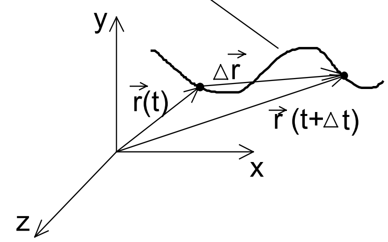
LECTURE #7 – SUMMARY

Let's consider an arbitrary position vector, $\vec{r}(t)$.



Position: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Displacement: $\Delta\vec{r}(t) = \vec{r}(t+\Delta t) - \vec{r}(t)$

Average velocity:

$$\vec{v}_{avg} = \frac{\Delta\vec{r}(t)}{\Delta t} = \frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \frac{1}{\Delta t}\left\{[x(t+\Delta t)-x(t)]\hat{i} + [y(t+\Delta t)-y(t)]\hat{j} + [z(t+\Delta t)-z(t)]\hat{k}\right\}$$

Instantaneous velocity:

$$\vec{v}(t) = \lim_{\Delta t\to 0}\vec{v}_{avg} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}(t)}{\Delta t} = \frac{d\vec{r}(t)}{dt}\vec{v}(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k} = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k}$$

Instantaneous acceleration:

$$\vec{a}(t) = \lim_{\Delta t\to 0}\vec{a}_{avg} = \lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}(t)}{dt} = \frac{dv_x(t)}{dt}\hat{i} + \frac{dv_y(t)}{dt}\hat{j} + \frac{dv_z(t)}{dt}\hat{k}$$

$$= \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} + \frac{d^2z(t)}{dt^2}\hat{k} = a_x(t)\hat{i} + a_y(t)\hat{j} + a_z(t)\hat{k}$$

Now, if the acceleration is constant, then these equations can be simplified.

$$\boxed{\vec{a}(t) = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}} \quad \text{where } a_x, a_y, a_z \text{ are constant}$$

Thus

$$\begin{aligned} v_x(t) &= v_{ox} + a_x(t-t_o) \\ v_y(t) &= v_{oy} + a_y(t-t_o) \\ v_z(t) &= v_{oz} + a_z(t-t_o) \end{aligned} \quad \Rightarrow \quad \boxed{\vec{v}(t) = \vec{v}_o(t) + \vec{a}(t-t_o)}$$

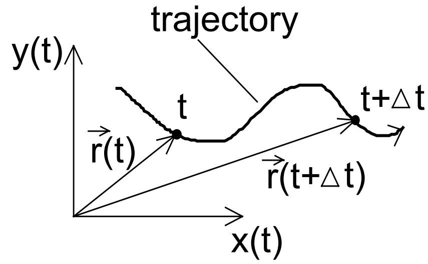
and

$$\begin{aligned} x(t) &= x_o + v_{ox}(t-t_o) + \tfrac{1}{2}a_x(t-t_o)^2 \\ y(t) &= y_o + v_{oy}(t-t_o) + \tfrac{1}{2}a_y(t-t_o)^2 \\ z(t) &= z_o + v_{oz}(t-t_o) + \tfrac{1}{2}a_z(t-t_o)^2 \end{aligned} \quad \Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_o + \vec{v}_o(t-t_o) + \tfrac{1}{2}\vec{a}(t-t_o)^2}$$

There are a number of aspects of these formulas that relate to the direction of the trajectory and that <u>are NOT obvious</u>.

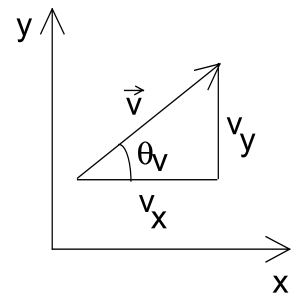
Let's look at the problem in more detail for the case of 2-D motion (i.e. motion in a plane). We will drop the z components of the equations.

(1) Position: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$



Note: This is not like a 1-D plot of x vs. t. Here the trajectory gives the actual position of the particle in space. Time is implied, not explicit.

(2) Velocity: $\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt} = \dfrac{dx(t)}{dt}\hat{i} + \dfrac{dy(t)}{dt}\hat{j} = v_x(t)\hat{i} + v_y(t)\hat{j}$



Magnitude $\quad |\vec{v}(t)| = \sqrt{\left(\dfrac{dx(t)}{dt}\right)^2 + \left(\dfrac{dy(t)}{dt}\right)^2}$

Direction $\quad \theta_v = \tan^{-1}\left(\dfrac{v_y}{v_x}\right)$

$\vec{v}$ is <u>parallel</u> to the trajectory because the slope of the tangent line at time t is

$$= \frac{dy(t)}{dx(t)} = \frac{dy(t)/dt}{dx(t)/dt} = \frac{v_y(t)}{v_x(t)} = \tan(\theta_{\text{tangent line}}) \qquad \text{so } \theta_{\text{tangent line}} = \theta_v$$

To summarize:

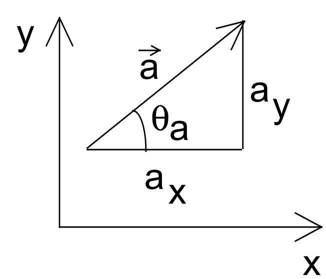
1-D	velocity || trajectory,	velocity || position vector
2-D and 3-D	velocity || trajectory,	velocity IS NOT || position vector

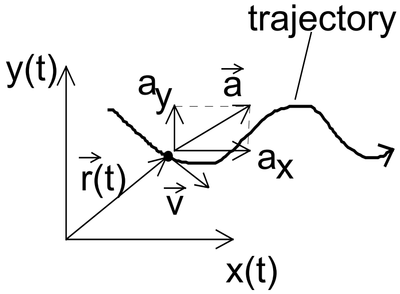
(3) Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{dv_x(t)}{dt}\hat{i} + \frac{dv_y(t)}{dt}\hat{j} = \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j} = a_x(t)\hat{i} + a_y(t)\hat{j}$$

Magnitude? $\quad |\vec{a}(t)| = \sqrt{\left(\dfrac{dv_x(t)}{dt}\right)^2 + \left(\dfrac{dv_y(t)}{dt}\right)^2} = \sqrt{\left(\dfrac{d^2x(t)}{dt^2}\right)^2 + \left(\dfrac{d^2x(t)}{dt^2}\right)^2}$

Direction? $\quad \theta_a = \tan^{-1}\left(\dfrac{a_y}{a_x}\right)$





In general, $\vec{a}$ NOT || $\vec{v}$
i.e. acceleration is not parallel to the velocity or to the trajectory.