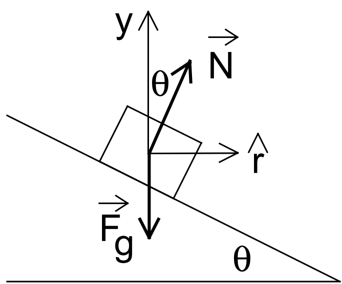
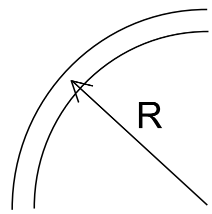
LECTURE #15 – SUMMARY

Example 6-7 - Application of Newton's Second Law to Circular Motion

At what angle should a road with a 150-m curvature radius (R) be banked for travel at speed v = 75 km/hr (21 m/s)? Assume a frictionless surface.

Want angle θ, such that the car stays at the same vertical level ("safe").





Apply Newton's Second Law: $\vec{F}_g + \vec{N} = m\vec{a}$

r direction (horizontal): $F_{r,net} = F_{g,r} + N_r = ma_r \quad \Rightarrow \quad N\sin\theta = m\dfrac{v^2}{R}$ (1)

y direction (vertical): $F_{y,net} = F_{g,y} + N_y = ma_y \quad \Rightarrow \quad N\cos\theta = mg$ (2)

Solve to get $N = \dfrac{mg}{\cos\theta}$ and hence $\boxed{\theta = \tan^{-1}\left(\dfrac{v^2}{gR}\right) = 17^\circ \quad \text{or} \quad v = \sqrt{Rg\tan\theta}}$

Note:

(1) The centripetal force is provided by the normal force.

(2) As v increases $\rightarrow$ θ increases, and as v decreases $\rightarrow$ θ decreases

i.e., At very low speeds, banking is not really needed, but as speed increases, the force needed to maintain circular motion increases and so does the banking angle.

(3) This is a very delicately balanced situation. Without friction, if R and θ are specified, there is only one speed v*, such that the car will take the turn safely.

Why is this? Let's define $v^* = C\sqrt{Rg\tan\theta}$ and substitute into the equations above to get $mC^2g - mg = ma_y$. Now, we can have three possibilities:

i) C=1, v*=v $\Rightarrow$ $m1^2g - mg = 0 = ma_y$ $\Rightarrow$ safe !

ii) C>1, v*>v $\Rightarrow$ $ma_y > 0$ $\Rightarrow$ if speed is greater than $\sqrt{Rg\tan\theta}$, then $N\cos\theta$ exceeds gravity and the car rises up the bank

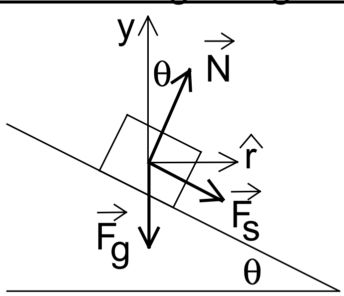
iii) C<1, v*<v $\Rightarrow$ $ma_y < 0$ $\Rightarrow$ if speed is less than $\sqrt{Rg\tan\theta}$, then gravity exceeds $N\cos\theta$ and the car slips down the bank

Example 6-7 Revisited (with Friction)

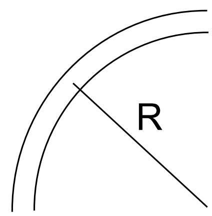
A car is travelling along a road that has radius of curvature R and is banked at angle θ. If the surface is NOT frictionless, what is the range of speeds such that the car will stay at the same level?

Static friction is acting because in the direction along the ramp (perpendicular to motion), the wheels are stationary w.r.t. ramp.
What is the direction of the force of static friction?
→ if speed is fast, then the car "wants" to go up the ramp and F_s opposes this
→ if speed is slow, then the car "wants" to go down the ramp and F_s opposes this

Case 1: The car is travelling at high speed.



OVERHEAD VIEW



r direction: $F_{r,net} = F_{g,r} + N_r + F_{s,r} = ma_r \quad \Rightarrow \quad 0 + N\sin\theta + F_s\cos\theta = m\frac{v^2}{R}$

y direction: $F_{y,net} = F_{g,y} + N_y + F_{s,y} = ma_y \quad \Rightarrow \quad -mg + N\cos\theta - F_s\sin\theta = 0$

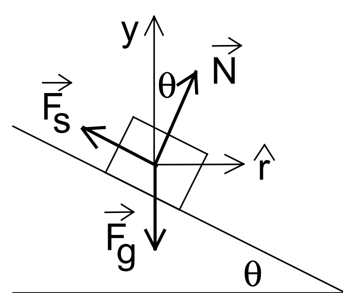
Use $F_s = \mu_s N$ to get: $N(\sin\theta + \mu_s\cos\theta) = \frac{m{v_{max}}^2}{R}$ and $N = \frac{mg}{\cos\theta - \mu_s\sin\theta}$

Solve to get the maximum velocity:

$$\boxed{{v_{max}}^2 = Rg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)}$$

Case 2: The car is travelling at low speed.

Just substitute $-\mu_s$ for μ_s, because static friction is now in the opposite direction.



The minimum velocity is:

$$\boxed{{v_{min}}^2 = Rg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right)}$$

As $\mu_s \to 0$ (i.e., frictionless):

- ${v_{min}}^2 = {v_{max}}^2 = Rg\tan\theta$, as derived before
- $\therefore$ if $v = \sqrt{Rg\tan\theta}$, then $F_s \to 0$ (no tendency to move up or down)

If $\sqrt{Rg\tan\theta} \le v \le v_{max}$, then static friction will increase as v increases and the car will NOT move up the ramp. If $v_{min} \le v \le \sqrt{Rg\tan\theta}$, then static friction will increase as v decreases and the car will NOT move down the ramp.