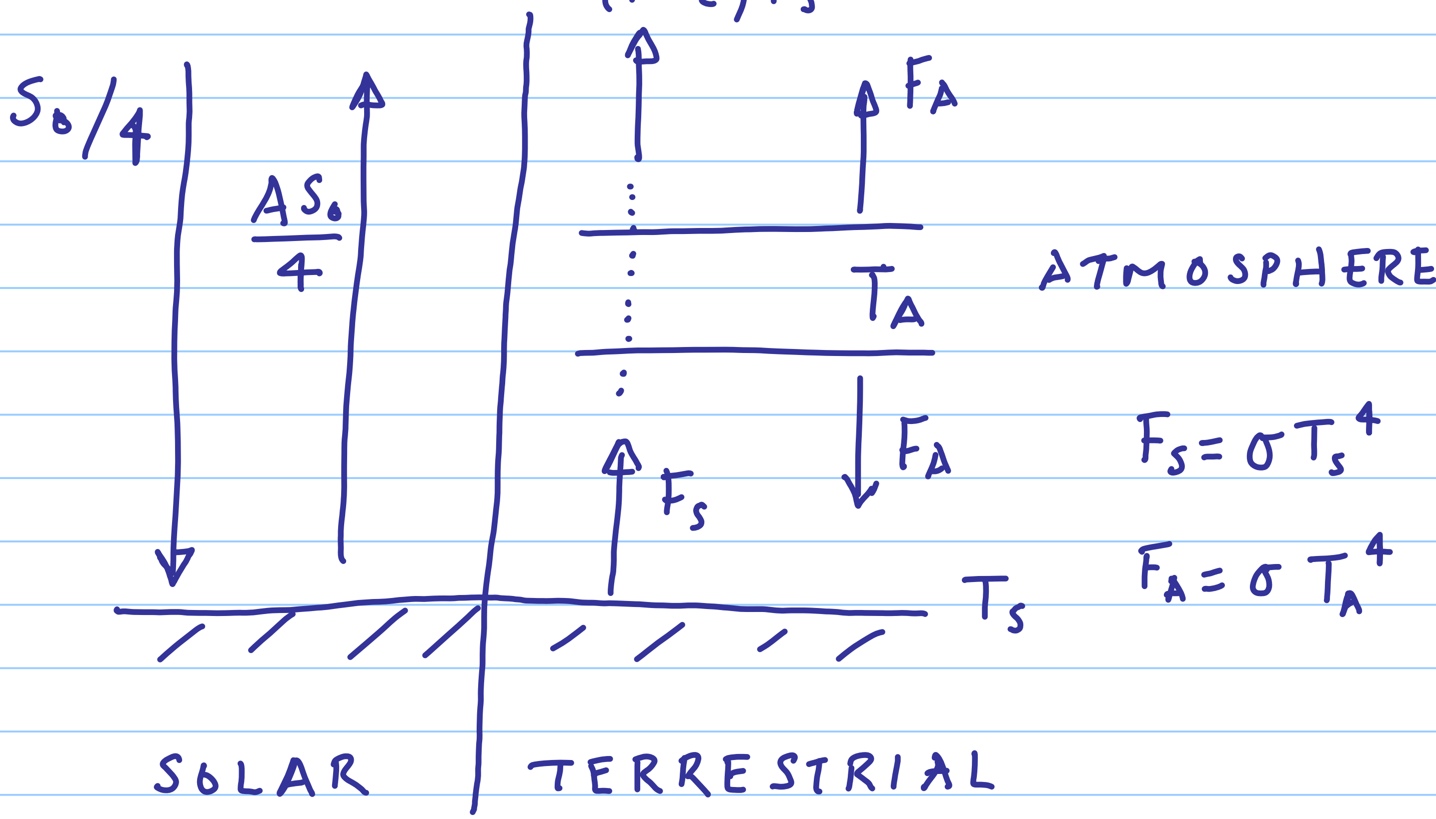
Leaky Greenhouse



$$F_s = \sigma T_s^4$$

$$F_A = \sigma T_A^4$$

ε = emissivity = absorptivity by Kirchoff's Law

= fraction of IR radiation absorbed

$\therefore (1-\varepsilon)$ = fraction of IR transmitted

① At the TOA

$$\frac{S_0}{4} = \frac{AS_0}{4} + (1-\varepsilon)F_s + F_A$$

② At the surface

$$\frac{S_0}{4} + F_A = \frac{A S_0}{4} + F_s .$$

Rearrange:

① $F_A = (1-A)\,S_0/4 - (1-\varepsilon)\,F_s$

② $F_s = (1-A)\,S_0/4 + F_A$

Substitute to eliminate F_A: ① into ②

$$F_s = 2(1-A)\,S_0/4 - (1-\varepsilon)\,F_s$$

$$(2-\varepsilon)\,F_s = 2(1-A)\,S_0/4$$

$$(2-\varepsilon)\,\sigma T_s^4 = 2\sigma T_e^4$$

[because $(1-A)\,S_0/4 = \sigma T_e^4$]

$$T_s = \left(\frac{2}{2-\varepsilon}\right)^{1/4} T_e$$

Transparent atmosphere (IR): $\varepsilon \to 0$
so $T_s \to T_e$.

Opaque atmosphere (IR): $\varepsilon \to 1$
so $T_s \to 2^{1/4}\,T_e$

If $0 < \varepsilon < 1$, then $T_s < 2^{1/4} T_e$

So the partial IR transparency of the atmosphere (leaky greenhouse) reduces the warming effect of the atmosphere.

$T_e = 255\,K$, $T_s = 288\,K$

$\rightarrow$ implies $\varepsilon = 0.77$

Now let's find T_A:

$$F_A = \varepsilon \sigma T_A^4 = \alpha_{IR} \sigma T_A^4$$

Kirchoff's Law

Substitute into eqn ①

$$\varepsilon \sigma T_A^4 = (1-A)\, S_0/4 - (1-\varepsilon) F_s$$

$$= \sigma T_e^4 - (1-\varepsilon) \frac{2\sigma}{2-\varepsilon} T_e^4$$

$$= \left(\frac{2 - \varepsilon - 2 + 2\varepsilon}{2-\varepsilon} \right) \sigma T_e^4$$

$$= \frac{\varepsilon}{2-\varepsilon} \sigma T_e^4$$

$$T_A = \left(\frac{1}{2-\varepsilon}\right)^{1/4} T_e$$

For $\varepsilon < 1$, $T_A < T_e$

and $T_A < T_s$ always.

The atmosphere is always cooler than the surface (leaky greenhouse).

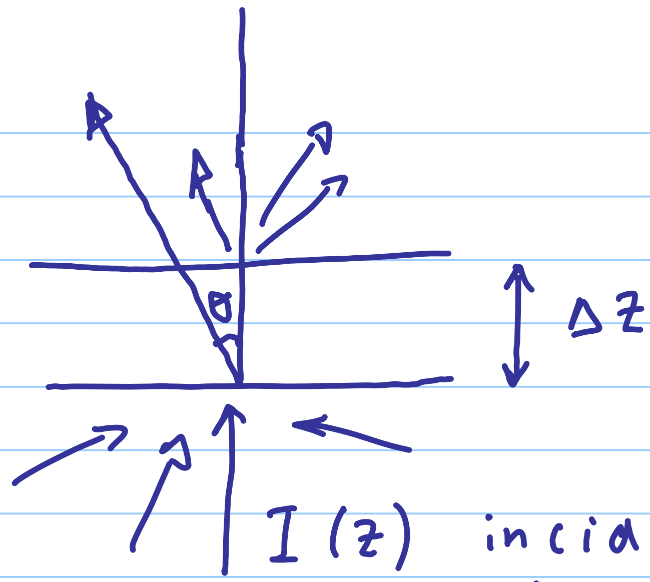
Terrestrial Fluxes

Unlike the solar radiation beam, the terrestrial radiation is a "Flux-like" field in all directions.

Although Schwarzchild's Eqn is valid, the calculation can be complicated because of the required integration over angle.

→ look for simplifications!

Consider an isotropic radiation field incident on the bottom of a thin layer of absorbing material in a plane-parallel atmosphere.



Ignore the emission for now.
Consider the absorption of energy in the layer.

In any particular direction, the <u>transmitted</u> energy is

$$dE = I \cos\theta \, e^{-k\rho\Delta z/\mu} \, d\Omega \, d\bar{\nu} \, dA \, dt$$

The total flux density through the upper surface is then

$$F = \int I \cos\theta \, e^{-k\rho\Delta z/\mu} \, d\Omega$$

$$= \int_0^{2\pi} \int_0^{\pi/2} I \cos\theta \, e^{-k\rho\Delta z/\mu} \sin\theta \, d\theta \, d\phi$$

where $\mu = \cos\theta$

$d\mu = -\sin\theta \, d\theta$

$$F = 2\pi \int_0^{\pi/2} I \mu e^{-k\rho\Delta z/\mu} \underbrace{\sin\theta \, d\theta}_{-d\mu}$$

$$= 2\pi I \int_0^1 e^{-k\rho\Delta z/\mu} \mu \, d\mu$$

$\theta \quad 0 \longrightarrow \pi/2$ but limits get

$\mu \quad 1 \longrightarrow 0$ reversed because

$1/\mu \quad 1 \longrightarrow \infty$ of $-d\mu$

$$F = 2\pi I \int_1^\infty e^{-k\rho\Delta z x} \frac{dx}{x^3}$$

where

$x = 1/\mu$

$\mu = 1/x$

$d\mu = -\frac{dx}{x^2}$

$\mu \, d\mu = -\frac{dx}{x^3}$

$$F = 2\pi I E_3(k\rho\Delta z)$$

where

$E_3(y) = \int_1^\infty e^{-yx} \frac{dx}{x^3} \Rightarrow$ defining equation for exponential integral

However E_3 is not easily integrated or differentiated compared with the exponential which results for vertical intensity ($I = I_0 e^{-k\rho \Delta z}$).

This third exponential integral is similar to

$$\frac{1}{2} e^{-ry} \quad \text{where} \quad r = 5/3$$

→ generally known as the <u>diffusivity factor</u>

i.e., can approximate

$$E_3(y) \approx \tfrac{1}{2} e^{-ry} = \frac{1}{2} e^{-5/3\, y}$$

This substitution allows the flux density to be written in the same form as the intensity:

$$F \approx 2\pi I \left(\frac{1}{2} e^{-5/3\, y} \right)$$

$$F \approx \pi I \exp\left(-\frac{5}{3} k \rho \Delta z \right)$$

transmitted flux density

We can follow the same approach to derive an expression for the emission from the layer.

The emitted flux density is then:

$$F = 2\pi B E_3 (k\rho \Delta z) \simeq \pi B \exp\left(-\frac{5}{3} k \rho \Delta z\right)$$

where we have substituted B for J - valid for lower atmosphere.

This allows us to write an approximate form of Schwarzchild's Equation for Fluxes:

$$\frac{dF}{d\chi^*} = F - \pi B \qquad \left(\text{vs. } \frac{dI}{d\chi} = I - J \right)$$

where

$$\chi^* = -r k \rho \Delta z \rightarrow \text{modified optical depth}$$
$$= -\frac{5}{3} k \rho \Delta z$$

Let's use this equation under these assumptions:

① atmosphere is transparent to solar radiation so that the problem is one of a system heated from below

2. atmosphere is grey, i.e. only has one value for k

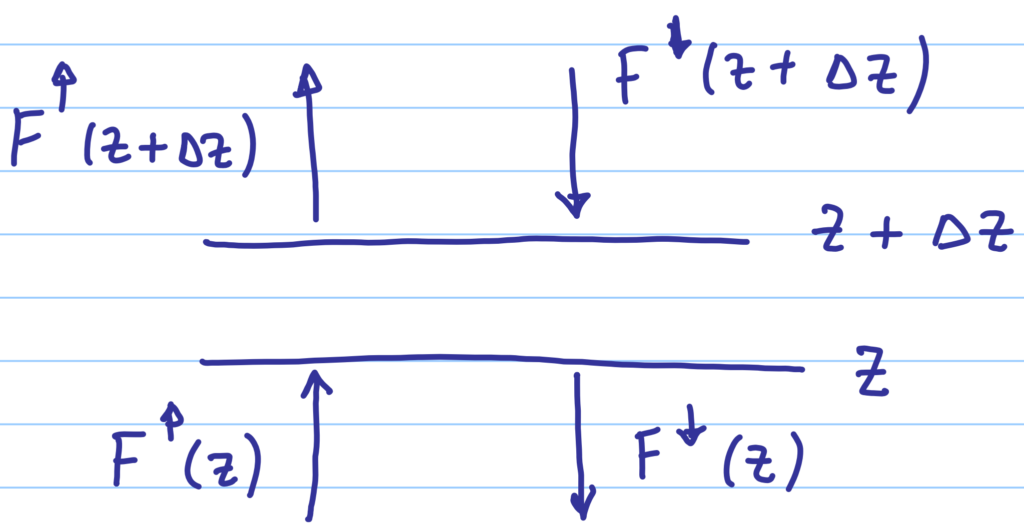
3. atmosphere is in radiative equilibrium at every level, so net heating rate $dT/dt = 0$.

4. atmosphere is plane parallel.

For these conditions we can write a simple balance solution for a layer in terms of

$F^{\uparrow}(z)$ = the upwards flux density

$F^{\downarrow}(z)$ = the downwards flux density



Upward flux density:

$$F^{\uparrow}(z+\Delta z) = F^{\uparrow}(z) + \frac{dF^{\uparrow}(z)}{dz}\Delta z$$

Let's use: $\dfrac{dF}{d\chi^*} = F - \pi B$

$$\Delta F = (F - \pi B)\Delta\chi^*$$

$$\Delta F^{\uparrow} = F^{\uparrow}\Delta\chi^* - \pi B \Delta\chi^*$$

$$= -F^{\uparrow} r\rho k \Delta z$$

$$+ \pi B r\rho k \Delta z$$

$$F^{\uparrow}(z+\Delta z) - F^{\uparrow}(z) = \frac{dF^{\uparrow}}{dz}\Delta z$$

$$= -F^{\uparrow}(z) r k \rho \Delta z + \pi B r k \rho \Delta z$$

So $\dfrac{dF^{\uparrow}}{d\chi^*} = F^{\uparrow} - \pi B$

where $d\chi^* = -rk\rho\, dz$
measured in opposite direction to z, hence negative sign.

Downward Flux density:

$$F^{\downarrow}(z+\Delta z) - F^{\downarrow}(z)$$

$$= F^{\downarrow}(z)\, r\, k\, \rho\, \Delta z - \pi B\, r\, k\, \rho\, \Delta z$$

$$= \frac{dF^{\downarrow}}{dz} \Delta z$$

So: $$-\frac{dF^{\downarrow}}{d\chi^*} = F^{\downarrow} - \pi B$$

We thus have two forms of Schwarzchild's Equation, with the difference due to the choice of axis.

Total energy balance for the layer can be related to the change in flux density and the heating rate.

$$\frac{dF^{\downarrow}(z)}{dz} \Delta z - \frac{dF^{\uparrow}(z)}{dz} \Delta z$$

$$= \rho\, c_p\, \Delta z \frac{dT}{dt}$$

$$\frac{d\,(F^{\downarrow} - F^{\uparrow})}{dz} = \rho c_p \frac{dT}{dt} = 0 \text{ for radiative equilibrium}$$

$$\therefore \quad F^{\uparrow} - F^{\downarrow} = \phi$$

where

ϕ = constant = net upward flux density

Define $\Psi = F^{\uparrow} + F^{\downarrow}$
= total flux density

The two differential equations then become

$$\frac{dF^{\uparrow}}{d\chi^*} = F^{\uparrow} - \pi B$$

$$-\frac{dF^{\downarrow}}{d\chi^*} = F^{\downarrow} - \pi B$$

Add: $\dfrac{d}{d\chi^*}(F^{\uparrow} - F^{\downarrow}) = F^{\uparrow} + F^{\downarrow} - 2\pi B$

Subtract: $\dfrac{d}{d\chi^*}(F^{\uparrow} + F^{\downarrow}) = F^{\uparrow} - F^{\downarrow}$

$$① \quad \frac{d\phi}{d\chi^*} = \psi - 2\pi B = 0$$

$$② \quad \frac{d\psi}{d\chi^*} = \phi = \text{constant}$$

radiative equilibruim

Solve ② to get

$$\psi = \phi \chi^* + c \overset{①}{=} 2\pi B$$

At the top of the atmosphere

$\chi^* = 0$ and $F^{\downarrow} = 0$ because we have assumed no solar flux in the terrestrial IR region.

At TOA: $\psi = c = \phi = F^{\uparrow}$

To maintain the overall energy balance, the outgoing terrestrial flux density must equal total absorbed solar radiation.

So $c = \phi = \sigma T_e^4$

The equations become

$$\phi = F^{\uparrow} - F^{\downarrow} = \sigma T_e^4$$

$$F^{\uparrow} = \frac{\phi}{2}(\chi^* + 2)$$

$$F^{\downarrow} = \frac{\phi}{2}\chi^*$$

$$\pi B = \frac{\phi}{2}(\chi^* + 1)$$

Let's plot these with χ^* on the y axis.