PHY 2506
Problem Set 1
Assigned Oct 18th, Due Nov 4th, 2015

1. Consider the following three distributions

Uniform:

$$p(x) = \begin{cases} \frac{1}{2\sqrt{3}}, & \text{if } -\sqrt{3} \le x \le \sqrt{3} \\ 0 & \text{otherwise} \end{cases}$$

Triangular:

$$p(x) = \begin{cases} \frac{\sqrt{6}+x}{6}, & \text{if } -\sqrt{6} \le x \le 0 \\ \frac{\sqrt{6}-x}{6}, & \text{if } 0 < x \le \sqrt{6} \\ 0 & \text{otherwise} \end{cases}$$

Gaussian:

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\}$$

(a) Plot these curves for x = [-4.0,4.0], with an interval of 0.1

(b) What is the mean and variance for each distribution? Calculate this by hand, not numerically.

(c) Use the intrinsic random number generator in any analysis package that you are familiar with (e.g. MATLAB, IDL, etc) to generate n samples of the uniform pdf. Plot the sample pdf for a number of different values of n. What is a value of n that will produce a good distribution of the pdf?

(d) Repeat Part (c) except for the normal distribution.

2. Filtering properties. In this problem, we are going to examine data assimilation as a form of filtering that is controlled by the error covariance matrices. Start by defining a grid over the interval $(-L_x, L_x]$ with J gridpoints. We will use the homogeneous and isotropic Gaussian correlation function in 1-D:

$$\rho(r = |x-y|) = \exp\left(-\frac{1}{2} \frac{(x-y)^2}{L_d^2} \right).$$

Here r is the distance between two points in the domain and L_d is the decorrelation length. Points in the discrete domain are defined by

$$x_j = j\Delta x$$

where $\Delta x = 2L_x/J$ for $j \in \{-J/2+1,\ J/2\}$ and $L_d = 0.2$. Form the homogeneous, isotropic correlation matrix $\mathbf{Q}$ using

$$\mathbf{Q}_{ij} = \rho(x_i, y_j).$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{Q}$. (You are free to use MATLAB, Mathematica, or IDL, or any statistical analysis package with which you are familiar.) Plot the eigenvalues as a function of index. Plot the eigenvectors corresponding to the 6 largest eigenvalues, as a function of x. Draw a zero-line and count the number of zero crossings for each eigenvectors plotted.

(b) Now compute

$$\mathbf{B} = \mathbf{S}_a\mathbf{K}^T(\mathbf{K}\mathbf{S}_a\mathbf{K}^T + \mathbf{S}_\varepsilon)^{-1} = \mathbf{S}_a(\mathbf{S}_a + \mathbf{S}_\varepsilon)^{-1}$$

where we have assumed $\mathbf{K} = \mathbf{I}$. Here $\mathbf{S}_\varepsilon = (\sigma^r)^2\mathbf{I}$, $\mathbf{S}_a = \mathbf{DQD}$ and $\mathbf{D}$ is a diagonal matrix with each diagonal element equal to σ^b. Let $\sigma^r = 1$ and $\sigma^b = 2$. Compute the eigenvalues and eigenvectors of $\mathbf{B}$ making sure to sort by magnitude. Compare the eigenvalues of $\mathbf{B}$ to $\left(1+\alpha/\lambda_q\right)^{-1}$ where $\alpha = (\sigma^r/\sigma^b)^2$ and λ_q is an eigenvalue of $\mathbf{Q}$. Plot the eigenvectors of $\mathbf{B}$. Are they the same as for $\mathbf{Q}$?

(c) Now let us create the innovations ($\mathbf{z} = \mathbf{y} - \mathbf{Kx}$). To avoid interpolation, place an observation at every grid. To see the filtering aspects, define an innovation by

$$z = \cos(c\pi x)$$

where the x-grid values are from (-1,1]. Parameter c determines the wavenumber and hence the spatial scale of the observation increment. Create an innovation vector and compute the analysis increment using

$$\mathbf{d} = \mathbf{Bz}.$$

Plot $\mathbf{z}$, $\mathbf{d}$ versus x for various values of c. Which waves are filtered the least? For which waves is the amplitude reduced by 80% or more?

(d) How are the filtering properties affected by a change in L_d?

3. Relate the eigenvectors and eigenvalues of the averaging kernel $\mathbf{A}$ to the singular vectors and values of the prewhitened Jacobian $\tilde{\mathbf{K}}$. Remember $\mathbf{A}$ may be asymmetric.

4. Using an eigenvector expansion of the averaging kernel **A**, give an interpretation of the following linearization in which the retrieval is expressed using scalar rather than matrix weights.

$$\hat{\mathbf{x}} = \mathbf{x}_a + \mathbf{A}(\mathbf{x} - \mathbf{x}_a) + \mathbf{G}_y \varepsilon_y = (\mathbf{I} - \mathbf{A})\mathbf{x}_a + \mathbf{A}\mathbf{x} + \mathbf{G}_y \varepsilon_y .$$

5. Obtain the two sample retrievals of the vertical profile of CO from the TES instrument. The TES retrieval can be described by the equation

$$\hat{\mathbf{x}} = \mathbf{x}_a + \mathbf{A}(\mathbf{x} - \mathbf{x}_a) = (\mathbf{I} - \mathbf{A})\mathbf{x}_a + \mathbf{A}\mathbf{x}$$

where **A** is the averaging kernel and $\mathbf{x}_a$ is the a priori profile used in the retrieval. For TES, the retrieval is performed with respect to the natural logarithm of the volume mixing ratio (VMR) of the trace gas. Therefore, the quantities $\hat{\mathbf{x}}$, $\mathbf{x}$, and $\mathbf{x}_\mathbf{a}$ should be expressed in terms of the natural log of the VMR. In interpreting the averaging kernels you may find it helpful to know that the TES retrievals are performed in the presence of clouds, but for clouds of large optical depths the retrievals can only extend down to the cloud top.

(a) Plot the first 20 rows of the averaging kernel matrices for the two retrievals of CO. The y-axis should be log-pressure and plot only from 1000 hPa to 10 hPa (with log-pressure on the y-axis, in decreasing order). Calculate the number of independent pieces of information obtained in the retrievals.

(b) Calculate and plot the three leading eigenvectors and eigenvalues of the averaging kernel matrices. How are the eigenvalues related to the number of independent pieces of information in the retrievals that you calculated in part (a)?

(c) Calculate and plot the three leading eigenvectors and eigenvalues of the observation error covariance matrices. What fraction of the total variance is captured by these three modes? How is the first mode different for the two retrievals?

(d) Obtain the a priori profile (mopitt_xa.dat) used in the MOPITT retrievals of CO. Using the equation given above, substitute the a priori used in the two TES retrievals with the MOPITT a priori such that the transformed TES retrievals are given by

$$\hat{\mathbf{x}}_{trans} = \hat{\mathbf{x}} - (\mathbf{I} - \mathbf{A})\mathbf{x}_a + (\mathbf{I} - \mathbf{A})\mathbf{x}_a^{MOP}$$

where $\mathbf{x}_a^{MOP}$ and $\hat{\mathbf{x}}_{trans}$ are the MOPITT a priori profile and the transformed TES CO retrieval, respectively. The trace gas abundances in the retrieval must be expressed as the natural log of the VMR and that 1 ppb is a VMR of 1×10^{-9}. For each of the TES retrievals, plot the original retrieved and a priori CO profile (only plot the levels between the surface and 10 hPa), the MOPITT a priori, and the transformed retrieved CO profile that you obtained using the MOPITT a priori. Comment on the differences between the original and transformed retrieved profiles.