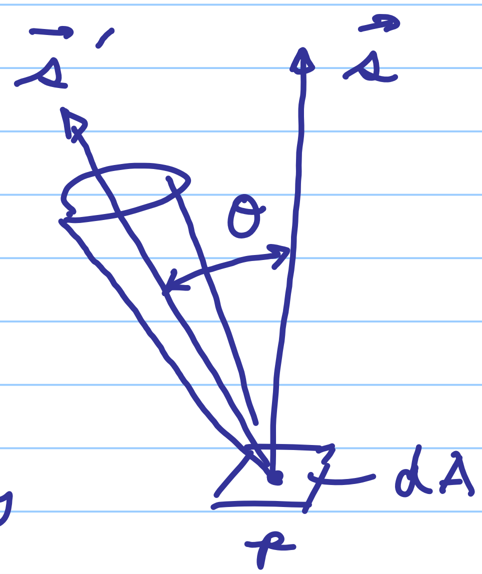
Radiant Flux Density and Radiance

Given plane with normal $\vec{s}$ and small area dA in the plane.
The energy passing through dA in direction $\vec{s}\,'$ is related to intensity $I(p, \vec{s}\,')$.



The area of dA projected onto direction $\vec{s}\,'$ is $\cos(\vec{s}, \vec{s}\,')\, dA$.

The energy passing through dA in direction $\vec{s}\,'$ is then

$$dE_\lambda = \int_\Omega I_\lambda(p, \vec{s}) \cos(\vec{s}, \vec{s}\,')\, d\Omega\, dA\, d\lambda\, dt$$

and the radiant flux density is

$$F_\lambda(p, \vec{s}) = \int_\Omega I_\lambda(p, \vec{s}) \cos(\vec{s}, \vec{s}\,')\, d\Omega$$

This is difficult to integrate unless I_λ is isotropic.

In spherical polar coordinates,

$$\text{angle}\,(\vec{s}, \vec{s}\,') = \theta$$

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$$F_\lambda = \int_\Omega I_\lambda(\theta, \phi) \cos\theta \, d\Omega$$

$$= \int_0^{2\pi} \int_0^{\pi/2} I_\lambda(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi$$

If isotropic $$= I_\lambda \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= 2\pi I_\lambda \left[\frac{\sin^2\theta}{2} \right]_0^{\pi/2} = \pi I_\lambda$$

$\boxed{F_\lambda = \pi I_\lambda}$ for isotropic radiance

We should distinguish between the radiant flux density in one direction and the net flux through area dA.

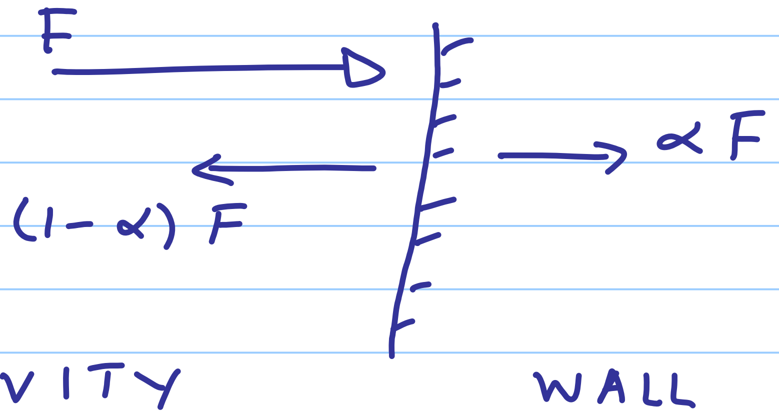
Usually, we are interested in F_λ in one direction, $\vec{s}$ or $\vec{s}\,'$.

However, in an isotropic field, $F(p, \vec{s}) = F(p, \vec{s}\,')$ so the <u>total</u> flux is zero with <u>no</u> net energy transport.

Blackbody Radiation

In our BB cavity, we have $F_\lambda = \pi I_\lambda$

Of this energy, a fraction "α" will be absorbed and a fraction $1-\alpha$ will be reflected.



Because of the isotropic field, the total energy/area emitted by the surface is F.

This requires an emission term εF such that

emissivity ε = absorptivity α
(emittance) (absorptance)

This is Kirchoff's Law : $\boxed{\varepsilon = \alpha}$

This is true for both the total radiation field and for each monochromatic interval.

i.e. $\boxed{\varepsilon_\lambda = \alpha_\lambda}$ or $\boxed{\varepsilon_{\bar{\nu}} = \alpha_{\bar{\nu}}}$

The isotropic field inside the cavity, B, is a function of temperature T. but is independent of cavity material.

$\longrightarrow$ blackbody radiation

If we cut a small opening in the cavity, then the radiation leaving it is isotropic with intensity $I = B$ and radiant flux density $F = \pi I$.

What is the spectral dependence of this BB radiation?

This was first determined experimentally:

$$B_{\bar{\nu}}(T)\, d\bar{\nu} = \frac{c_1 \bar{\nu}^3}{e^{c_2 \bar{\nu}/T} - 1}\, d\bar{\nu}$$

$\hookrightarrow$ blackbody function describing the radiance emitted by a blackbody.

where

$c_1 = 1.1911 \times 10^{-8}\ \mathrm{W\,m^{-2}\,sr^{-1}\,(cm^{-1})^{-4}}$
$\quad =$ First radiation constant

$c_2 = 1.438 \ K\,(cm^{-1})^{-1}$
$= $ second radiation constant

In terms of λ:

$$B_\lambda(T)\,d\lambda = \frac{c_1}{\lambda^5\left(e^{c_2/\lambda T} - 1\right)}\,d\lambda$$

The theoretical derivation of B begins with a consideration of the modes of vibration or density of states of the cavity per unit volume.

Because B is independent of the cavity wall material, we can assume that it consists of dipoles oscillating at λ, ν, or $\tilde{\nu}$ and emitting according to EM theory.

Then the density of states or number of oscillators at wavenumber between $\bar{\nu}$ and $\bar{\nu} + d\bar{\nu}$ per unit volume is given by EM theory as

$$N(\bar{\nu})\, d\bar{\nu} = 8\pi \bar{\nu}^2\, d\bar{\nu}$$

The total energy per unit volume is (density of states) $\times$ $\underbrace{\text{(energy per state)}}_{\text{need to determine}}$.

① First approach : Rayleigh-Jeans which assumes classical thermodynamic equipartition of energy:

average energy per state $= kT$

Then energy density is:

$$E(\bar{\nu})\, d\bar{\nu} = 8\pi \bar{\nu}^2 kT\, d\bar{\nu}$$

This expression is valid at small $\bar{\nu}$ (long λ) but predicts infinite radiation in the cavity at large $\bar{\nu}$, ("the UV catastrophe").

② Second approach : Boltzmann statistics for which the energy per state is the product (state energy) × (probability of occupation)

$hc\bar{\nu} \times \exp(-hc\bar{\nu}/kT)$

Then the energy density is:

$$E(\bar{\nu})\,d\bar{\nu} = 8\pi hc\bar{\nu}^3 \exp\left(\frac{-hc\bar{\nu}}{kT}\right) d\bar{\nu}$$

This is valid at large $\bar{\nu}$ (short λ) but does not work at small $\bar{\nu}$.

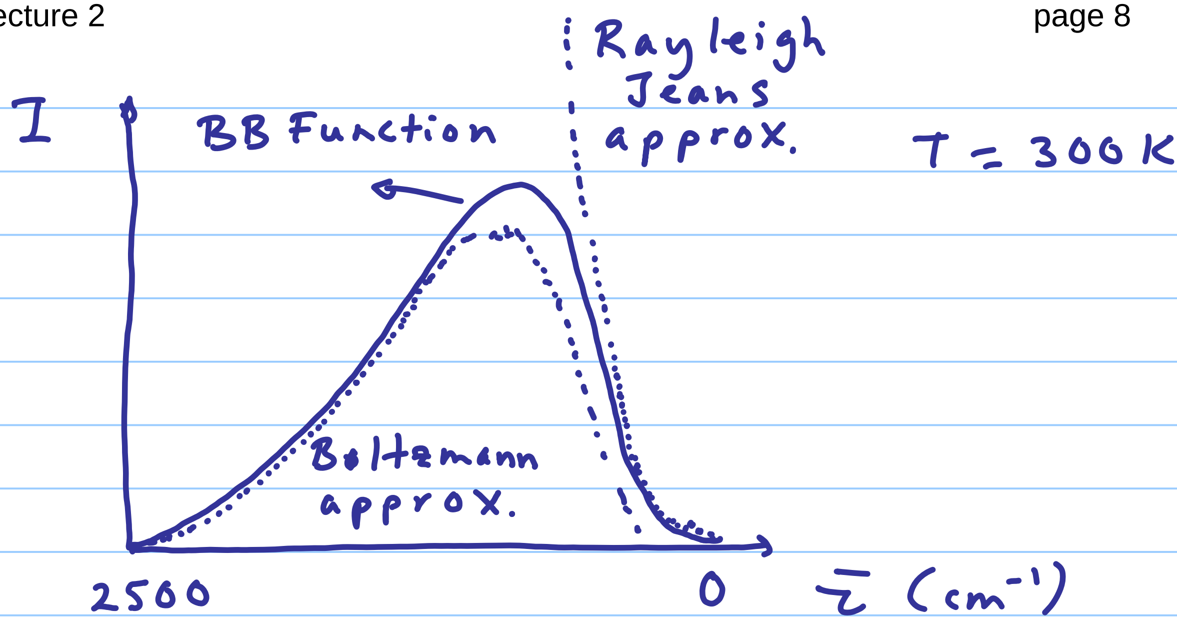
③ The correct solution requires the use of Bose-Einstein statistics (since photons are bosons). The probability of occupation is

$$\frac{1}{(e^{hc\bar{\nu}/kT} - 1)}$$

Thus the energy density becomes:

$$E(\bar{\nu})\,d\bar{\nu} = \frac{8\pi hc\bar{\nu}^3}{e^{hc\bar{\nu}/kT} - 1}\,d\bar{\nu}$$

The Rayleigh-Jeans formula approximates this function at small $\bar{\nu}$ (long λ). The Boltzmann formula approximates this at large $\bar{\nu}$ (short λ).



The energy per unit solid angle is the rate of transport of the energy divided by the total solid angle 4π

$$B_{\bar{\nu}}(T) = B(\bar{\nu}, T) = \frac{c}{4\pi} E(\bar{\nu}, T)$$

or

$$B_{\bar{\nu}}(T) = \frac{2hc^2\bar{\nu}^3}{e^{hc\bar{\nu}/kT} - 1}$$

$W\, m^{-2}\, sr^{-1}\, (cm^{-1})^{-1}$

k = Boltzmann's constant
c = speed of light.
h = planck's constant
T = absolute temp. (K)

This gives us:

$$c_1 = 2hc^2$$
$$c_2 = hc/k$$

$B_{\bar{\nu}}(T)$ or $B_\lambda(T)$ or $B_\nu(T)$ is the Planck blackbody function.

Additional Radiation Laws

① Wien's Displacement Law

The λ or $\tilde{\nu}$ at which $B_\lambda(T)$ or $B_{\tilde{\nu}}(T)$ is a maximum can be found using:

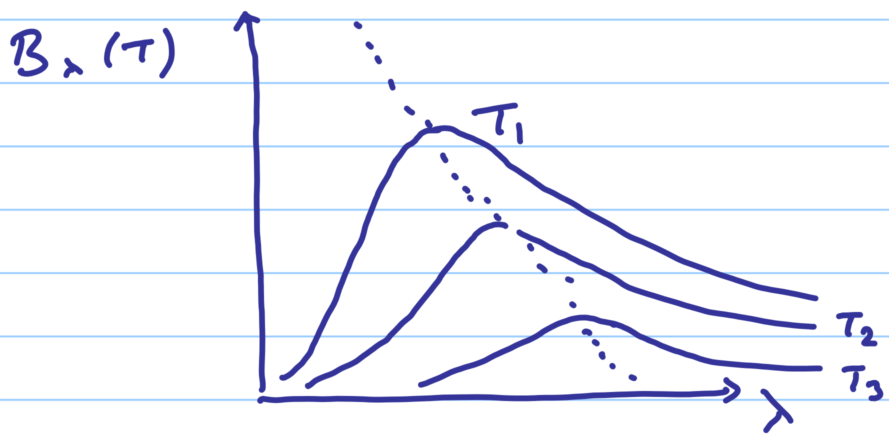
$$\frac{\partial B_\lambda(T)}{\partial \lambda} = 0$$

This gives $\lambda_{max} = \dfrac{2897.9}{T}$ μm for T in K

or $\tilde{\nu}_{max} = 1.962\, T$ cm^{-1}

$\simeq 2T$ (T in K)

This can be used to determine BB T
→ from the position of the maximum B_λ or $B_{\tilde{\nu}}$.



$T_1 > T_2 > T_3$

2 Stefan-Boltzmann Law

Monochromatic radiant exitance

$$M_\lambda(T) = F_\lambda(T) = \pi I_\lambda(T) = \pi B_\lambda(T)$$

because BB radiance is isotropic.

The total radiant exitance from a BB is:

$$M(T) = \int_0^\infty M_\lambda(T)\, d\lambda$$

$$= \int_0^\infty \pi B_\lambda(T)\, d\lambda$$

... integrate to get

$$M(T) = \frac{\pi^5}{15} \frac{c_1}{c_2} T^4 = \sigma T^4$$

$$\boxed{M(T) = \sigma T^4}$$

where $\sigma = \frac{\pi^5}{15} \frac{c_1}{c_2} = 5.67 \times 10^{-8}\ W m^{-2} K^{-4}$

$=$ Stefan-Boltzmann constant

3 Rayleigh-Jeans Approximation

Can simplify B_λ for longer λ, e.g. in the microwave region (λs in mm, cm, ...)

For longer λ, $\frac{c_2}{\lambda T} << 1$ for T relevant to Earth.

$\therefore \quad e^{c_2/\lambda T} \simeq 1 + c_2/\lambda T$

We get $\boxed{B_\lambda(T) \approx \frac{c_1}{c_2} \frac{T}{\lambda^4}}$

At long λ, $B_\lambda(T) \propto T$. Often the radiance will be scaled by $\left(\frac{c_1}{c_2}\right)\frac{1}{\lambda^4}$ to get a brightness temperature T_b, which is a radiance expressed in units of temperature.

T_b is the T required to match the measured intensity to the Planck BB function at a given λ.

T_b is also used in the infrared as an <u>equivalent blackbody temperature</u>, but must be derived from the Planck function.

4 Kirchoff's Radiation Law

To quantify how closely a real material approximates a BB, we use the emissivity ε of the material.

$$\varepsilon_\lambda \equiv \frac{\text{emitted radiance at } \lambda}{B_\lambda}$$

For a BB, $\varepsilon_\lambda = 1$.
For other materials, $0 \leq \varepsilon_\lambda < 1$ grey bodies

We can define

α_λ or A_λ = absorptivity or absorptance = $\dfrac{\text{absorbed radiance at } \lambda}{\text{incident radiance at } \lambda}$

ρ_λ or R_λ = reflectance = $\dfrac{\text{reflected radiance at } \lambda}{\text{incident radiance at } \lambda}$

τ_λ = transmittance = $\dfrac{\text{transmitted radiance at } \lambda}{\text{incident radiance at } \lambda}$

Three possibilities for incident radiation
All lie between 0 and 1.

By conservation of energy

$$\alpha_\lambda + \rho_\lambda + \tau_\lambda = 1 .$$

Kirchoff's Law states that for a body in local thermodynamic equilibrium (LTE) (i.e. a body having a single uniform T, true below ~ 40 km in atmosphere)

$$\varepsilon_\lambda = \alpha_\lambda$$

For a blackbody, $\varepsilon_\lambda = \alpha_\lambda = 1$

For a grey body, $0 \leq \varepsilon_\lambda = \alpha_\lambda < 1$