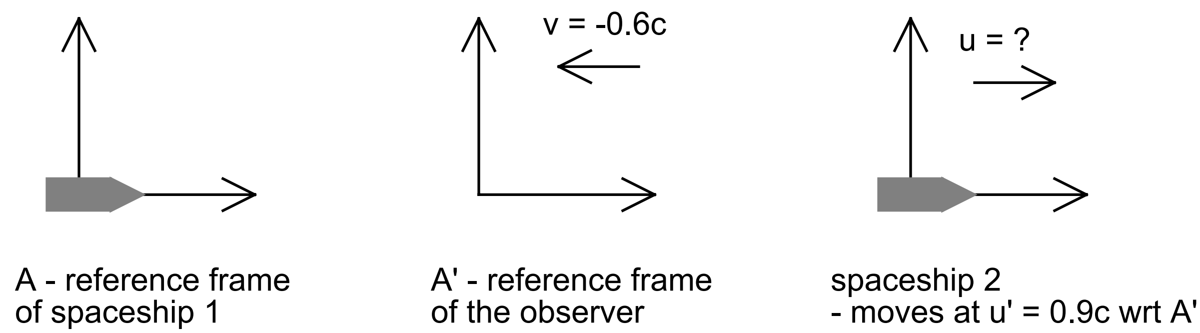
PHY 140Y – FOUNDATIONS OF PHYSICS
2001-2002
Tutorial Questions #12 – Solutions
December 3/4

Lorentz Velocity Addition, Time Dilation, and Lorentz Contraction

1. Two spaceships are travelling with velocities of 0.6c and 0.9c relative to a third observer.
(a) What is the speed of one spaceship relative to the other spaceship if they are going in the same direction?
(b) What is their relative speed if they are going in opposite directions?

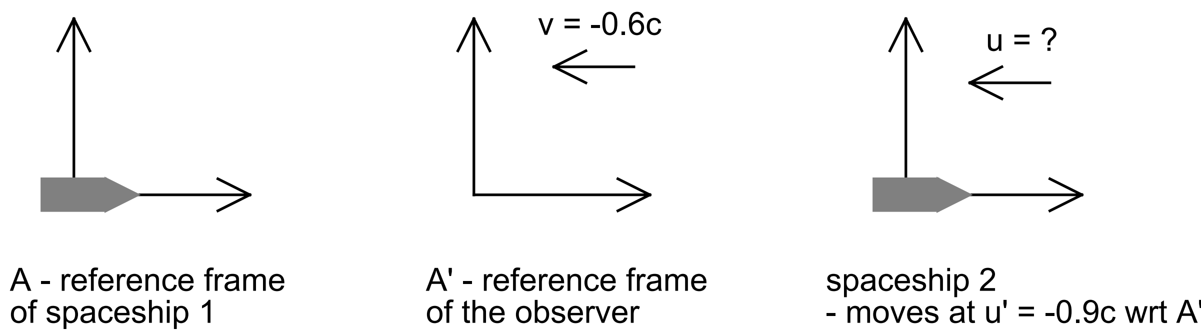
Solution:
(a) Define reference frame A for one of the spaceships, say the one travelling at 0.6c (call it #1). Then the observer moves at velocity v = –0.6c with respect to this reference frame.



The speed of spaceship #2 with respect to spaceship #1 is then given by Lorentz velocity addition:

$$u = \frac{u'+v}{1+\frac{v}{c^2}u'} = \frac{0.9c - 0.6c}{1+\frac{-0.6c}{c^2}(0.9c)} = 0.652c = 2.0\times10^{8}\ \text{m/s}$$

(b) Now the spaceships are going in opposite directions, so u’ = -0.9c.



Lorentz velocity addition gives:

$$u = \frac{u' + v}{1 + \frac{v}{c^2} u'} = \frac{-0.9c - 0.6c}{1 + \frac{(-0.6c)}{c^2}(-0.9c)} = -0.974c = -2.9 \times 10^8 \text{ m/s}$$

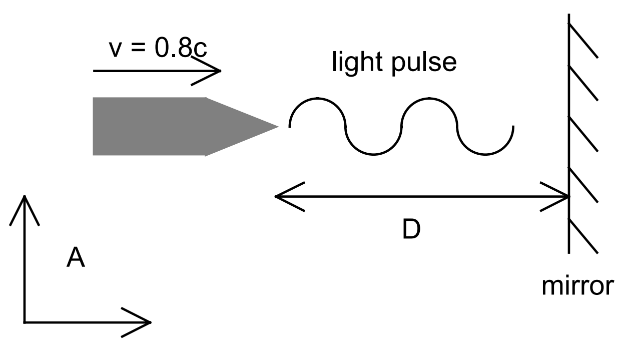
The negative sign indicates that the space ships are moving in opposite directions.

2. A rocket moves towards a mirror at 0.8c relative to the reference frame A, shown below. The mirror is stationary relative to A. A light pulse emitted by the rocket travels to the mirror and is reflected back to the rocket. The front of the rocket is 1.8×10^{12} m from the mirror (as measured in A) at the moment that the light pulse leaves the rocket. What is the total travel time of the pulse as measured by an observer in
(a) reference frame A, and
(b) the front of the rocket?

Solution:

Define $D = 1.8 \times 10^{12}$ m = distance from the front of the rocket to the mirror

$v = 0.8c = \beta c$



(a) In reference frame A:

travel time to the mirror: $\Delta t_1 = \frac{D}{c}$

travel time back to the rocket: $\Delta t_2 = \frac{D - v\Delta t}{c} = \frac{D - \beta c \Delta t}{c}$

Thus the total travel time is: $\Delta t = \Delta t_1 + \Delta t_2 = \frac{D}{c} + \frac{D - \beta c \Delta t}{c} = \frac{2D - \beta c \Delta t}{c}$

Rearrange and solve for Δt

$$\Delta t = \frac{2D}{(1+\beta)c} = \frac{2(1.8 \times 10^{12}\text{ m})}{(1+0.8)(3.00 \times 10^8 \text{ m/s})} = 6667\text{ s} = 1.8(5)\text{ hours}$$

(b) We can apply time dilation to calculate the travel time measured by an observer on the front of the rocket because the two events (light emitted from rocket and light returned to rocket) occur at the same place for this observer (at the front of the rocket). Define A' as the reference frame of the rocket.

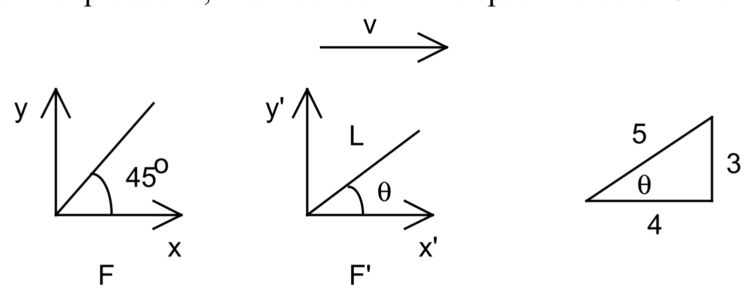
$$\Delta t' = \frac{\Delta t}{\gamma} = \frac{2D}{(1+\beta)c}\sqrt{1-\frac{v^2}{c^2}} = \frac{2D}{(1+\beta)c}\sqrt{1-\beta^2}$$

Thus:

$$= \frac{2D}{c}\frac{\sqrt{1-\beta^2}}{1+\beta} = \frac{2D}{c}\sqrt{\frac{1-\beta}{1+\beta}}$$

$$= \frac{2(1.8\times 10^{12}\text{ m})}{3.00\times 10^{8}\text{ m/s}}\sqrt{\frac{1-0.8}{1+0.8}} = 4000\text{ s} = 1.1(1)\text{ hours}$$

3. A rod of proper length L is at rest in reference frame F'. The rod lies in the (x', y') plane and makes an angle of θ with the x' axis. F' moves with a constant velocity v parallel to the x-axis of another frame F.
(a) What is the value of v if, as measured in F, the rod is at 45° to the x axis?
(b) What is the length of the rod as measured in F under the situation given in part (a)? Derive general expressions, and also look at the special case of $\theta = \sin^{-1}(3/5)$.



Solution:
(a) In reference frame F':

x' component of length: $x' = L\cos\theta$

y' component of length: $y' = L\sin\theta$

We can apply length contraction to convert these to reference frame F:

x component of length: $x = \frac{x'}{\gamma} = \frac{L\cos\theta}{\gamma}$ (length contraction applies)

y component of length: $y = y' = L\sin\theta$ (no motion in the y direction)

If the rod is at angle 45° to the x axis, as measured in F, then the x and y components must be equal. Note that this angle is not the same as θ in frame F'. Thus:

$$L\sin\theta = \frac{L\cos\theta}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$1-\frac{v^2}{c^2} = \tan^2\theta$$

The general expression for the velocity in F is then: $v = c\sqrt{1 - \tan^2\theta}$

For the special case of $\theta = \sin^{-1}(3/5)$:

$\theta = \sin^{-1}(3/5) = 36.87°$ (in reference frame F') and $\tan\theta = 3/4$

So: $$v = c\sqrt{1 - \tan^2\theta} = c\sqrt{1 - \tan^2 36.87°} = \frac{\sqrt{7}}{4}\,c = 1.98 \times 10^8 \text{ m/s}$$

(b) The length of the rod, as measured in reference frame F, is:

$$L' = \sqrt{x'^2 + y'^2} = \sqrt{\left(\frac{L\cos\theta}{\gamma}\right)^2 + (L\sin\theta)^2}$$

$$= L\sin\theta\sqrt{\left(\frac{1}{\gamma^2}\right)\left(\frac{\cos^2\theta}{\sin^2\theta}\right) + 1}$$

$$= L\sin\theta\sqrt{\frac{1}{\gamma^2\tan^2\theta} + 1}$$

From part (a), we have $\gamma = \dfrac{1}{\tan\theta}$, so the general expression is:

$$L' = L\sin\theta\sqrt{\frac{1}{\gamma^2\tan^2\theta} + 1} = L\sqrt{2}\sin\theta$$

For the special case of $\theta = \sin^{-1}(3/5)$: $L' = L\sqrt{2}\sin\theta = \dfrac{3\sqrt{2}}{5}L$

Lorentz Transformations

4. Spaceship A of proper length L is travelling east at speed v_A, and spaceship B of proper length 2L is travelling west at speed v_B, both as seen from Earth. The pilot of spaceship A sets a clock to zero when the front of spaceship B passes by. (The spaceship pilots sit in the nose cones.) Use Lorentz transformations to derive an expression for the time at which, according to the pilot of spaceship A, the tail of spaceship B passes by.

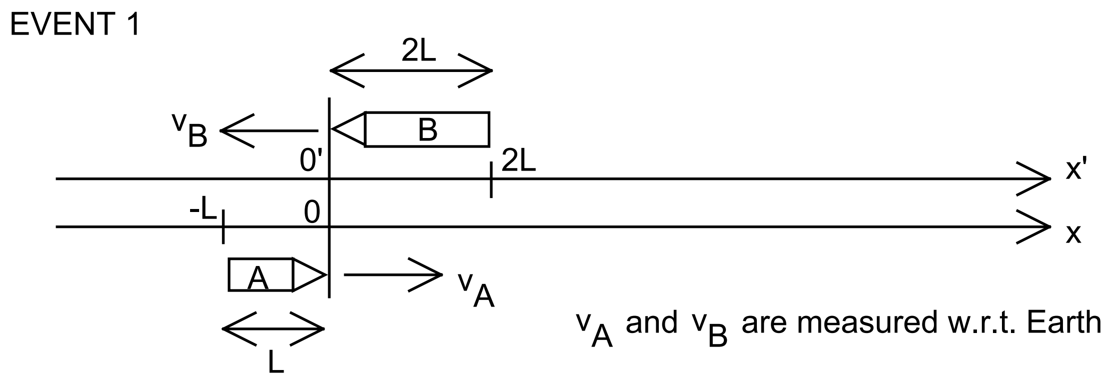
Solution:

Define reference frame S at rest with respect to spaceship A, and reference frame S' at rest with respect to spaceship B. Set up the x and x' axes as shown below.

Define the two events and the relevant x and x' coordinates for these events.

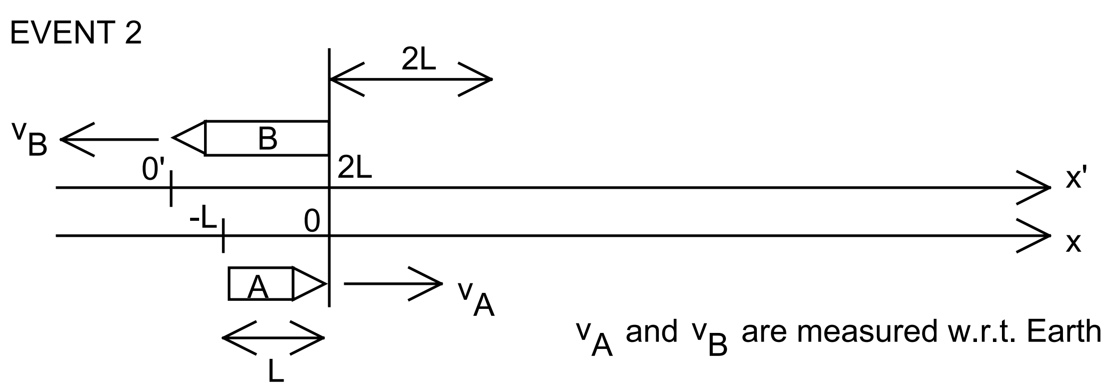
Event #1:

- the front of spaceship B passes by the front of spaceship A
- the front of A is at x = 0, the back of A is at x = -L in reference frame S
- the front of B is at x' = 0, the back of B is at x' = 2L in reference frame S'
- set t = t' = 0 as the time when event 1 occurs and x = x' = 0



Event #2:

- the back of spaceship B passes by the front of spaceship A
- the front of A remains at x = 0, and the back of A at x = -L in reference frame S
- the front of B remains at x' = 2L, and the back of B at x' = 2L in reference frame S'
- this occurs at time t in reference frame S, when x = 0



We need t apply the Lorentz transformations to find time when Event 2 occurs. These require the relative velocity of spaceship B with respect to A, which we first find using Lorentz velocity addition:

$$u = \frac{u' + v}{1 + \frac{v}{c^2}u'} = \frac{v_A + v_B}{1 + \frac{v_B v_A}{c^2}}$$

where

u = velocity of spaceship B with respect to spaceship A

v_A = velocity of spaceship A with respect to Earth

v_B = velocity of spaceship B with respect to Earth

Now, apply the Lorentz transformation for position:

$$x' = \gamma(x - ut)$$
$$2L = \gamma(0 - ut) = -\gamma ut$$
$$t = -\frac{2L}{\gamma u}$$

But note that velocity u is in the negative x direction in reference frame S, so: $t = \frac{2L}{\gamma|u|}$

This is the time for Event 2 as recorded by the pilot in spaceship A. So pilot A sees the length of spaceship B contracted from 2L to $2L/\gamma$.

Because spaceship B is moving at speed $|u|$ relative to spaceship A, the time it takes for B to pass A is the length of B ($2L/\gamma$) divided by the relative speed $|u|$.

Energy in STR

5. Solar energy reaches the Earth at the rate of about 1400 W/m^2 of surface area perpendicular to the direction to the Sun. By how much does the mass of the Sun decrease in each second? The mean radius of Earth's orbit is 1.5×10^8 km. Would the real mass loss of the Sun be greater than or less than your calculated answer, and why?

Solution:

The total energy output of the Sun (per unit time) is:

$$\begin{aligned} E &= 1400\ \text{W/m}^2 \times (\text{surface area of a sphere whose radius is Earth's orbit}) \\ &= 1400\ \text{W/m}^2 \times 4\pi R^2 \\ &= 1400\ \text{W/m}^2 \times 4\pi(1.50 \times 10^8 \times 10^3\ \text{m})^2 \\ &= 3.96 \times 10^{26}\ \text{W} = 3.96 \times 10^{26}\ \text{Joules/second} \end{aligned}$$

This amount of energy is released by the Sun every second, resulting in a loss of mass given by:

$$E = mc^2$$
$$m = \frac{E}{c^2} = \frac{3.96 \times 10^{26}\ \text{J}}{(3.00 \times 10^8\ \text{m/s})^2} = 4.40 \times 10^9\ \text{kg}$$

So the Sun loses 4.4 billion kg per second.
The real mass loss is actually greater than this because the Sun also emits the solar wind (which consists of charged particles), which add to the mass loss due to radiant energy just calculated.

6. A proton (mass = 1.67×10^{-27} kg) is moving at speed v = 0.900c.
(a) What is the proton's total relativistic energy?
(b) What is the proton's kinetic energy?
(c) What is the proton's rest energy?
(d) What is the magnitude of the proton's relativistic momentum?

Solution:

(a) First evaluate the relativistic factor: $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} = \dfrac{1}{\sqrt{1-0.900^2}} = 2.29$

The proton's total relativistic energy is:

$$\begin{aligned} E &= \gamma mc^2 \\ &= 2.29 \times (1.67 \times 10^{-27}\,\mathrm{kg}) \times (3.00 \times 10^8\,\mathrm{m/s})^2 \\ &= 3.44 \times 10^{-10}\,\mathrm{Joules} \end{aligned}$$

(b) The proton's kinetic energy is:

$$\begin{aligned} K &= \gamma mc^2 - mc^2 \\ &= (\gamma - 1)mc^2 \\ &= (2.29 - 1) \times (1.67 \times 10^{-27}\,\mathrm{kg}) \times (3.00 \times 10^8\,\mathrm{m/s})^2 \\ &= 1.94 \times 10^{-10}\,\mathrm{Joules} \end{aligned}$$

(c) The proton's rest energy is:

$$\begin{aligned} E_o &= E - K = mc^2 \\ &= (1.67 \times 10^{-27}\,\mathrm{kg}) \times (3.00 \times 10^8\,\mathrm{m/s})^2 \\ &= 1.50 \times 10^{-10}\,\mathrm{Joules} \end{aligned}$$

(d) The magnitude of the proton's relativistic momentum is:

$$\begin{aligned} p &= \gamma mv \\ &= 2.29 \times (1.67 \times 10^{-27}\,\mathrm{kg}) \times (0.900 \times 3.00 \times 10^8\,\mathrm{m/s}) \\ &= 1.03 \times 10^{-18}\ \mathrm{kg\,m/s} \end{aligned}$$

or

$$\begin{aligned} p &= \frac{v}{c^2}E = \frac{v}{c}\frac{E}{c} \\ &= 0.900 \times \frac{3.44 \times 10^{-10}\ \mathrm{J}}{3.00 \times 10^8\,\mathrm{m/s}} \\ &= 1.03 \times 10^{-18}\ \mathrm{kg\,m/s} \end{aligned}$$