Scattering of Sunlight

A simple introductory model
- to compare properties of absorption and scattering.

Recall the defn of optical depth with absorption coefficient k_a and scattering coefficient k_s:

$$d\chi = -(k_a + k_s)\rho\, dz$$

Following the reasoning used in developing the two-stream:
- scattered radiation will not be lost from the system, but will change direction
- since (in two-stream) the only directions are up and down, some scattered radiation remains in the beam and some joins the opposite beam

For the downward flux density:

$$dF^{\downarrow} = +F^{\downarrow}(k_a + k_s)\rho\, dz^* - \pi B k_a \rho\, dz^*$$
$$- f F^{\downarrow} k_s \rho\, dz^* - (1-f) F^{\uparrow} k_s \rho\, dz^*$$

where
fraction f continues in the same direction
fraction $1-f$ changes direction (by 180°)

Introduce the <u>single scattering albedo</u>

$$SSA = \tilde{\omega} = \frac{k_s}{k_a + k_s} = \begin{cases} 0 & \text{for no scattering} \\ 1 & \text{for all scattering} \end{cases}$$

Then we can substitute in $d\chi^*$ and $\tilde{\omega}$ and rearrange to get:

$$\frac{dF^{\downarrow}}{d\chi^*} + F^{\downarrow} = \pi B(1-\tilde{\omega}) + \tilde{\omega}\left[f F^{\downarrow} + (1-f)F^{\uparrow}\right]$$

Similarly, we can show that:

$$-\frac{dF^{\uparrow}}{d\chi^*} + F^{\uparrow} = \pi B(1-\tilde{\omega}) + \tilde{\omega}\left[f F^{\uparrow} + (1-f)F^{\downarrow}\right]$$

The easiest case to solve is for $\tilde{\omega} = 1$ (i.e., a cloud) for which

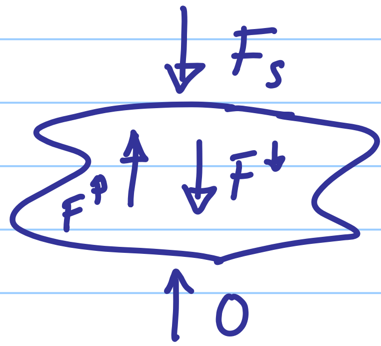
$$\frac{dF^{\downarrow}}{d\chi^*} + (F^{\downarrow} - F^{\uparrow})(1-f) = 0$$

and

$$\frac{dF^{\uparrow}}{d\chi^*} + (F^{\downarrow} - F^{\uparrow})(1-f) = 0$$

so that $F^{\downarrow} - F^{\uparrow} = \text{constant}$

Now, for a cloud of thickness χ^*_0, the downward solar flux density on the cloud is F_s and the upward solar flux density on the bottom of the cloud is 0.



Therefore, we can solve the equations for albedo A and transmission τ (from Lecture 7 - see slide 37):

$$A = \frac{\chi_0^* (1-f)}{1 + \chi_0^* (1-f)} \qquad \tau = \frac{1}{1+\chi_0^*(1-f)}$$

For isotropic scattering (equal in both directions), $f = 1/2$, so $\tau = \dfrac{1}{1+\chi_0^*/2}$

Compare with $\tau = \exp(-\chi_0^*)$ for absorption, which decays faster.

χ_0^*	scattering	absorption
0	1.00	1
1	0.67	0.37
2	0.50	0.14
5	0.27	0.007
10	0.17	4×10^{-5}
20	0.09	0
50	0.04	0
100	0.02	0

We can also solve the equations for non-zero absorption:

$$A = \beta \frac{1 - \exp(-2\alpha \chi_0^*)}{1 - \beta^2 \exp(-2\alpha \chi_0^*)}$$

$$\tau = \frac{(1-\beta^2)\exp(-\alpha \chi_0^*)}{1 - \beta^2 \exp(-2\alpha\chi_0^*)}$$

where

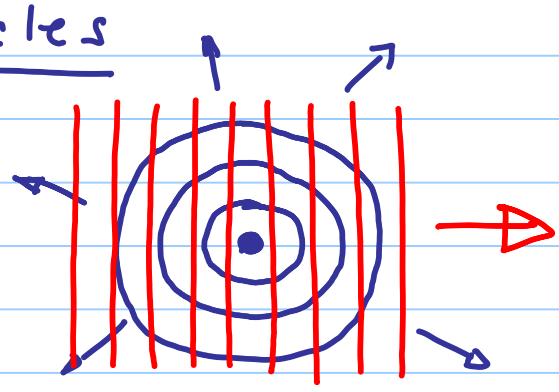
$$\beta = \frac{\alpha - 1 + \tilde{\omega}}{\alpha + 1 - \tilde{\omega}}, \quad \alpha = (1-\tilde{\omega})(1+\tilde{\omega}-2f)$$

This shows that a small absorption has a large effect because the scattering multiplies the effective path length of each ray significantly.

Thus
- "a little bit of scattering goes a long way"
- then "a little bit of absorption becomes significant".

Scattering from Particles

Consider a plane wave of intensity I_0 incident on a particle.



After the interaction,

1. the plane wave continues almost unchanged except for the loss of some energy
2. a scattered wave will be generated, centred on the particle.

When the particle size and λ are small wrt viewing distance (common in the atmosphere), the scattered wave can be considered to be a spherically expanded wavefront centred on the particle.

The scattered wave can be described by

$$\frac{I_o}{r^2} \; \frac{f(\theta,\phi)}{k^2}$$

where

$k = \dfrac{2\pi}{\lambda}$ = wave vector for radiation of wavelength λ (not k_a, or k_s or Boltzmann's constant!)

$\dfrac{I_o}{r^2}$ term $\Rightarrow$ shows that the radiation is proportional to input intensity and is spherically expanding

$\dfrac{f(\theta,\phi)}{k^2}$ term $\Rightarrow$ contains the physics of the scattering process

Integrate the scattered intensity over the surface of any sphere surrounding the particle to get the total energy scattered $= I_0 C_{sca} = I_0 k_s$

with

$$k_s = \frac{1}{k^2} \int F(\theta, \phi)\, d\omega$$

$=$ scattering cross section (area)

We can also define

$C_{abs} = k_a =$ absorbing cross section

$C_{ext} = k_e = k_a + k_s$ = extinction cross section

k_s represents the amount of incident energy which is removed from the original direction due to a single scattering event such that the energy is redistributed isotropically over the area of a sphere whose centre is the scatterer.

The efficiencies corresponding to these cross sections are

$$Q = \frac{C}{\text{Area}_{\text{particle}}} = \frac{C}{\pi r^2} \quad \text{for a sphere}$$

$$Q_s = \frac{k_s}{\pi r^2} \quad \text{from Lecture 11}$$

Rayleigh Scattering (Rayleigh, 1871)

This is the simplest form of scattering, but is very important in the atmosphere. It applies when $\lambda \gg$ particle diameter.

$\therefore$ For visible light, it is restricted to molecular scattering.
For radio waves, it can also apply to droplets and particles for which $\lambda \gg d$.

Under this assumption, the incident electric field of the radiation, $\vec{E}_0$, induces a dipole moment in the particle:

$$\vec{p} = \alpha \vec{E}_0$$

where

α = polarizability of the particle

Dimensions: $p \sim$ charge × length
$E_0 \sim$ charge / area
$\alpha \sim$ volume

The oscillating dipole in turn produces a plane-polarized EM wave, the scattered wave, at the same frequency as $\vec{E}_0$.

The total electric field (incident + emitted) is given by classical EM theory as

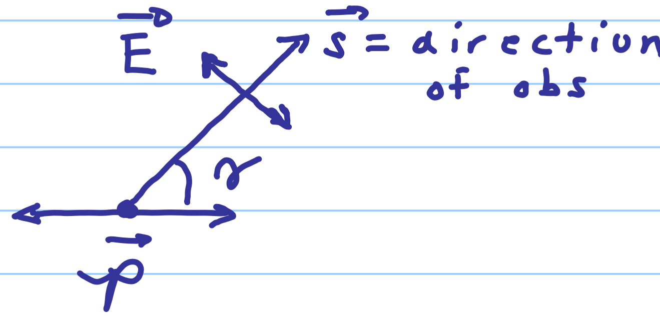
$$\vec{E} = -\vec{E}_0 e^{-ik(r-ct)} \frac{k^2\alpha}{r} \sin\gamma$$

where

γ = angle between $\vec{p}$ and the direction of observation

r = distance between the dipole and the observer

$k = 2\pi/\lambda$



Now consider the scattering of unpolarized sunlight.

Let the plane defined by the direction of the incident and scattered waves be the reference plane.

The perpendicular and parallel components are:

$$(\perp)\quad E_r = -E_{0r}\, e^{-ik(r-ct)} \frac{k^2\alpha}{r} \sin\gamma_1$$

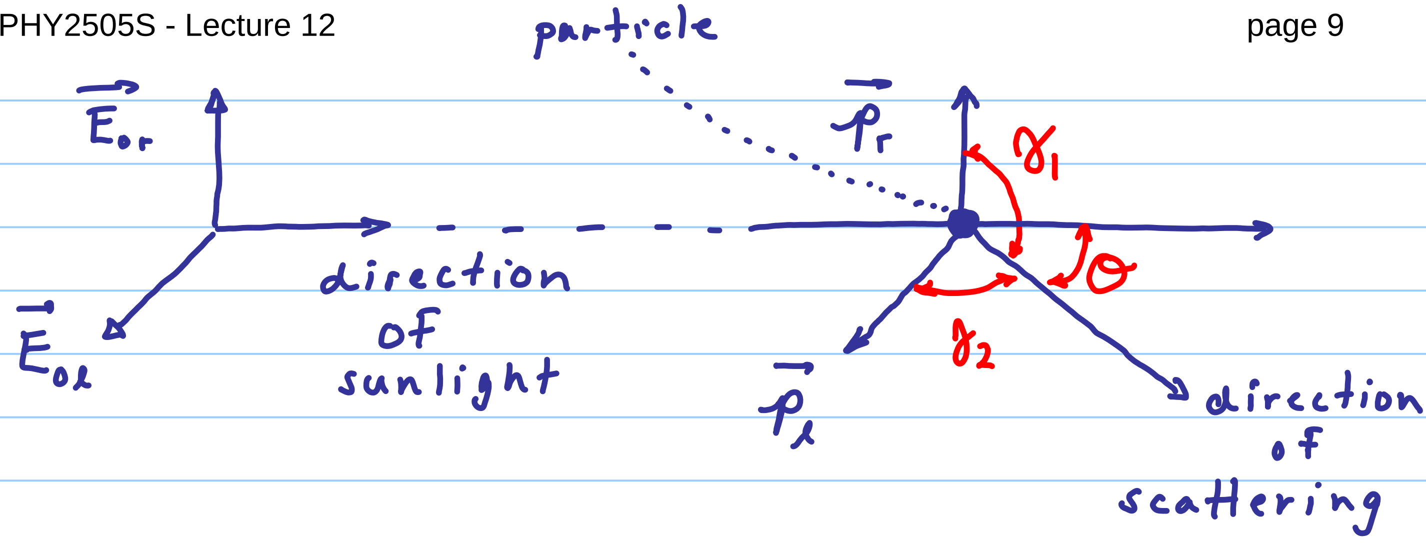
$$(\parallel)\quad E_l = -E_{0l}\, e^{-ik(r-ct)} \frac{k^2\alpha}{r} \sin\gamma_2$$

where

$\gamma_1 = \pi/2$, $\quad \gamma_2 = \pi/2 - \Theta$

and

Θ = scattering angle between incident and scattered waves



γ_1 has to be 90° because the scattered dipole moment $\vec{p}_r$ (and $\vec{E}_r$) in the r direction is $\perp$ to the reference plane.

The intensities of the incident and scattered light (per unit solid angle) are:

$$I_0 = \frac{1}{\Delta\Omega}\frac{c}{4\pi}|E_0|^2 \quad , \quad I = \frac{1}{\Delta\Omega}\frac{c}{4\pi}|E|^2$$

So the two components of the scattered intensity are:

$$I_r = I_{0r}\, k^4 \alpha^2 / r^2$$

$$I_l = I_{0l}\, k^4 \alpha^2 \cos^2\theta / r^2$$

The total scattered intensity is:

$$I = I_r + I_l = (I_{0r} + I_{0l}\cos\theta)\frac{k^4\alpha^2}{r^2}$$

But for unpolarized sunlight,

$$I_{0r} = I_{0l} = I_0/2$$

Also recall that $k = 2\pi/\lambda$, so we have:

$$I = \frac{I_0}{r^2}\alpha^2\left(\frac{2\pi}{\lambda}\right)^4\frac{1+\cos^2\theta}{2}$$

This is the original <u>Rayleigh scattering</u> formula.

Note the dependence of the scattered intensity on $\frac{1}{r^2}$, I_0, α^2, $\frac{1}{\lambda^4}$, $\cos^2\theta$

The angular distribution of the scattered energy is described using the phase function $P(\cos\theta)$, with

$$\int_0^{2\pi}\int_0^{\pi/2}\frac{P(\cos\theta)}{4\pi}\sin\theta \, d\theta \, d\phi = 1$$

For Rayleigh scattering, $P(\cos\theta) = \frac{3}{4}(1+\cos^2\theta)$ so that

$$I(\theta) = \frac{I_0}{r^2}\alpha^2\,\frac{128\,\pi^5}{3\,\lambda^4}\,\frac{P(\theta)}{4\pi}$$

The scattered flux (energy/time) is

$$f = \int_\Omega \underbrace{(I\,\Delta\Omega)}_{\text{flux density}}\;\underbrace{r^2\,d\Omega}_{\text{area}}$$

integrating scattered flux density $I\,\Delta\Omega$ over area r^2 away from the particle

Substituting for I and Ω gives

$$f = \frac{F_o \alpha^2 \, 128 \pi^5}{3 \lambda^4}$$

where

$F_o = I_o \Delta\Omega =$ incident flux density

Then we can define the <u>scattering cross section</u> per molecule as

$$k_s = \frac{f}{F_o} = \frac{128 \alpha^2 \pi^5}{3 \lambda^4} \qquad (cm^2/molec)$$

This represents the amount of incident energy which is removed from the original direction due to a single scattering event.

Then

$$\boxed{I(\theta) = I_o \frac{k_s}{r^2} \frac{P(\theta)}{4\pi}}$$

This is the general expression for scattered intensity.

The polarizability can be derived from the principle of dispersion of EM waves:

$$\alpha = \frac{3}{4\pi N_s} \left(\frac{m^2 - 1}{m^2 + 2} \right)$$

see, e.g. Liou

where

$N_s =$ number of molecules per unit volume

m = refractive index of molecules
$= m_r + i m_i$
corresponds to
m_r scattering properties (real)
m_i absorption properties (imaginary)

- m_r depends on λ and is usually given by an empirical expression.
- m_i is negligible for this discussion of scattering by air molecules in atmosphere

Since $m_r \approx 1$ for practical purposes, we have

$$\alpha \approx \frac{1}{4\pi N_s}(m_r^2 - 1)$$

and

$$k_s \cong \frac{8\pi^3 (m_r^2 - 1)^2}{3\lambda^4 N_s^2} f(\delta)$$

where
$f(\delta)$ = correction factor for anisotropy of molecules (i.e., refractive index varies in x, y, z directions)

$$f(\delta) = \frac{6 + 3\delta}{6 - 7\delta}$$

with
$\delta = 0.035$ = depolarization factor

The optical depth of the entire molecular atmosphere can be obtained from the scattering cross section:

$$\chi(\lambda) = k_s(\lambda) \int_0^{z_{top}} N(z)\, dz \quad \Big\} \text{ extinction of light due to scattering}$$

where
$N(z)$ = vertical profile of number density
z_{top} = top of the atmosphere

Mie Scattering (Mie, 1908)

Scattering by larger particles of any size and shape can, in theory, be solved using Maxwell's Equations with appropriate boundary conditions. However, in practice, this is difficult for all but simple geometrical shapes.

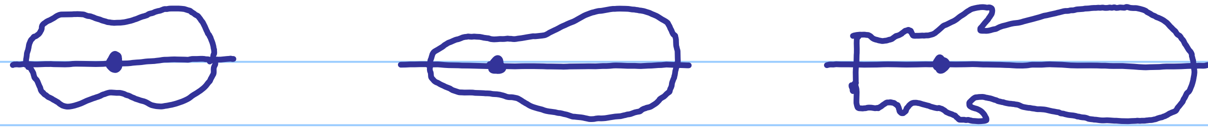
Mie solved the general case for homogeneous spherical particles
→ Mie scattering

Two results of these calculations:

1. Scattering diagrams - show how energy is distributed upon scattering, which is usually strongly forward or backward

scattering phase function P

e.g.



(2) Extinction efficiency factor, Q_s, as a function of the size parameter
$\chi = 2\pi r / \lambda$
$r \rightarrow$ particle radius

For $m_i = 0$, a perfect reflector, there is no absorption.

Q_s shows minima and maxima, and tends to 2 as χ increases
→ see Lecture 11, slides 19-20.

Scattering of Solar Radiation in Plane Parallel Atmospheres

Now let's include scattering effects in radiative transfer -- this is primarily important for solar radiation since UV-VIS λ are scattered more than IR λ.

Need a way to express the source function in Schwarzchild's Equation due to scattering in the beam.

two parts {
- scattering from the solar beam (scattered once)
- scattering from other directions (scattered more than once)

Use a scalar approximation, i.e., ignore polarization, and assume all radiation is given by intensity I.

Energy scattered from direction Ω' into direction Ω is proportional to

$$I(\Omega') \frac{P(\Omega,\Omega')}{4\pi}$$

where $P(\Omega,\Omega')$ describes the angular pattern of scattering, normalized by 4π.

The contribution due to scattering is therefore contained in P.

The scattered energy is then:

$$k_s \frac{dz}{\mu} \int I(\Omega') \frac{P(\Omega,\Omega')}{4\pi} d\Omega'$$

$$+ k_s \frac{dz}{\mu} I_{sun} \Delta\Omega_{sun} \frac{P(\Omega,\Omega_{sun})}{4\pi} e^{-\chi/\mu_{sun}}$$

Insert this into Schwarzchild's Equation to get: (Liou, Eqn. 6.5)

$$\mu \frac{dI(\Omega)}{d\chi} = I(\Omega) - \frac{\tilde{\omega}}{4\pi} \int_{4\pi} I(\Omega') P(\Omega,\Omega') d\Omega'$$

$$- \frac{\tilde{\omega}}{4\pi} I_{sun} \Delta\Omega_{sun} P(\Omega,\Omega_{sun}) e^{-\chi/\mu_{sun}}$$

where $\tilde{\omega} = SSA = \frac{k_s}{k_a + k_s}$ again.

This is the equation to solve. A number of techniques have been developed to do this.

These approaches include:

1. Two-stream model
2. Eddington's Approximation
3. Order of scattering approximation.

but we'll stop here.