \phi_{21}}{i_1} = \frac{\pi \mu_0 N_1 N_2 R_2^2}{2 R_1}$$

b – What is the value of M for $N_1 = N_2 = 1200$ Turns, $R_2 = 1.1$ cm, $R_1 = 15$ cm?

Sol.

$$M = \frac{\pi (4\pi \times 10^{-7}\, H/m)(1200)(1200)(0.011 m)^2}{2(0.15 m)} = 2.29 \times 10^{-3}\, H$$

Consider the situation if we reverse the roles of the two coils, that is if we set up a current i_2 in the smaller coil and try to calculate M from

$$M_{12} = \frac{N_1 \Phi_{12}}{i_2}$$

The calculation of Φ_{12} is not simple.

But if we solve it numerically, we get $M_{12} = M_{21}$.