# Chapter 25
# Gauss' law

25-1

Gauss' law in electrostatic is equivalent to Coulomb's law but it tremendously simplifies the problem whenever there exists a symmetry.

## 25-2 What Gauss' law is all about

Gauss' law relates the fields at a Gaussian surface and the charges enclosed by the surface.

Gaussian Surface: is a hypothetical closed surface.

## 25-3 Flux

$\Phi$: Volume flow rate

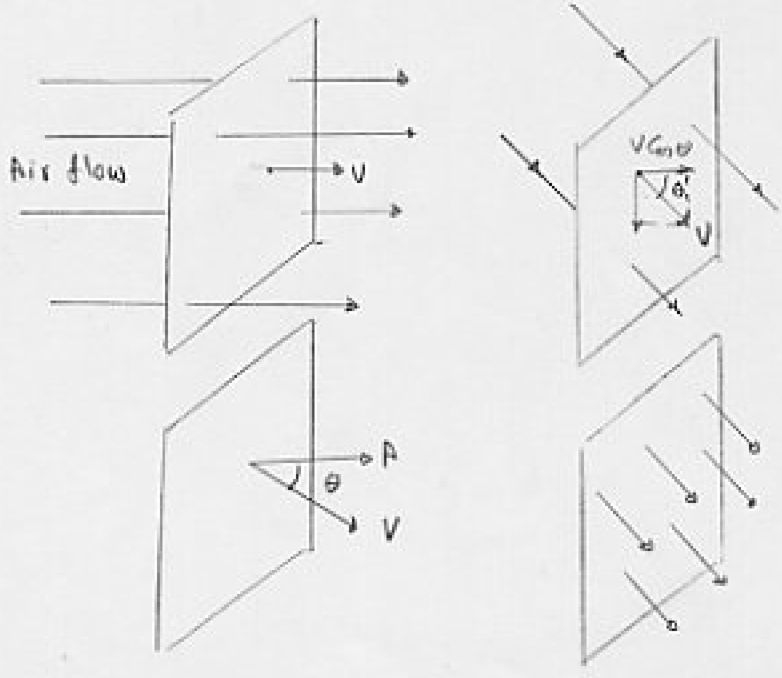
(i.e. volume per unit time)

If $V \perp$ plane

$\longrightarrow \Phi = VA$

If $V \parallel$ Plane

$\longrightarrow \Phi = 0$

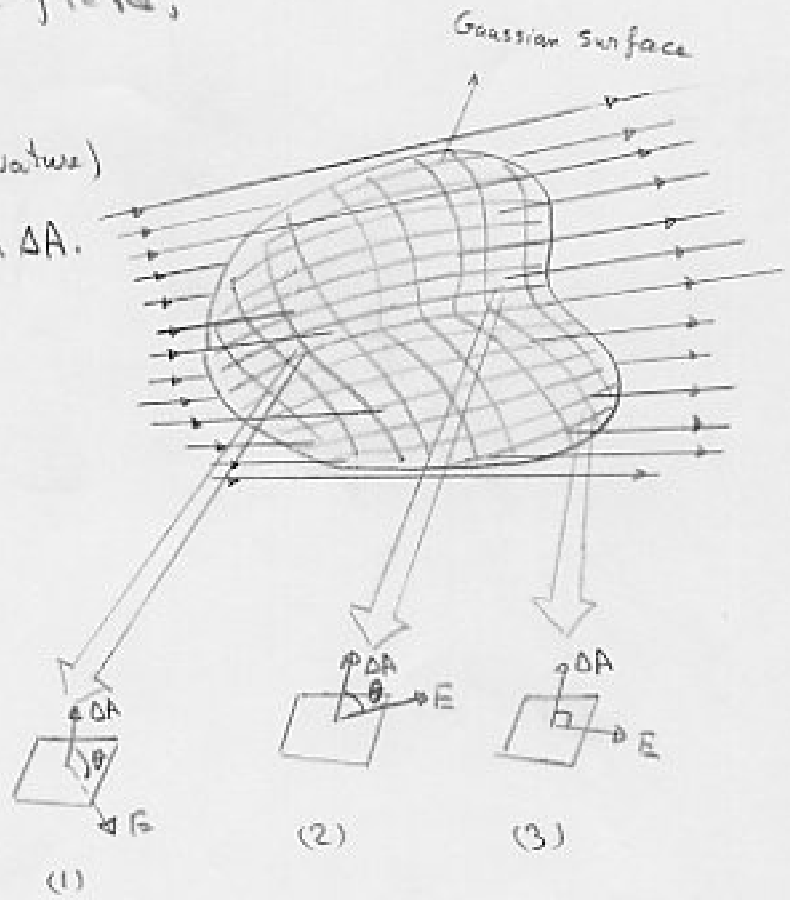


If V and Plane have an angle $\theta$:

$$\Phi = (V\cos\theta)\,A$$

$$\rightarrow \Phi = VA\cos\theta = \bar{V}\cdot\bar{A}$$ <u>Flux</u> ( A is $\perp$ to Plane )

## 25-4 Flux of the Electric field;




$\Delta A$: Small area (without curvature)
We assume E remains const. on $\Delta A$.

$$\Phi = \sum E\cdot\Delta A$$

$\Delta A \rightarrow dA$

$\Sigma \rightarrow \int$

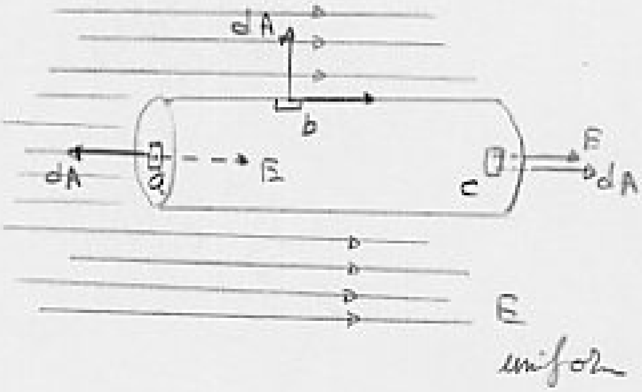
$$\Phi = \oint E\cdot dA$$ electric flux through a Gaussian surface

unit: $N.m^2/C$

| Square | $\theta$ | Sign of $E.\Delta A$ |
|---|---|---|
| 1 | $>90$ | $-$ |
| 2 | $<90$ | $+$ |
| 3 | $=90$ | 0 |

Ex

$\Phi = ?$ through the cylinder




Sol.

$$\Phi = \oint E \cdot dA$$

$$\Phi = \int_a E \cdot dA + \int_b E \cdot dA + \int_c E \cdot dA$$

$$\int_a E \cdot dA = \int E(\cos 180)\, dA = -E \int dA = -EA$$

$$\int_c E \cdot dA = \int E(\cos 0)\, dA = E \int dA = EA$$

$$\int_b E \cdot dA = \int E(\cos 90)\, dA = 0$$

$$\Phi = -EA + 0 + EA = 0$$

25-5 Gauss' Law

Gauss' law relates the (total) flux $\Phi$ of an electric field through a closed surface (a Gaussian surface) to the net charge $q$ that is enclosed by that surface.

$$\epsilon_0 \Phi = q \qquad \text{Gauss' law}$$

$\epsilon_0$: permittivity const. $\quad \epsilon_0 = 8.85 \times 10^{-12}\ C^2/N.m^2$

$$\Phi = \oint E \cdot dA \longrightarrow \epsilon_0 \oint E \cdot dA = q \quad \text{(Gauss' law)}$$

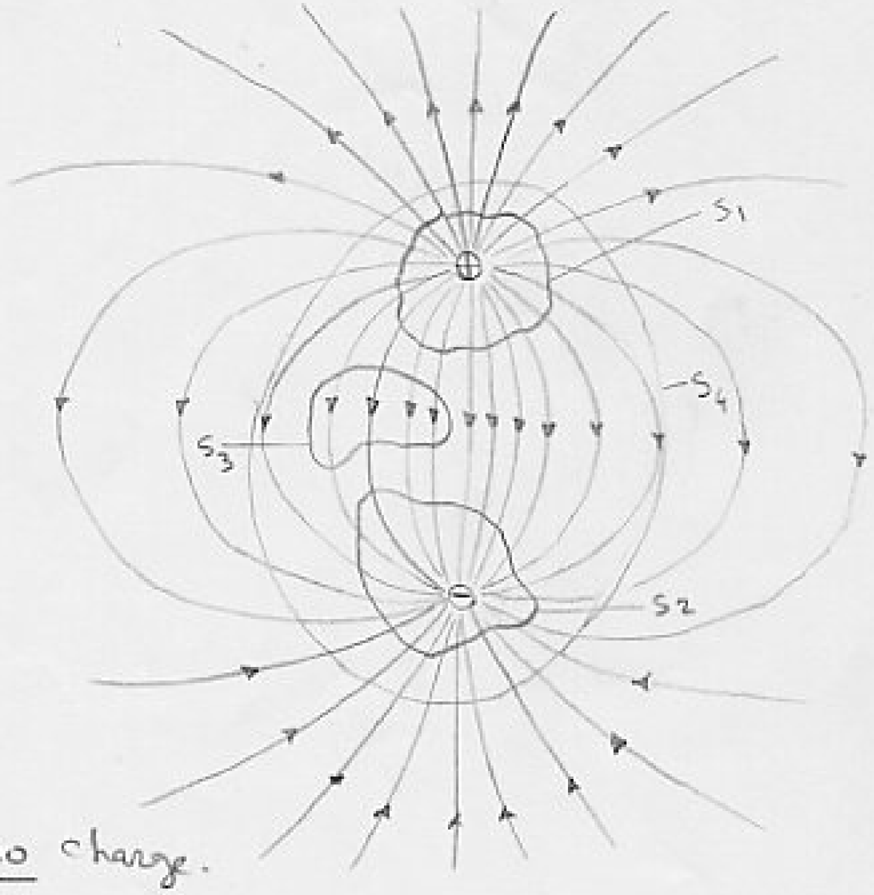
$q$: net charge enclosed by the Gaussian surface.

$q > 0 \longrightarrow \Phi$: outward
$q < 0 \longrightarrow \Phi$: inward

Charge <u>outside</u> the surface <u>is not</u> included in the term $q$ in Gauss law.

The exact <u>form</u> or <u>location</u> of the charges inside the Gaussian surface is also of <u>no</u> concern.

<u>Ex.</u>




S1: The electric field outward $\longrightarrow \Phi > 0$
$\xrightarrow{\exists}$ net charge $q > 0$

S2: The electric field inward $\longrightarrow \Phi < 0$
$\xrightarrow{\exists}$ net charge $q < 0$

S3: This surface contains <u>no</u> charge.
$q = 0 \rightarrow \Phi = 0$ (Comming $E$ = leaving $E$)

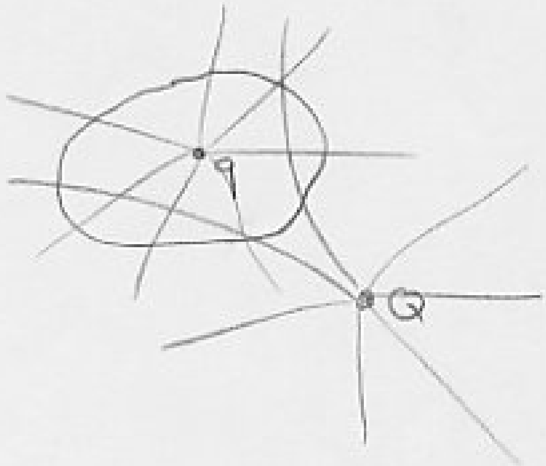
S4: This surface encloses no <u>net</u> charge ($+q - q = 0$)
$\rightarrow \Phi = 0$ (Comming $E$ = leaving $E$)

If we bring charge Q up close to $S_4$ (not inside), the pattern of the field lines changes, but the net flux for the four surfaces would not change.

$$\epsilon_0 \oint E \cdot dA = q$$

$\downarrow$

field of both q and Q



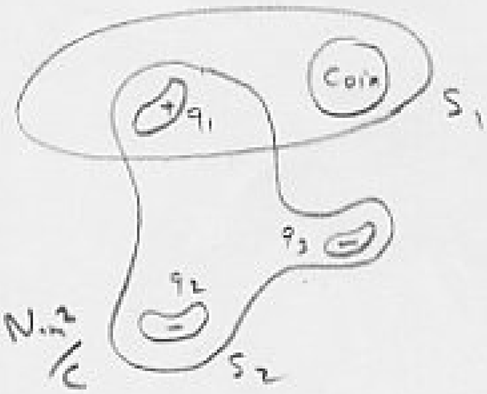

Ex.

$q_1 = +3.1\ nC$ (Can = neutral)

$q_2 = -5.9\ nC$

$q_3 = -3.1\ nC$

$\Phi_{S_1} = ?\ \Phi_{S_2} = ?$




Sol.

$$\Phi_{S_1} = \frac{q_1}{\epsilon_0} = \frac{+3.1 \times 10^{-9}\ C}{8.85 \times 10^{-12}\ C^2/N.m^2} = +350\ N.m^2/C$$

+sign $\longrightarrow$ $\begin{cases} \text{enclosed charge is positive} \\ \text{the flux through the surface is outward} \end{cases}$

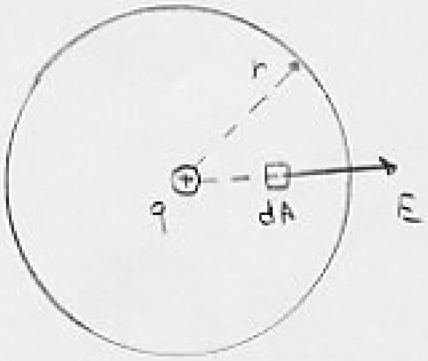
$$\Phi_{S_2} = \frac{q_1 + q_2 + q_3}{\epsilon_0} = -670\ N.m^2/C$$

## 25-6 Gauss' law and Coulomb law

If Gauss' law $\underset{\text{equivalent}}{\sim}$ Coulomb law

$\longrightarrow$ One can be derived from the other.

Point charge:




$$\epsilon_0 \oint E \cdot dA = \epsilon_0 \oint E\, dA \cos(0) = q$$

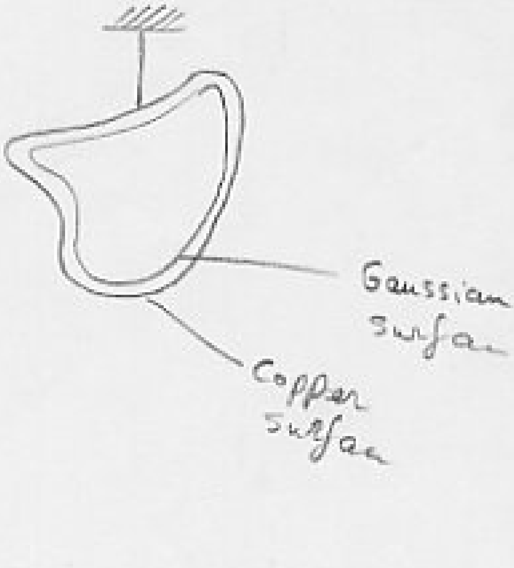
E = const. on the sphere

$$\rightarrow \epsilon_0 E \oint dA = q \rightarrow \epsilon_0 E(4\pi r^2) = q$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

## 25-7 A charged isolated Conductor

If an excess charge is placed on an isolated conductor, the charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of conductor.



Proof: Inside the Gaussian surface $E=0$

otherwise we would have currents. (there is no such a current except at the moment of adding charge to the conductor which quickly gets its equilibrium state)

$E=0 \longrightarrow q=0$ (inside the surface)

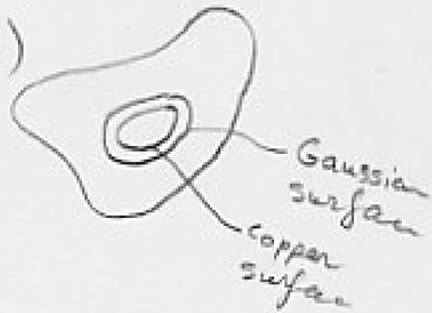
$\longrightarrow$ the charge must be distributed on the surface.

This is <u>natural</u>, because like charges repel each other and get far away from each other.

An isolated Conductor with a Cavity:

There is <u>no</u> charge on the inner surface.

Proof: Since $E=0$ (inside the conductor)




$\longrightarrow$ there can be <u>no</u> flux through this surface.

$\longrightarrow$ From Gauss law, this surface can enclose <u>no net</u> charge

$\longrightarrow$ There is <u>no</u> charge on the cavity wall.

The conductor removed!

Suppose by some way, the excess charge could be frozen into the position on the conductor surface.

Distribution of charge on the surface is not uniform in general

Now we remove the conductor by enlarging the cavity.

$\xrightarrow{\text{still}}$ $\underline{E} = 0$ for inner region

$\underline{E}$ = the same as before for outer region

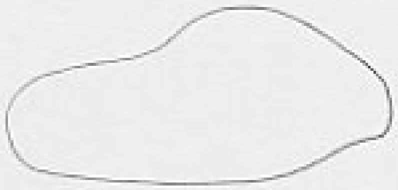
$\xrightarrow{\text{This shows}}$ $\underline{E}$ is set up by the charges and not conductor

The conductor only provides an initial pathway for the charges to take their positions.

The External Electric Field:

spherical conductor having excess charge Q

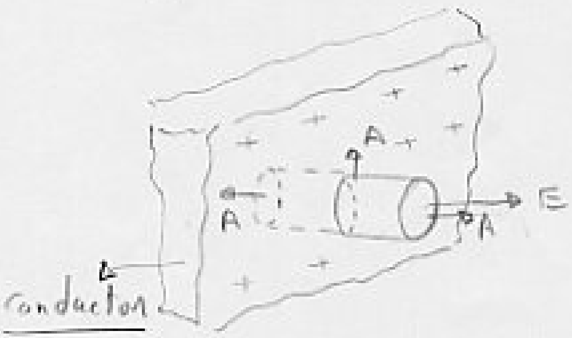
$\sigma$ = const.

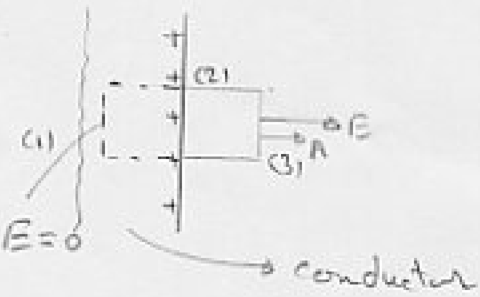


conductor having excess charge Q

$\sigma = \sigma(r)$

However in both cases it is easy to determine $\bar{E}$ just outside surface.




$\epsilon_0 \oint E \cdot dA = q$ $\quad\quad \int_1 E \cdot dA + \int_2 E \cdot dA + \int_3 E \cdot dA = q/\epsilon_0$

$\int_1 = 0 \quad (E=0) \quad\quad \int_2 = 0 \quad (E \perp A)$

$\int_3 = \int E\, dA \cos 0 = E \int dA = EA$

$\oint E \cdot dA = EA \quad\quad EA = \frac{q}{\epsilon_0} \quad\quad E = \frac{q/A}{\epsilon_0} = \frac{\sigma}{\epsilon_0}$

Just outside surface: $E \perp A$

otherwise E would have a component <u>along</u> the conductor surface

surface that would exert forces on the surface charges, causing them to move. This is in contradiction with our basic assumption (electrostatic equilibrium).

Why there is no field inside the conductor?

Answer: The configration of charges inside is in such a way $\Sigma \vec{E} = 0$ inside

Ex.

$|E| = 150\,N/C$ just above the surface of the Earth (downward)

$Q = ?$ total net surface charge

Assume the Earth as conductor with uniform charge density.

Sol.

$$\sigma = \epsilon_0 E = (8.85\times10^{-12}\,C^2/N.m^2)(150\,N/C) = 1.33\times10^{-9}\,C/m^2$$

$$|Q| = (4\pi R^2)\sigma = 4\pi(6.37\times10^{6}m)^2(1.33\times10^{-9}\,C/m^2) = 6.8\times10^{5}\,C$$

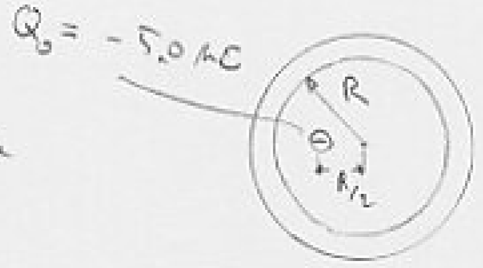
Since $\vec{E}$ is downward $\longrightarrow$ $Q = -6.8\times10^{5}\,C$

Alternatively:

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} \qquad \left(= \frac{\sigma}{\epsilon_0}\frac{4\pi R^2}{4\pi R^2}\right)$$

Point charge concentrated at the center.

Ex.

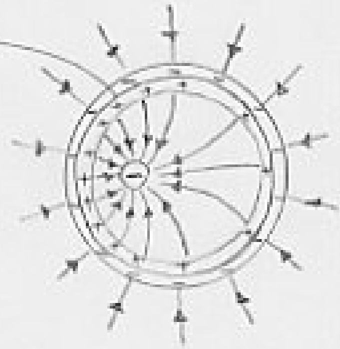



$Q = ?$ induced charge on inner surface

$Q' = ?$ " " " outer "

The distribution of charge ?

Field pattern outside the shell ?




spherical metal (electrically neutral)

Sol.

$E_{\substack{\text{inside}\\ \text{metal}}} = 0 \longrightarrow E = 0$ on Gaussian surface

$\rightarrow \epsilon_0 \oint E \cdot dA = q \qquad 0 = q$

$q = Q_0 + Q \rightarrow Q = +5.0\,\mu C$

Since total charge of metal $= 0 \rightarrow Q' = -5.0\,\mu C$

Since negative charge located at $R/2$ is off-center

$\longrightarrow$ the distribution of inner induced charge is <u>not</u> uniform.

On outer surface the distribution <u>is uniform</u>.

(because the metal is spherical and the skewed distribution of the positive charge on inner surface <u>cannot</u> produce an electric field in the shell to affect the outer distribution)

25-8 A Sensetive Test of Coulomb's Law

If an excess charge on an insolated conductor does not move entirely to the conductor's surface

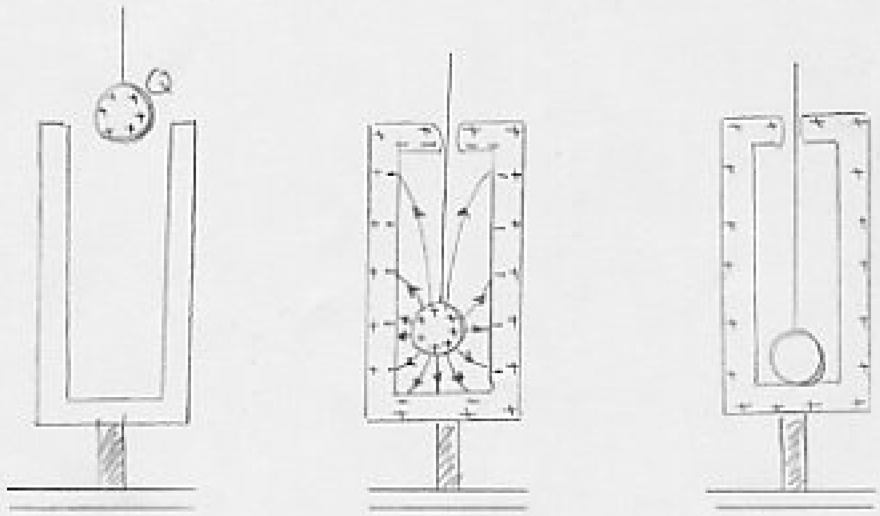
⟶ the Gauss's law can not be true $\left\{ \begin{array}{l} \text{i.e. inside conductor} \\ E = 0, \text{ but } q \neq 0! \\ \underset{0}{\epsilon_0 \oint E \cdot dA} = \underset{\neq 0}{q} \end{array} \right.$

⟶ the Coulomb's law cannot be true

In particular the exponent 2 in the inverse square law might not be exactly 2.

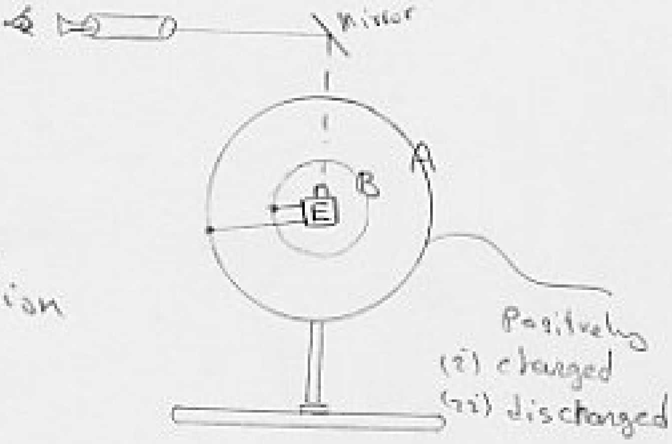
$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^{2 \pm \delta}}$$

( consider the connection between Coulomb's law and Gauss' law and the role of $r^2$ )



No charge on the ball with a high accuracy conductor spheres A and B are connected via the electrometer E.

When A is positively charged and later discharged, no deflection is recorded in either cases.




A modern and more precise apparatus

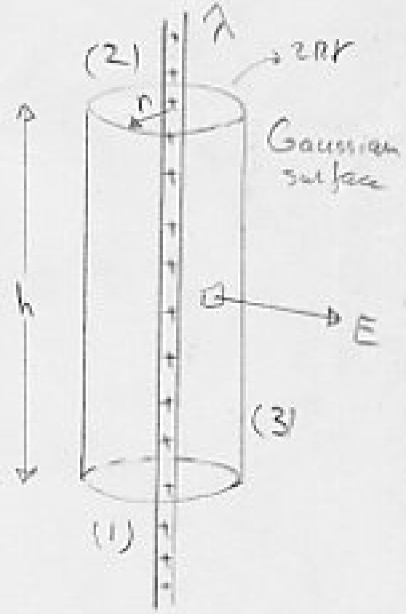
$\longrightarrow \delta < 3.0 \times 10^{-16}$

25-9. Gauss' Law; Sylindrical symmetry

An infinitely long charged cylindrical plastic rod.

Uniform charge density $= \lambda$

$E = ?$




$\epsilon_0 \oint E \cdot dA = q$

$\int_1 E \cdot dA + \int_2 E \cdot dA + \int_3 E \cdot dA = q/\epsilon_0$

$0 + 0 + \int_3 E \cdot dA = \frac{q}{\epsilon_0}$     ($E \perp A$ on (2) and (1))

$E \int dA = \frac{q}{\epsilon_0}$     $E(2\pi r h) = \frac{q}{\epsilon_0}$     $q = \lambda h$

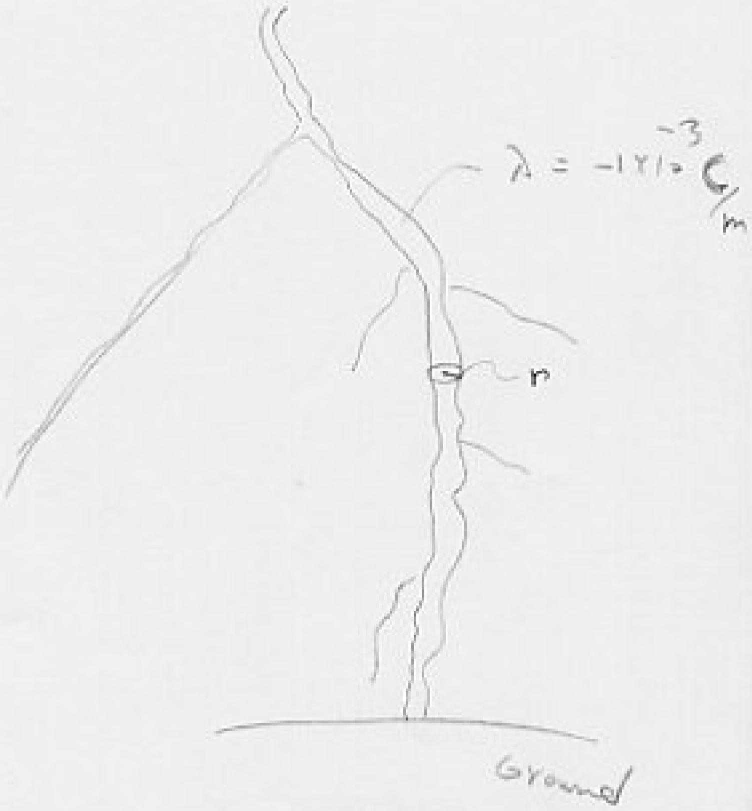
$E = \frac{\lambda}{2\pi \epsilon_0 r}$ (outward)

Ex.

Lightning strike:

Air molecules are ionized in an electric field exceeding $3 \times 10^6$ N/C

$r = ?$ radius of brilliant flash




Sol.

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$r = \frac{\lambda}{2\pi\epsilon_0 E} = \frac{1 \times 10^{-3}\ C/m}{(2\pi)(8.85 \times 10^{-12}\ C^2/N \cdot m^2)(3 \times 10^6\ N/C)} = 6m$$

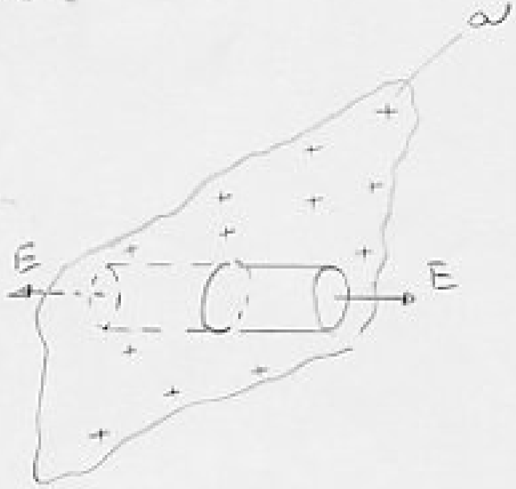
25-10 Gauss. law; Planar symmetry

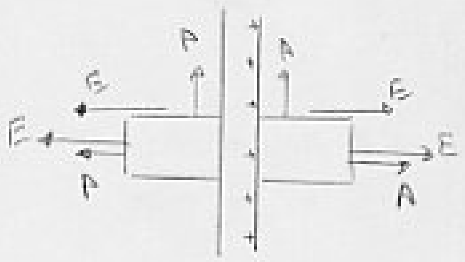
Noncondubting Sheet

$$\epsilon_0 \oint E \cdot dA = q$$

$$\epsilon_0 (EA + EA) = \sigma A$$

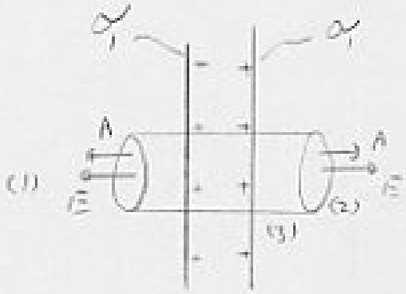
$$E = \frac{\sigma}{2\epsilon_0}$$




an infinite insulating sheet with uniform charge density σ

## Conducting plate

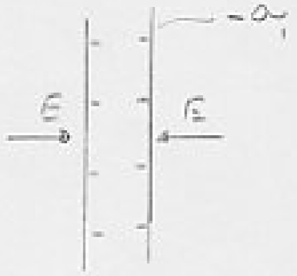



$$\epsilon_0 \oint E \cdot dA = q \qquad \int_1 + \int_2 + \int_3 = q/\epsilon_0$$

$$EA + EA + 0 = \frac{A(2\sigma_1)}{\epsilon_0}$$

$$E = \frac{\sigma_1}{\epsilon_0}$$

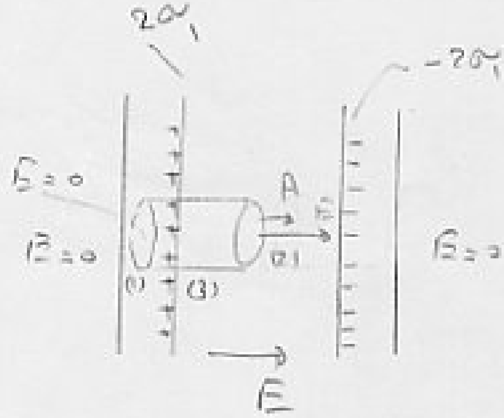
similarly $E = \dfrac{-\sigma_1}{\epsilon_0}$




$$\epsilon_0 \oint E \cdot dA = q$$

$$\int_1 + \int_2 + \int_3 = q/\epsilon_0$$

$$0 + EA = \frac{A(2\sigma_1)}{\epsilon_0} \qquad E = \frac{2\sigma_1}{\epsilon_0}$$




new distribution of charges

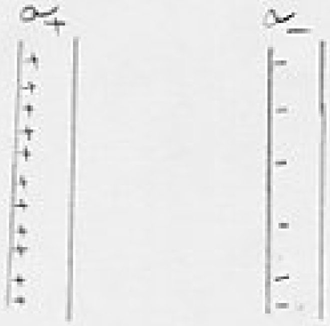
### Ex.

Two large nonconducting sheets;

$\sigma_+ = 6.8\ \mu C/m^2$

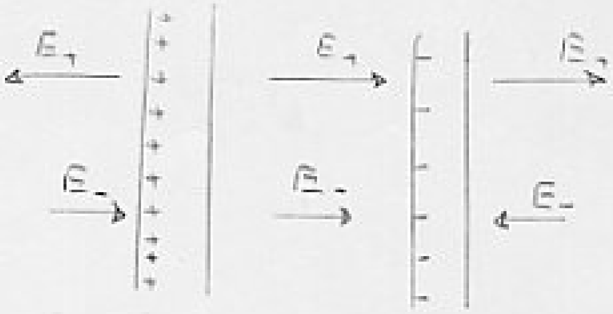
$\sigma_- = 4.3\ \mu C/m^2$

$E_I = ?$ $E_{II} = ?$ $E_{III} = ?$

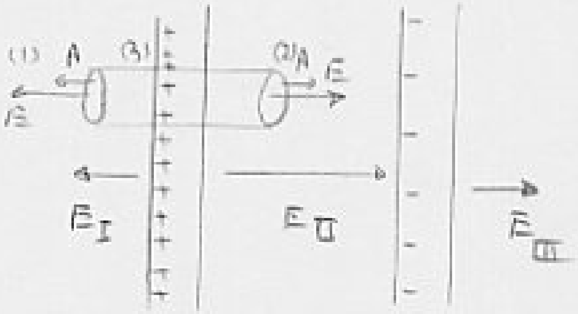



### Sol.

Since the charges are fixed (their distributions don't change under the influence of each other) we can use superposition rule.



$$\oint E \cdot dA = \frac{q}{\epsilon_0}$$




$$\int_1 + \int_2 + \int_3 = \frac{q}{\epsilon}$$

$$EA + EA + 0 = \frac{A\sigma_+}{\epsilon_0} \qquad E_+ = \frac{\sigma_+}{2\epsilon_0}$$

Similarly $E_- = \frac{\sigma_-}{2\epsilon_0}$

$$E_+ = \frac{\sigma_+}{2\epsilon_0} = \frac{6.8 \times 10^{-6}\ C/m^2}{2(8.85 \times 10^{-12}\ C^2/N.m^2)} = 3.84 \times 10^5\ N/C$$

$$E_- = \frac{\sigma_-}{2\epsilon_0} = 2.43 \times 10^5\ N/C$$

$$E_I = E_+ - E_- = 1.4 \times 10^5\ N/C$$

$$E_{II} = E_+ + E_- = 6.3 \times 10^5\ N/C$$

$$E_{III} = E_- - E_+ = -1.4 \times 10^5\ N/C$$

25-11 Gauss law; Spherical Symmetry

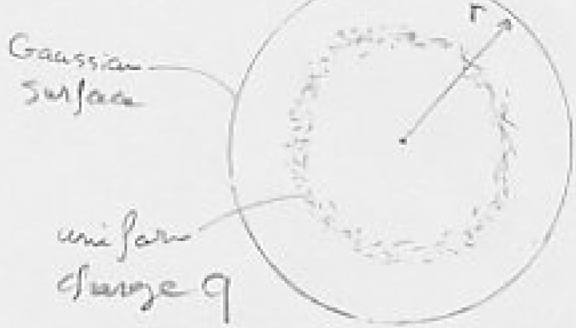
Using Gauss law we prove the two shell theorems:

?) A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell.

ii) A shell of uniform charge exerts no electrostatic force on a charged particle that is located inside the shell.

Proof:

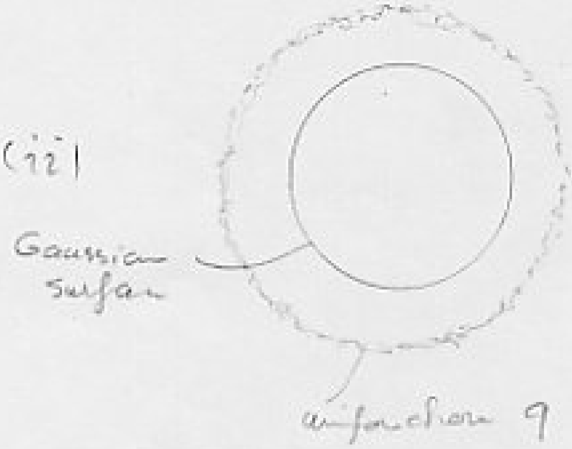
(i)




$$\oint E \cdot dA = \frac{q}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \quad E = \frac{q}{4\pi\epsilon_0 r^2}$$

(ii)




$$\oint E \cdot dA = 0 \rightarrow E = 0$$

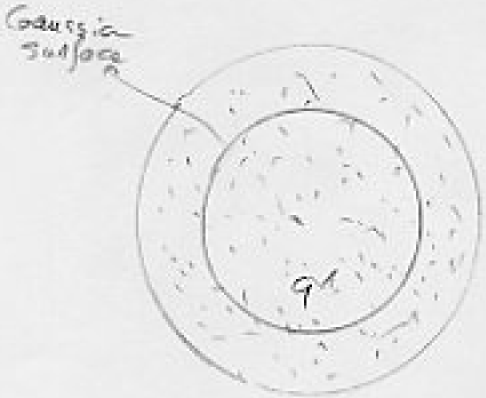
Ex.

Nucleus of an atom of gold:

$R = 6.2 \times 10^{-15}$ m  $q = Ze$ $(Z = 79)$

Assume the charge is uniformally distributed

plot $|E|$ from the center of the nucleus outward to a distance $2R$.




Sol.

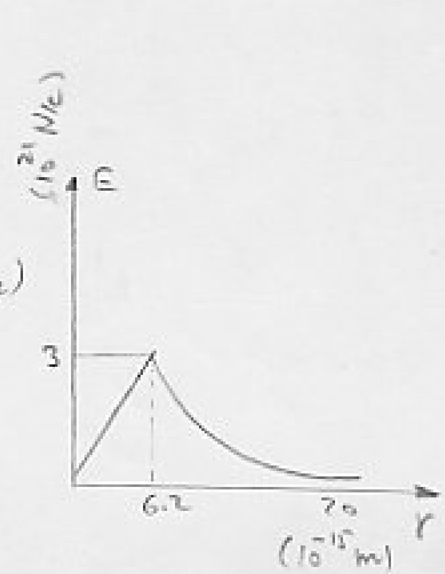
$$q = Ze = (79)(1.60 \times 10^{-19}\,C) = 1.264 \times 10^{-17}\,C$$

$$\frac{q'}{\frac{4}{3}\pi r^3} = \frac{q}{\frac{4}{3}\pi R^3} \rightarrow q' = q\frac{r^3}{R^3}$$

$$\oint E \cdot dA = q'/\epsilon_0 \rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{q'}{r^2} = \left(\frac{q}{4\pi\epsilon_0 R^3}\right) r \text{ (inside)}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2} = 3.0 \times 10^{21}\ N/C$$

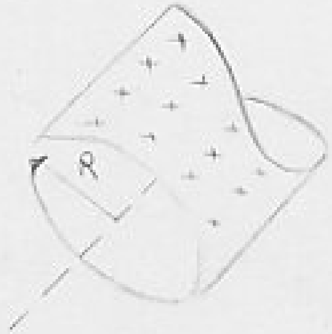
$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{outside } r > R$$



23P — charge density $= \lambda$

$E = ?$

Sol.



long thin metal Tube

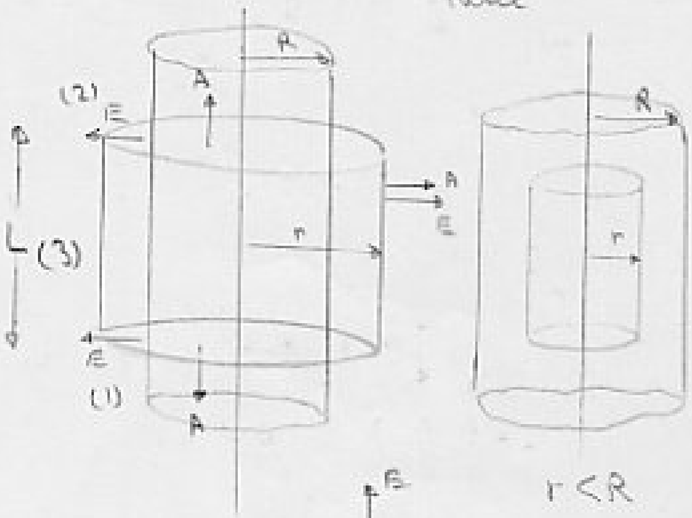
a) $r < R$ $\quad \epsilon_0 \oint E \cdot dA = q$

$q = 0 \longrightarrow E = 0$ for $r < R$

b) $r > R$ $\quad \oint = \int_1 + \int_2 + \int_3$

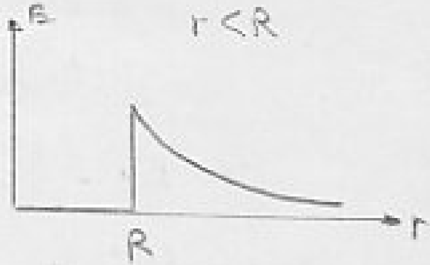
$\int_1 = \int_2 = 0 \quad (E \perp A)$

$\int_3 E \cdot dA = E \int dA = E(2\pi r L)$




$\epsilon_0 E (2\pi r L) = q \qquad q = L\lambda$

$E = \dfrac{\lambda}{2\pi\epsilon_0} \dfrac{1}{r}$




24P — |Charge density| $= \lambda$

$E = ?$

Sol.


nonconductor

a) for $r < a$, similar to (23P)

b) " $a < r < b$, " " "

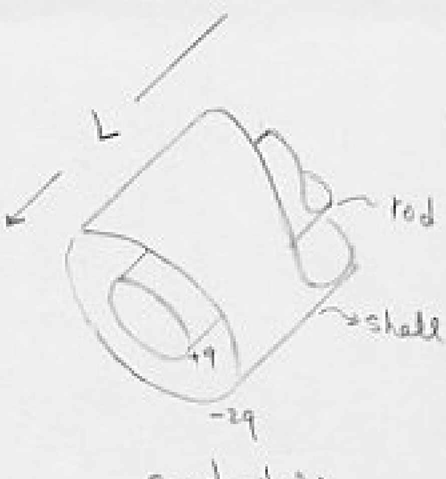
c) " $r > b$ $\quad \oint E \cdot dA = q \qquad q = L(\lambda - \lambda) = 0$

$\longrightarrow E = 0$

27P - a) $E = ?$ outside the shell

b) Distribution of charge on shell

c) $E = ?$ between the rod and shell




Sol.

a) $q_{tot} = q - 2q = -q$ for a Gaussian surface outside the shell

$\epsilon_0 \oint E \cdot dA = q \quad \rightarrow \epsilon_0 E (2\pi r L) = -q \quad E = \dfrac{-q}{2\pi \epsilon_0 L} \dfrac{1}{r}$ (inward field)

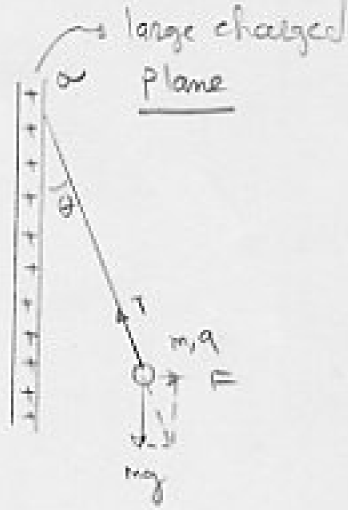
b) As shown in the fig.


c) $\epsilon_0 \oint E \cdot dA = q \quad \epsilon_0 E (2\pi r L) = +q$

$E = \dfrac{q}{2\pi \epsilon_0 L} \dfrac{1}{r} \qquad a < r < b$

34P - $\theta = 30$, $m = 1.0\,mg$, $q = 2.0 \times 10^{-8}\,C$

$\sigma = ?$ of the sheet




Sol.

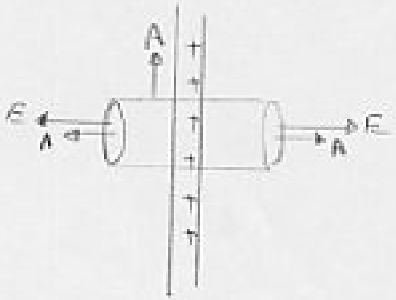
$\bar{F} + m\bar{g} + \bar{T} = 0$

$\rightarrow \tan\theta = \dfrac{F}{mg} = \dfrac{qE}{mg} = \dfrac{q \frac{\sigma}{2\epsilon_0}}{mg}$

$\sigma = \dfrac{2 m g \epsilon_0 \tan\theta}{q}$

$\sigma = \dfrac{2 \times 10^{-6} \times 9.8 \times 8.85 \times 10^{-12} \tan 30°}{2 \times 10^{-8}}$

$\sigma = 5.0 \times 10^{-9}\,C/m^2$

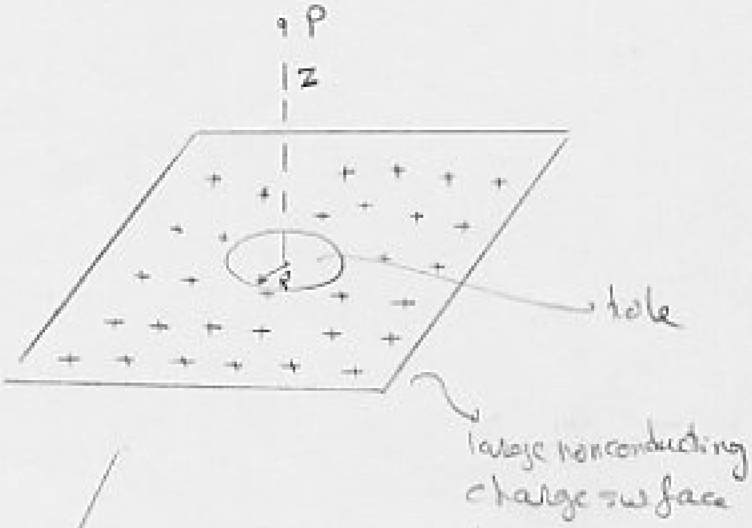



$E = \dfrac{\sigma}{2\epsilon_0}$

33E - Charge density = $\sigma$
Ignore fringing of field
lines around all edges.

$E_p = ?$ (superposition)

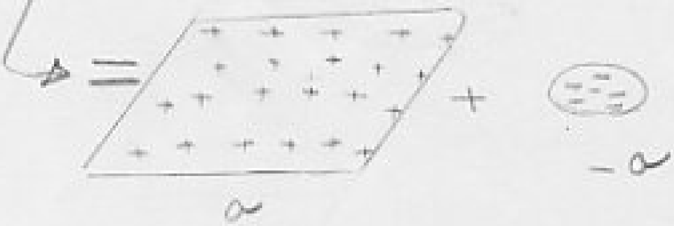
Sol.




We have obtained for a
charged disk:

$$E = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

$$E_{sheet} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$
$R \to \infty$ $\quad \to \infty$




$$E_{sheet} = \frac{\sigma}{2\epsilon_0}$$

$$E_{hole} = \frac{-\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

$$E = E_{sheet} + E_{hol} = \frac{\sigma}{2\epsilon_0}\frac{z}{\sqrt{z^2+R^2}}$$

48P —

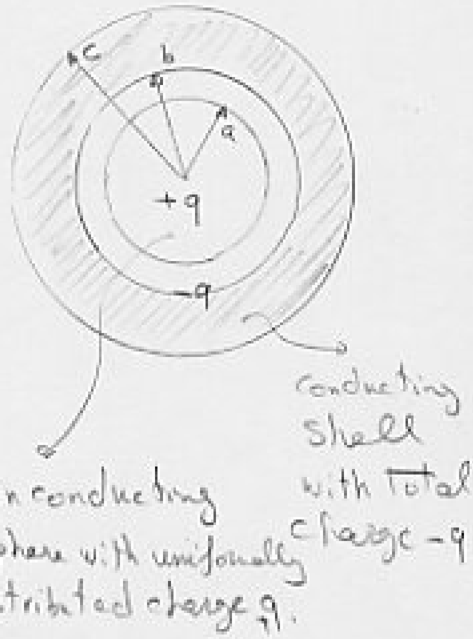
a) $E = ?$ $\quad r < a$

b) $E = ?$ $\quad a < r < b$

c) $E = ?$ $\quad b < r < c$

d) $E = ?$ $\quad r > c$

e) Charge on inner and outer surface of shell.




Sol.

a) $\epsilon_0 \oint E . da = Q$

$$Q = \frac{q}{\frac{4\pi}{3} a^3} \left(\frac{4\pi}{3} r^3\right) = \left(\frac{r}{a}\right)^3 q$$

$$\epsilon_0 (4\pi r^2) E = \left(\frac{r}{a}\right)^3 q \qquad E = \frac{1}{4\pi\epsilon_0} \frac{q}{a^3} r$$

b) $\epsilon_0 (4\pi r^2) E = q \qquad E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$

c) $E = 0$ inside the conductor (since it is conductor)

d) $\oint E . dA = Q \qquad Q = q - q = 0 \Rightarrow E = 0$

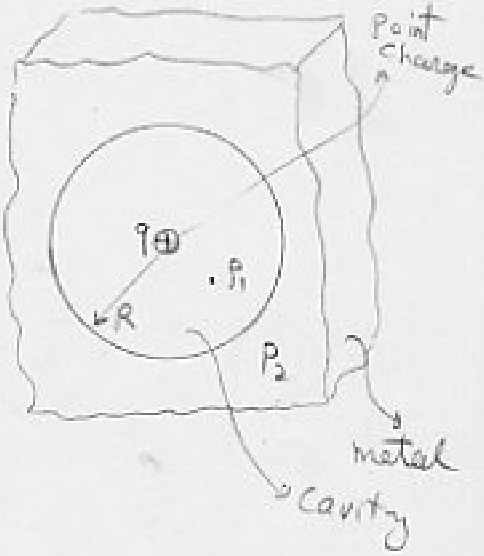
Charge on inner surface $= -q$

" " outer " $= 0$

50P- $E_{P_1} = ?$ $E_{P_2} = ?$

$q = 1.0 \times 10^{-7}\ C$ $R = 3.0\ cm$

$OP_1 = \frac{R}{2}$




Sol.

$\epsilon_0 \oint E \cdot dA = q$ $\epsilon_0 E (4\pi r^2) = q$

$E_{P_1} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$ $E_{P_1} = \frac{1}{4\pi\epsilon_0} \frac{q}{\frac{R^2}{4}} = 4.0 \times 10^6\ N/C$

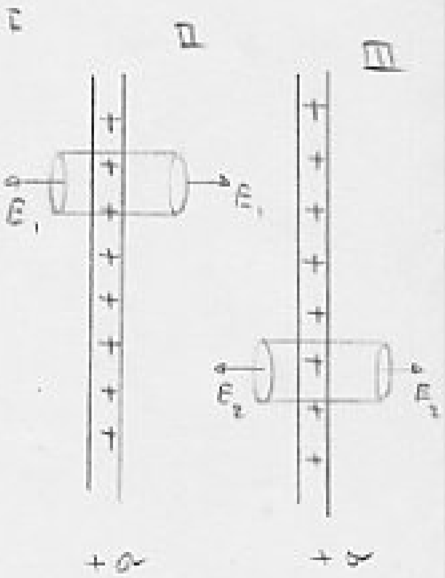
$E_{P_2} = 0$ (since it is conductor)

or since induced charge on inner surface of cavity $= -q$

$\epsilon_0 \oint E \cdot dA = Q$ $Q = q - q = 0 \rightarrow E = 0$

31E- Two large charged nonconducting sheets;

$E = ?$




Sol.

$\epsilon_0 \oint E \cdot dA = q$

$E_1 A + E_1 A + 0 = \frac{q}{\epsilon_0}$ $E_1 = \frac{q}{2A\epsilon_0}$ $E_1 = \frac{A\sigma}{2A\epsilon_0} = \frac{\sigma}{2\epsilon_0}$

similarly: $E_2 = \frac{\sigma}{2\epsilon_0}$

I) $E = E_1 + E_2 = \frac{\sigma}{\epsilon_0}$

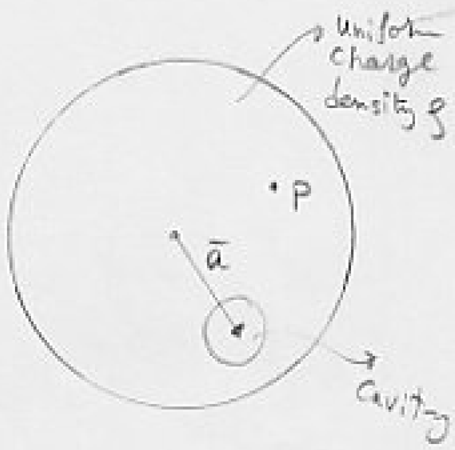
II) $E = E_1 - E_2 = 0$

III $E = E_1 + E_2 = \frac{\sigma}{\epsilon_0}$

54P —

a) Show that $\bar{E}_p = \rho \bar{r}/3\epsilon_0$

b) " " $\bar{E} = \rho \bar{a}/3\epsilon_0$ inside the cavity (uniform field)

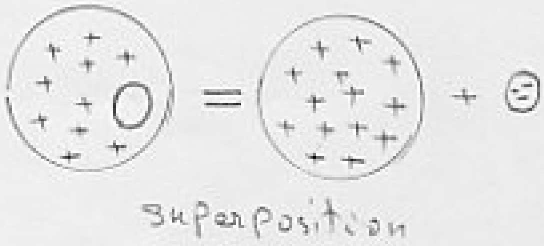



Sol.

a) $\oint E \cdot dA = \frac{q}{\epsilon_0}$ $\quad 4\pi r^2 E = \frac{q}{\epsilon_0}$

$E = \frac{q}{4\pi\epsilon_0 r^2}$ $\quad q = V\rho = \frac{4}{3}\pi r^3 \rho \longrightarrow \bar{E} = \frac{\rho}{3\epsilon_0}\bar{r}$

b)




$\bar{E}_{\oplus} = \frac{\rho}{3\epsilon_0}\bar{r}$

$\bar{E}_{\ominus} = \frac{-\rho}{3\epsilon_0}\bar{r}'$

$\bar{E} = \bar{E}_{\oplus} + \bar{E}_{\ominus} = \frac{\rho}{3\epsilon_0}(r - r')$

$\bar{r} - \bar{r}' = \bar{a}$ $\quad \bar{E} = \frac{\rho}{3\epsilon_0}\bar{a}$