Gravity Waves in 4D Data Assimilation

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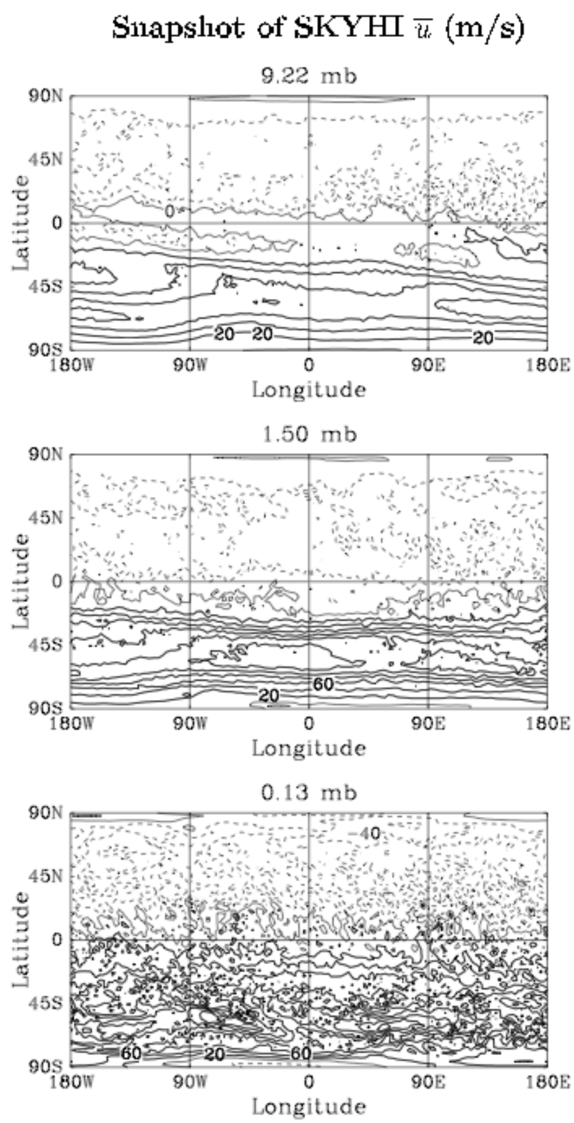
Outline

- Dynamical context & difficulties
- Experiments with a simplified model
- Gravity Waves in the EnKF
- Gravity Waves in 4D-Var
- Conclusions and Implications

Data Assimilation in the Middle Atmosphere



From Koshyk et al. 1999

Middle atmosphere: unbalanced motion is significant.

Improved DA: What does it take to capture a fast time scale?

Start very simply: can we capture a fast wave given observations on a slow timescale?

4D Data Assimilation: 2 approaches

The Kalman Filter

$$\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_k^f)$$

= forecast + **gain*** (obs - forecast)

Ensemble Kalman Filter (EnKF): evolve and update errors using ensemble statistics.

Nonlinear error evolution, but sampling error, gaussianity assumptions.

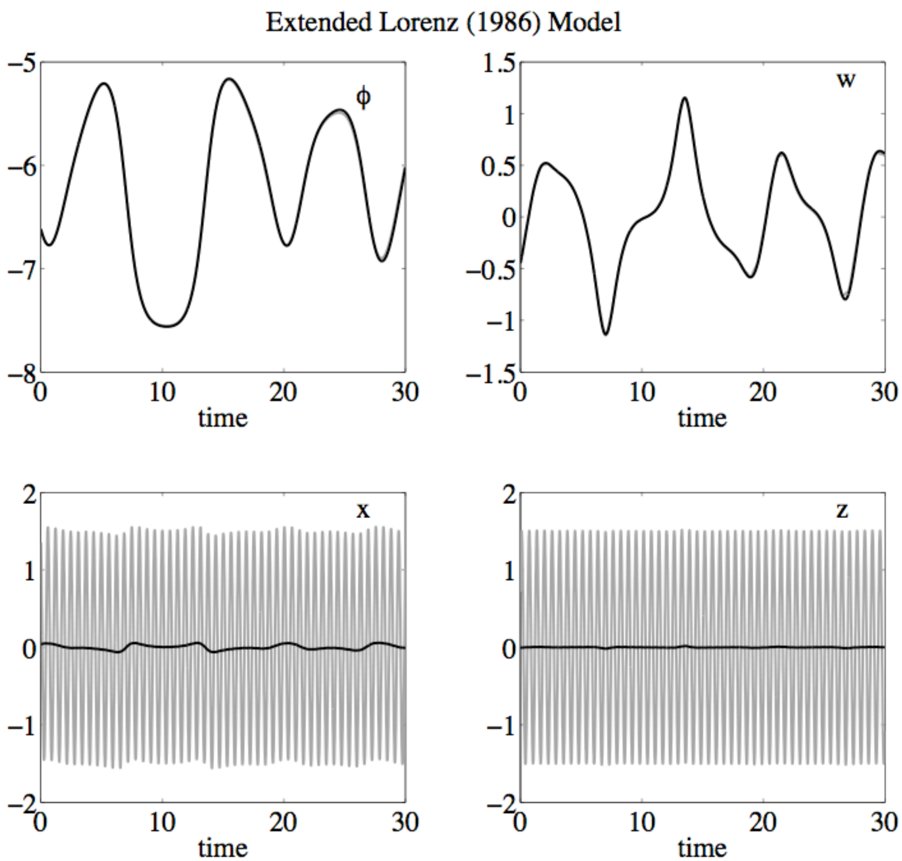
4 Dimensional Variational Assimilation (4DVAR)

$$J(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x}_0 - \mathbf{x}_0^{\mathbf{b}}\right)^{\mathbf{T}} \mathbf{B}^{-1}\left(\mathbf{x}_0 - \mathbf{x}_0^{\mathbf{b}}\right) + \frac{1}{2}\sum_{i=0}^{N}\left(\mathbf{z}_i - H(\mathbf{x}_i)\right)^{\mathbf{T}} \mathbf{R}^{-1}\left(\mathbf{z}_i - H(\mathbf{x}_i)\right)$$

No ensemble, but (subtle) linearity / gaussianity assumptions.

The Littlest Balance Model

Chaotic slow mode + slaved fast mode + free GW



$$\frac{d\phi}{dt} = w$$

$$\frac{dw}{dt} = -\frac{C}{2}\sin\left(2\phi + 2\epsilon b x\right)$$

$$\frac{dx}{dt} = -\frac{z}{\epsilon}$$

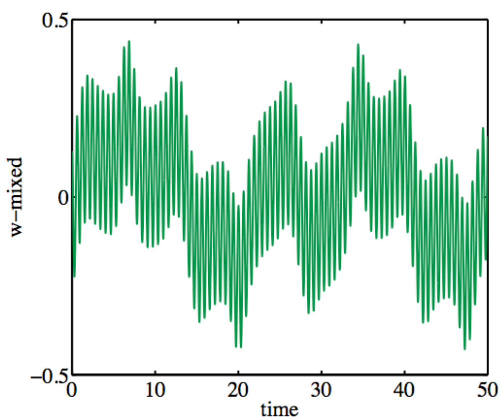
$$\frac{dz}{dt} = \frac{x}{\epsilon} + \frac{bC}{2}\sin\left(2\phi + 2\epsilon b x\right)$$

$$\epsilon = \frac{\mathrm{Ro}\ b}{\sqrt{1+b^2}} = \frac{U}{\sqrt{gH + f^2L^2}}$$

The fast wave has a componend which is **slaved** to the slow

$$\begin{aligned} x &= U_x(\phi;\epsilon) = -\frac{\epsilon}{2} Cb\sin 2\phi + O(\epsilon^3) \\ z &= U_z(\phi, w;\epsilon) = \epsilon^2 (Cbw\cos 2\phi + \frac{C'}{2} b\sin 2\phi) + O(\epsilon^3) \end{aligned}$$

For observations, transform to **mixed timescale variables**



$$w \equiv w' + bz'$$

(Like vorticity to PV)

$$z \equiv z' - bw'$$

(Like geopot. height to geostrophic imbalance)

Slaving means that there is a **covariance** between fast and slow.

$$\begin{aligned} c_{wz} &= \langle e_w e_{U_z} \rangle + \langle e_w e_z \rangle \\ &= \langle e_w e_{U_z} \rangle \end{aligned}$$

← Nongaussian!

Numerical Experiments

initial truth with $I_t = 1.5$

initial forecast is balanced.

initial ensemble has random GW amplitudes & phases.

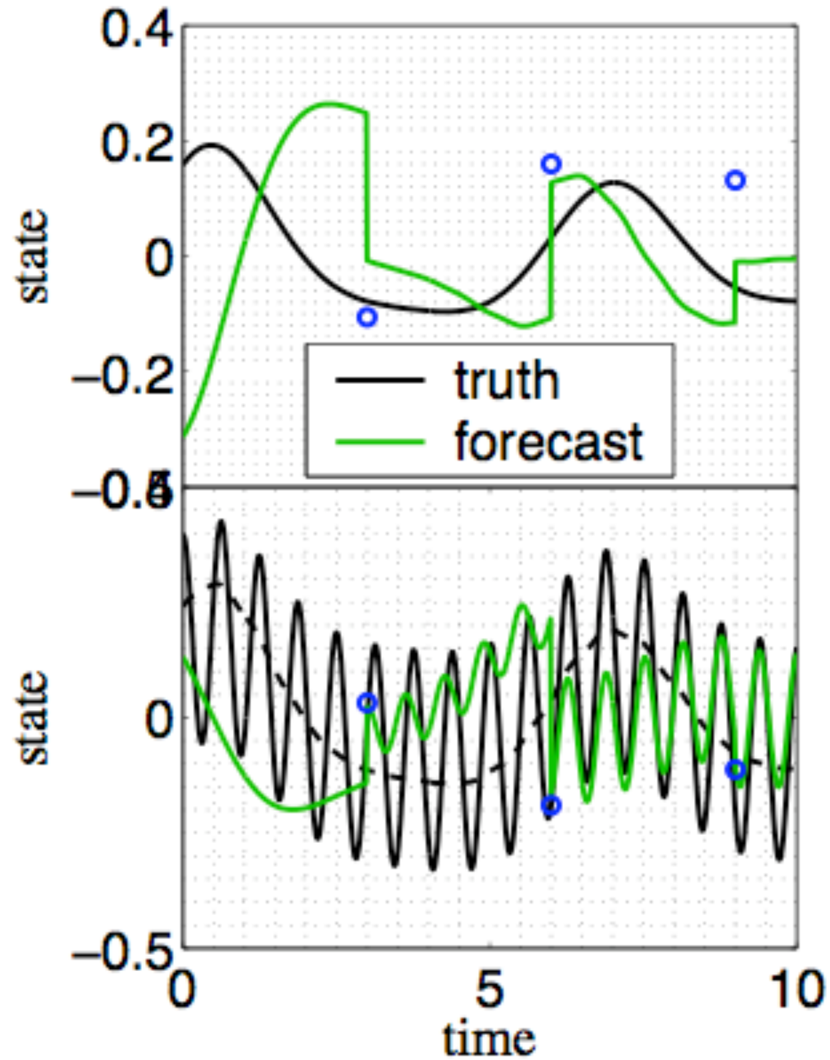
Observe a mixed state

$$\mathbf{y} = (\phi, w')^{\mathrm{T}}$$

Observe a slow ("filtered") state

$$\mathbf{y} = (\phi, w)^{\mathrm{T}}$$

Assimilation experiments with the Extended Lorenz 1986 model



KF Analysis Increments

$$\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k^f)$$

(1) slow observations:

$$\begin{aligned} \delta w^a &= k_{\phi w} \delta \phi^{obs} + k_{ww} \delta w^{obs} \\ \delta z^a &= k_{\phi z} \delta \phi^{obs} + k_{wz} \delta w^{obs} \end{aligned}$$

Observations only constrain the slaved part of the fast wave.

✓ *Need to get the slaving relationship right in order not to force a spurious fast wave.*

(2) mixed observations:

$$\begin{aligned} \delta w^a &= k_{\phi w} \delta \phi^{obs} + k_{w' w} \delta w^{obs} \\ \delta z^a &= k_{\phi x} \delta \phi^{obs} + k_{w' z} \delta w^{obs}. \end{aligned}$$

Observations now also contain the GW signal.

✓ *Need to get the slaving relationship right in order to correctly interpret the fast signal for the slow mode.*

Ensemble Kalman Filter

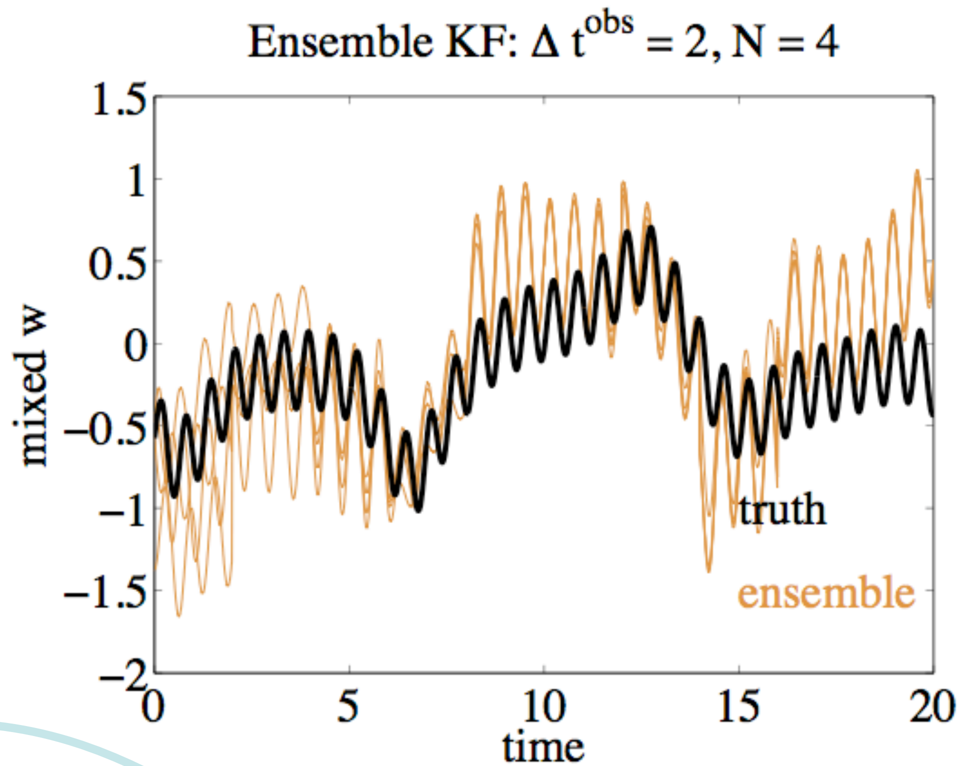
example: small ensemble

$$\mathbf{K}_k = \mathbf{P}_k^f \mathbf{H}_k^{\mathrm{T}} (\mathbf{H}_k \mathbf{P}_k^f \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}$$

Fast-slow covariances

$$\mathbf{P} = \begin{pmatrix} \sigma_\phi^2 & c_{\phi w} & c_{\phi x} & c_{\phi z} \\ c_{\phi w} & \sigma_w^2 & c_{wx} & c_{wz} \\ c_{\phi x} & c_{wx} & \sigma_x^2 & c_{xz} \\ c_{\phi z} & c_{wz} & c_{xz} & \sigma_z^2 \end{pmatrix}.$$

Fast variances



Evolve forward

$$\mathbf{P}_k^f = \left\langle \left(\mathbf{x}_i^f - \langle \mathbf{x}^f \rangle\right) \left(\mathbf{x}_i^f - \langle \mathbf{x}^f \rangle\right)^{\mathrm{T}} \right\rangle$$

Update with observations

Ensemble Kalman Filter

$$\mathbf{K}_k = \mathbf{P}_k^f \mathbf{H}_k^{\mathrm{T}} (\mathbf{H}_k \mathbf{P}_k^f \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}$$

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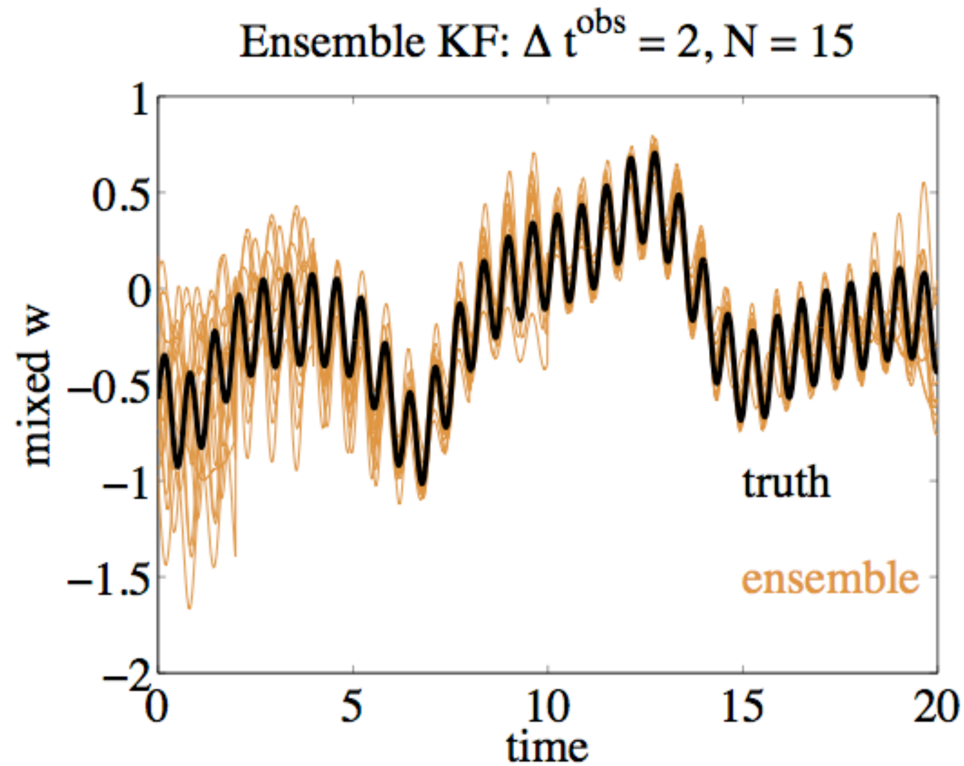
Fast variances

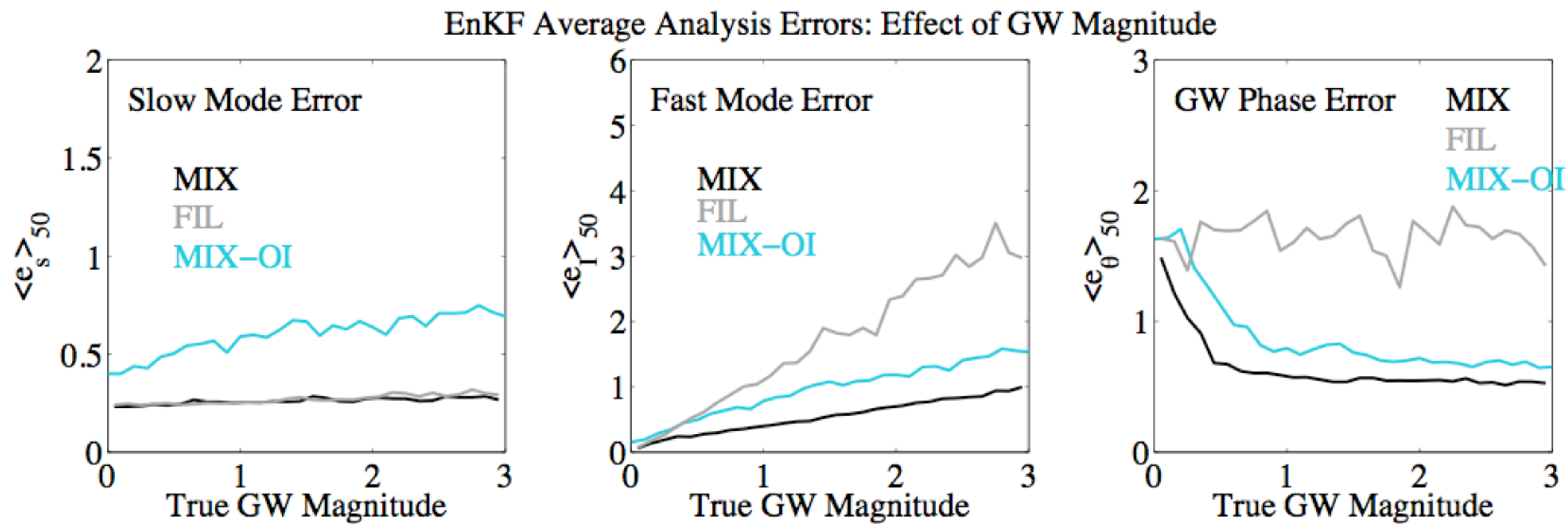
example: large ensemble



In both cases the slow component is captured, but the ensemble phase-locks around different GW magnitudes.

Ensemble Kalman Filter : capturing a Gravity Wave

As the magnitude of the true-state gravity wave increases:



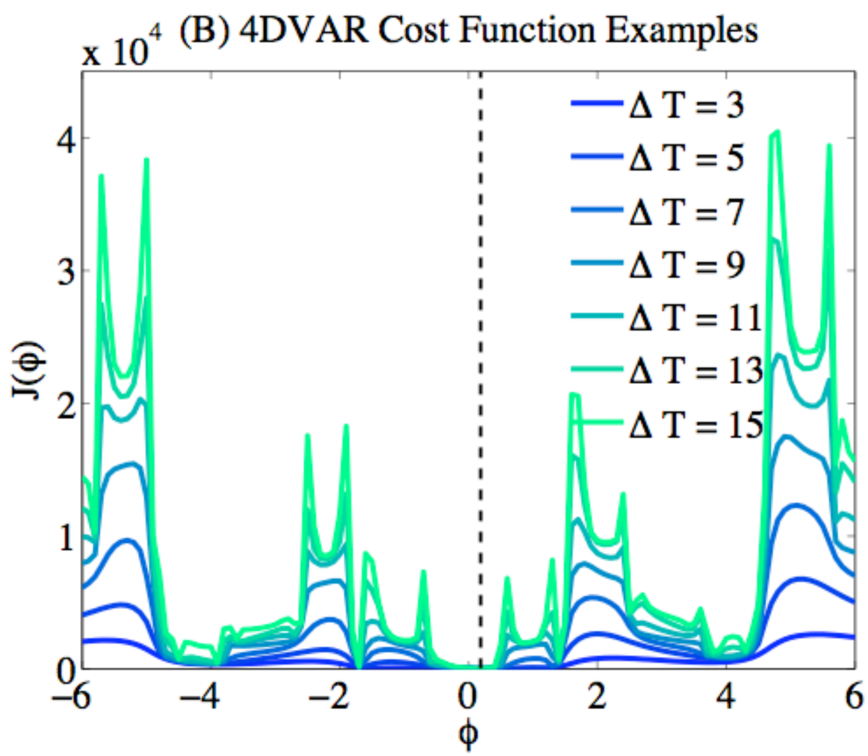
OI becomes worse for capturing the slow mode.

It takes longer (more cycles) to capture the **fast magnitude**.

It becomes easier to capture GW **phase**.

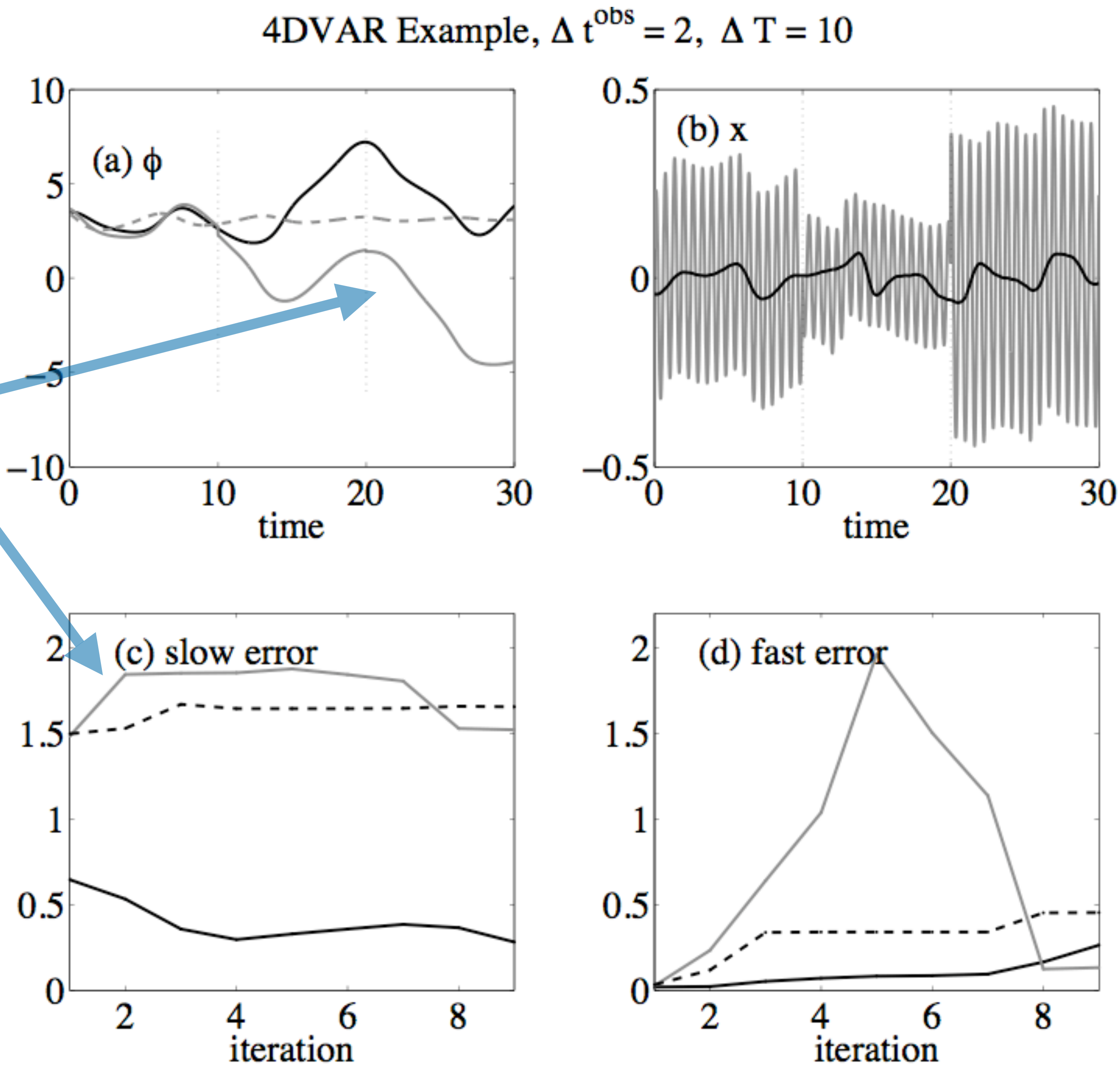
4D-Var

$$J(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x_0} - \mathbf{x_0^b}\right)^{\mathbf{T}} \mathbf{B}^{-1}\left(\mathbf{x}_0 - \mathbf{x_0^b}\right) + \frac{1}{2}\sum_{i=0}^{N}\left(\mathbf{z}_i - H(\mathbf{x}_i)\right)^{\mathbf{T}} \mathbf{R}^{-1}\left(\mathbf{z}_i - H(\mathbf{x}_i)\right)$$



As the assimilation window is increased, the cost function becomes more jagged: the 4D-Var problem is more difficult, and issues like balance become more of a problem.

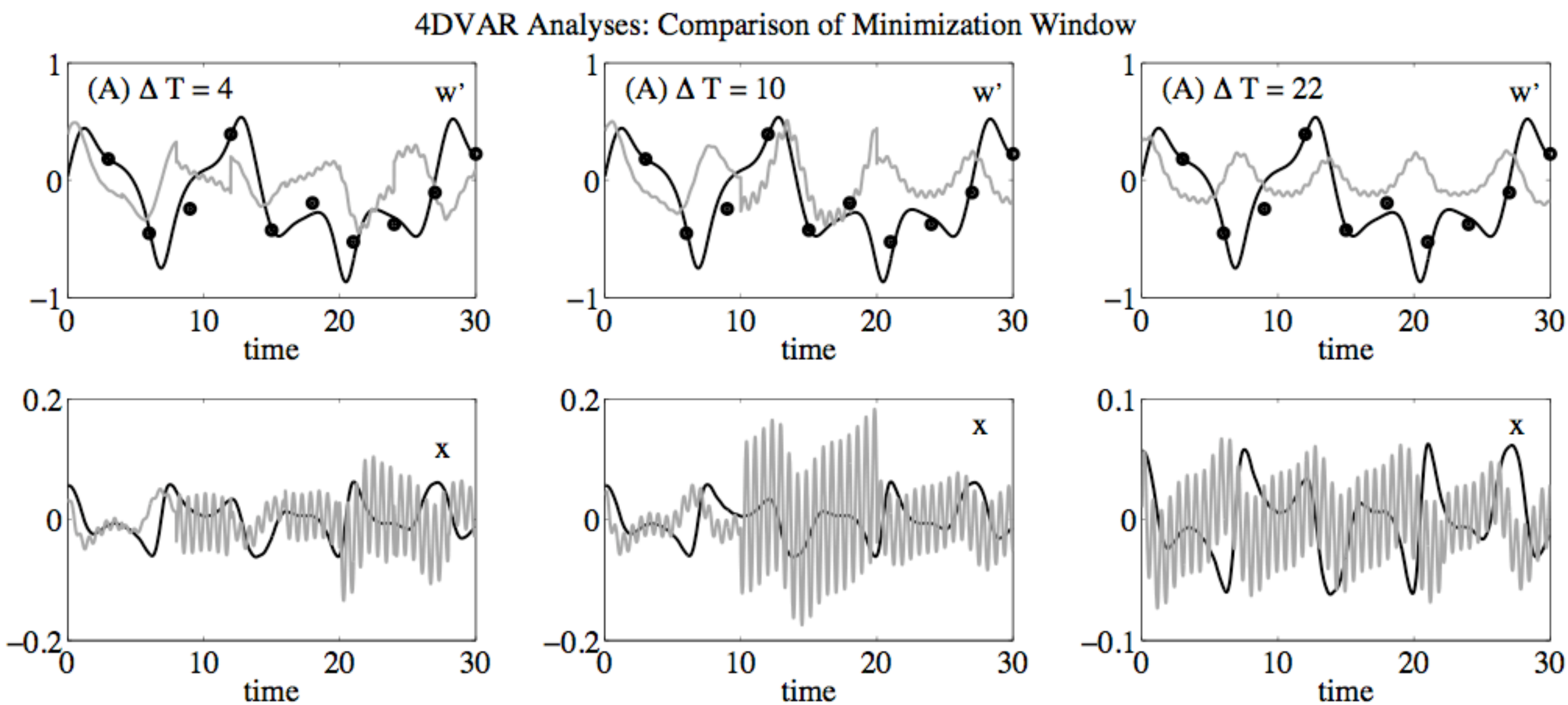
Aside: 4D-Var for a Balanced Truth



A poor fit.

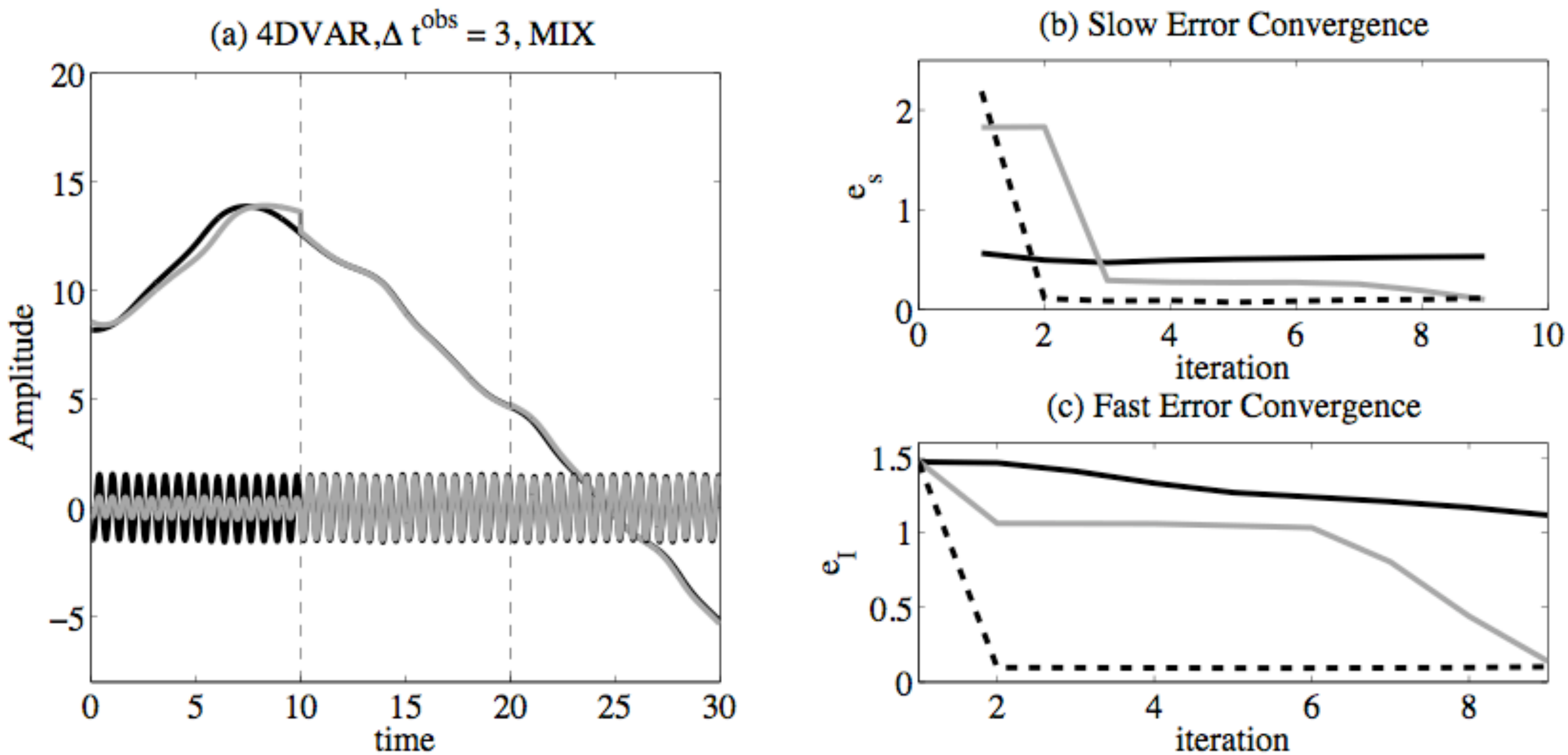
Spurious imbalance in different windows

Aside: 4D-Var for a Balanced Truth



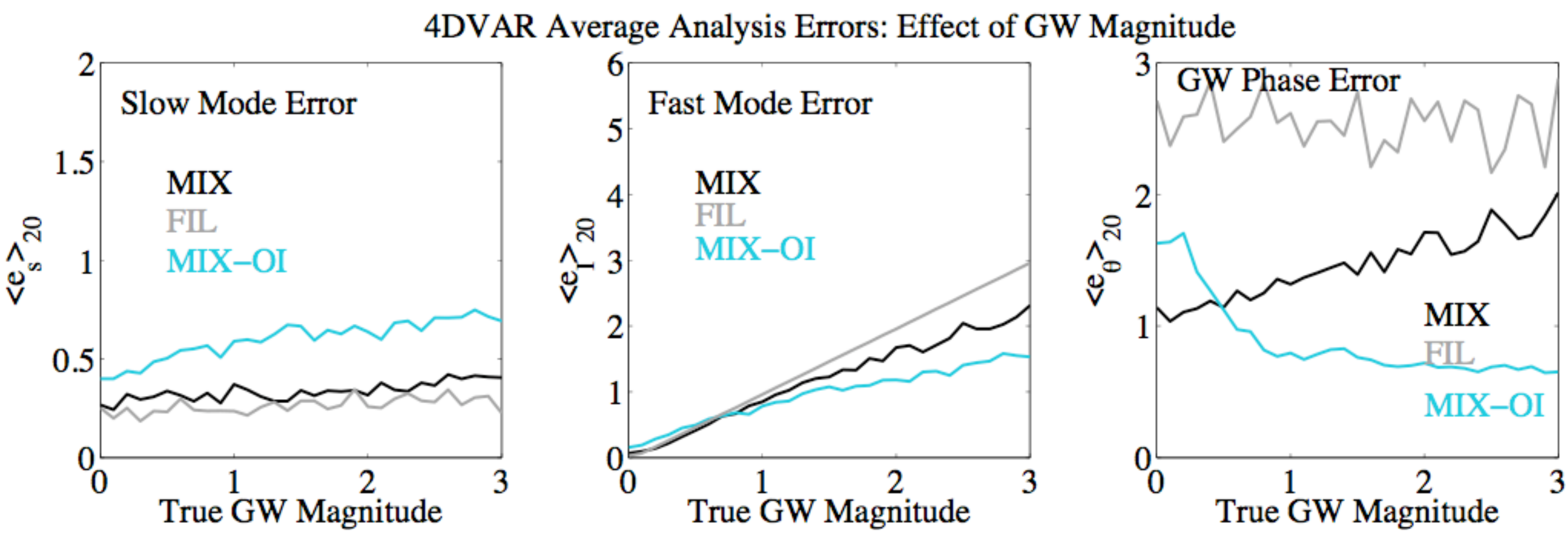
Increasing the window makes it harder to fit the slow mode. Spurious imbalance is generated as Var searches around for the minimum.

4D-Var for an Unbalanced Truth



In all three windows, the 4D-Var converges in the slow mode -- but it doesn't always generate the right GW.

Gravity Waves in 4D-Var



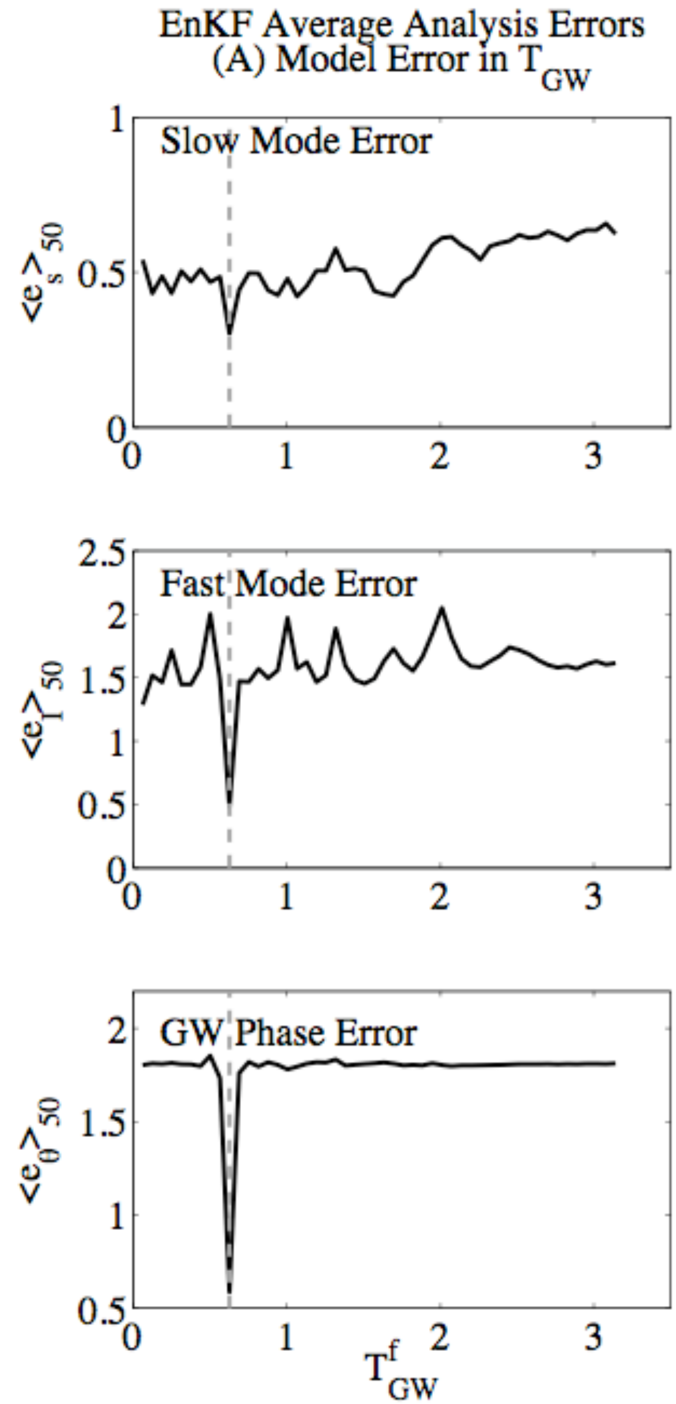
Ability to find the slow mode (esp. relative to OI).

It's difficult to capture the fast magnitude without more iterations. (Artifact? Even so…)

4D-Var also has trouble capturing GW phase

Back to the EnKF: what if the GW frequency isn't perfectly known?

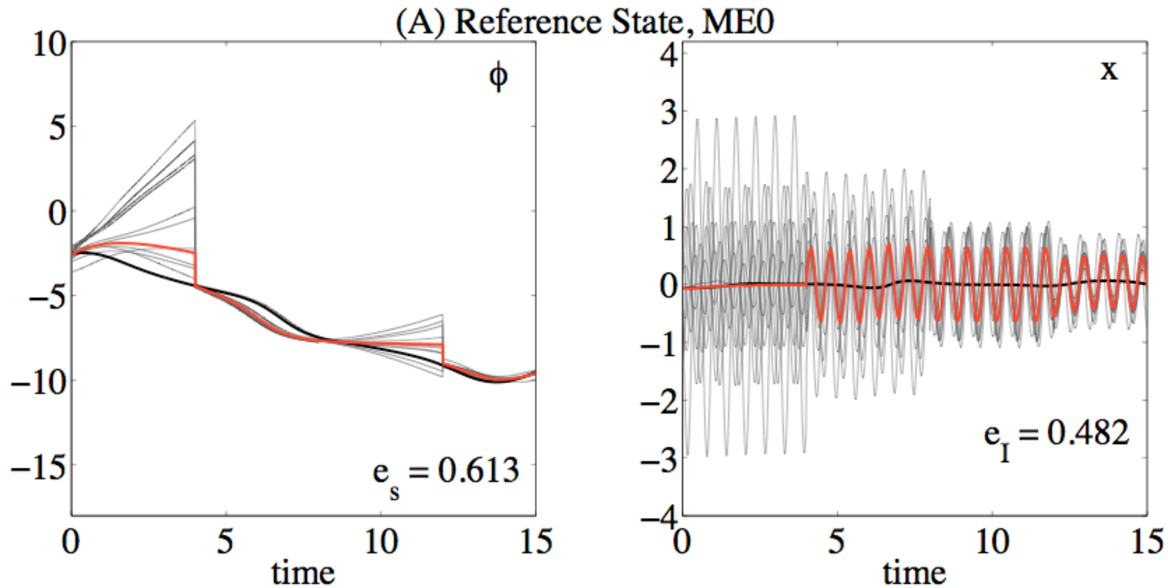
Introduce an error in the modeled GW frequency (ε): EnKF now really has trouble recovering the fast wave.



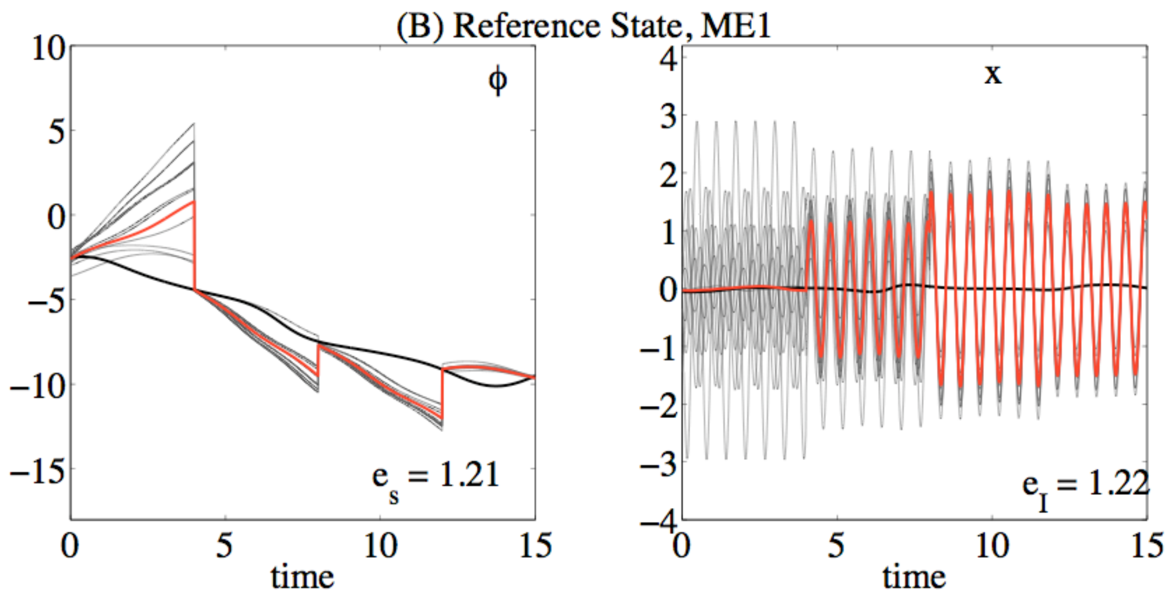
EnKF: What if we add model error?

(balanced truth)

No ME: Slow mode ok, but slight loss of balance.



Now the EnKF is even more likely to phase-lock onto the wrong GW magnitude.



Summary . Conclusions . Implications

- Both EnKF and 4D-Var generally improved upon OI
- The problem of capturing *slaving* remains in both contexts.
- EnKF does have the ability to capture both modes, especially GW phase, but experiments are idealized.
- 4D-Var has difficulty recovering the fast motion. Somewhat mysteriously, it's generated as we iterate further.
- "Unbalanced truth" suggests that 4D assimilation is well worth the effort.

More Questions! Some points for Further Research

- EnKF or 4D-Var? No decisive answer on which is better…
- Fitting GW *parameters*?
- Larger-dimensional models with GW spectra, spatial dimension, or a chaotic fast mode.
- How do EnKF/4D-Var differ when tropical waves are added to the mix?
- More finely-tuned 4D-Var implementation
- Testing alternative algorithms: the Lorenz-86 model (extended) is a good testing / teaching environment.

Fast-Slow Error Covariances

correlation between
ϕ and x

$$\rho_{\phi x} = \frac{c_{\phi x}}{\sigma_\phi \sigma_x}$$

covariance
resulting from
slaving

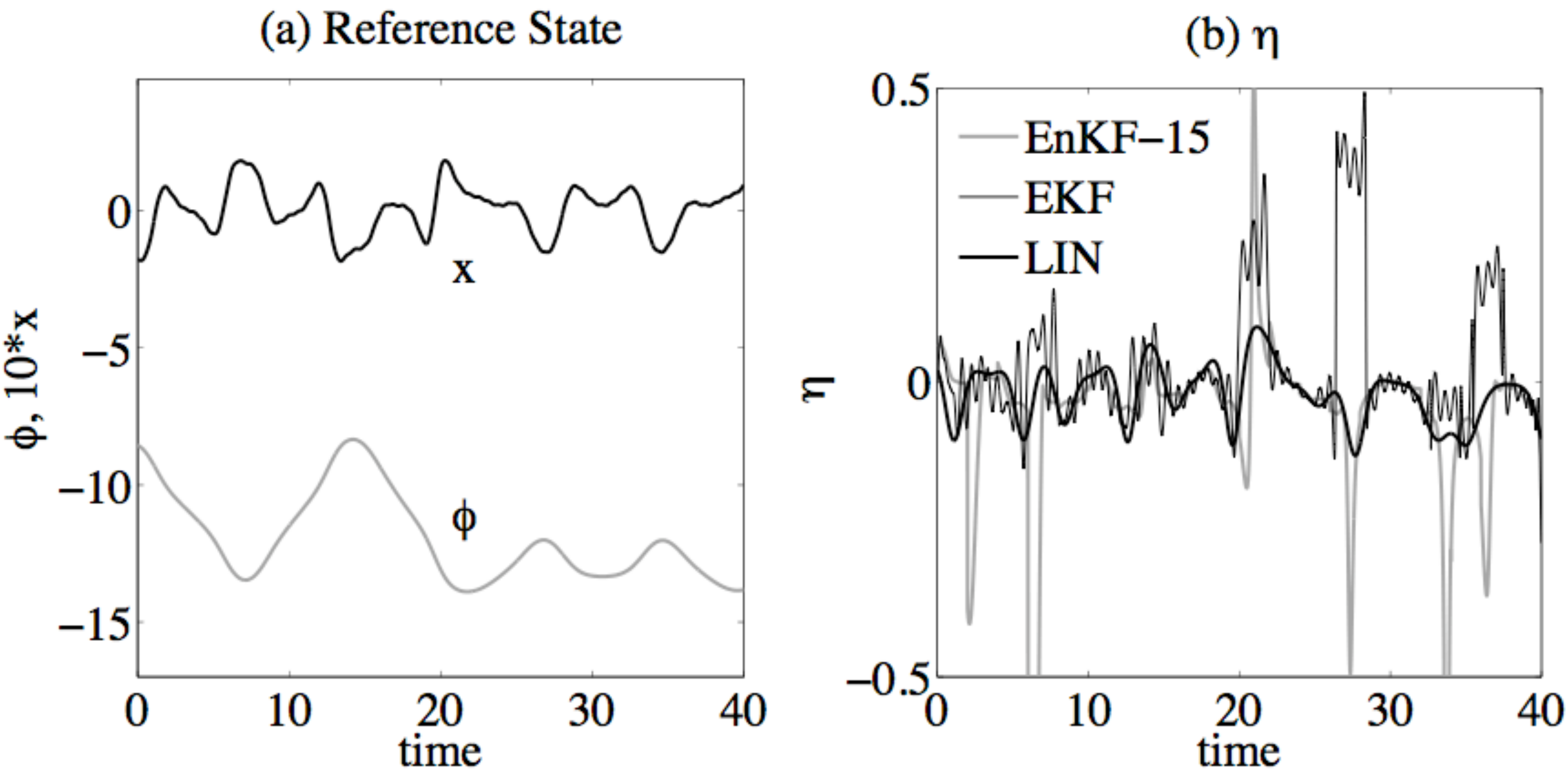
$$c_{\phi x} = \langle e_\phi \frac{\partial U_x}{\partial \phi} e_\phi \rangle = -\epsilon C b \cos 2\phi (\sigma_\phi^2)$$

$$\rho_{\phi x} = -\epsilon C b \cos 2\phi \frac{\sigma_\phi}{\sigma_x} \equiv \eta_{\mathrm{LIN}}(t) \frac{\sigma_\phi}{\sigma_x}$$

The ability of each
filter to capture
this term depends
on the accuracy of
assumptions made.

$$\eta = \rho_{\phi x} \frac{\sigma_x}{\sigma_\phi}$$

Example: fast-slow



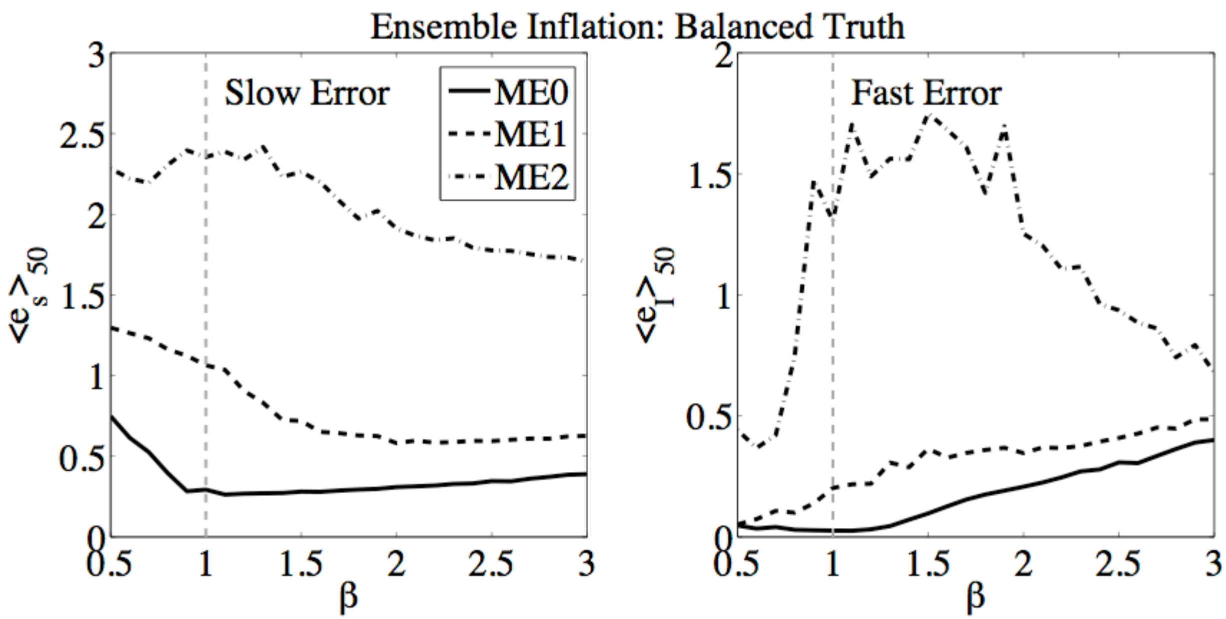
The correlation is state-dependent, so 4DDA should be more useful.

Both filters incur estimation error: let's look at the consequences in various regimes.

Modifications to the Standard Algorithms

Assimilation error comes not just from model error but also from accumulated analysis error.

How do filter divergence remedies affect these results?



Ensemble inflation

Extra terms tend to reduce error in the slow mode…

But increase error in the fast mode.

Balance and Timescale Separation

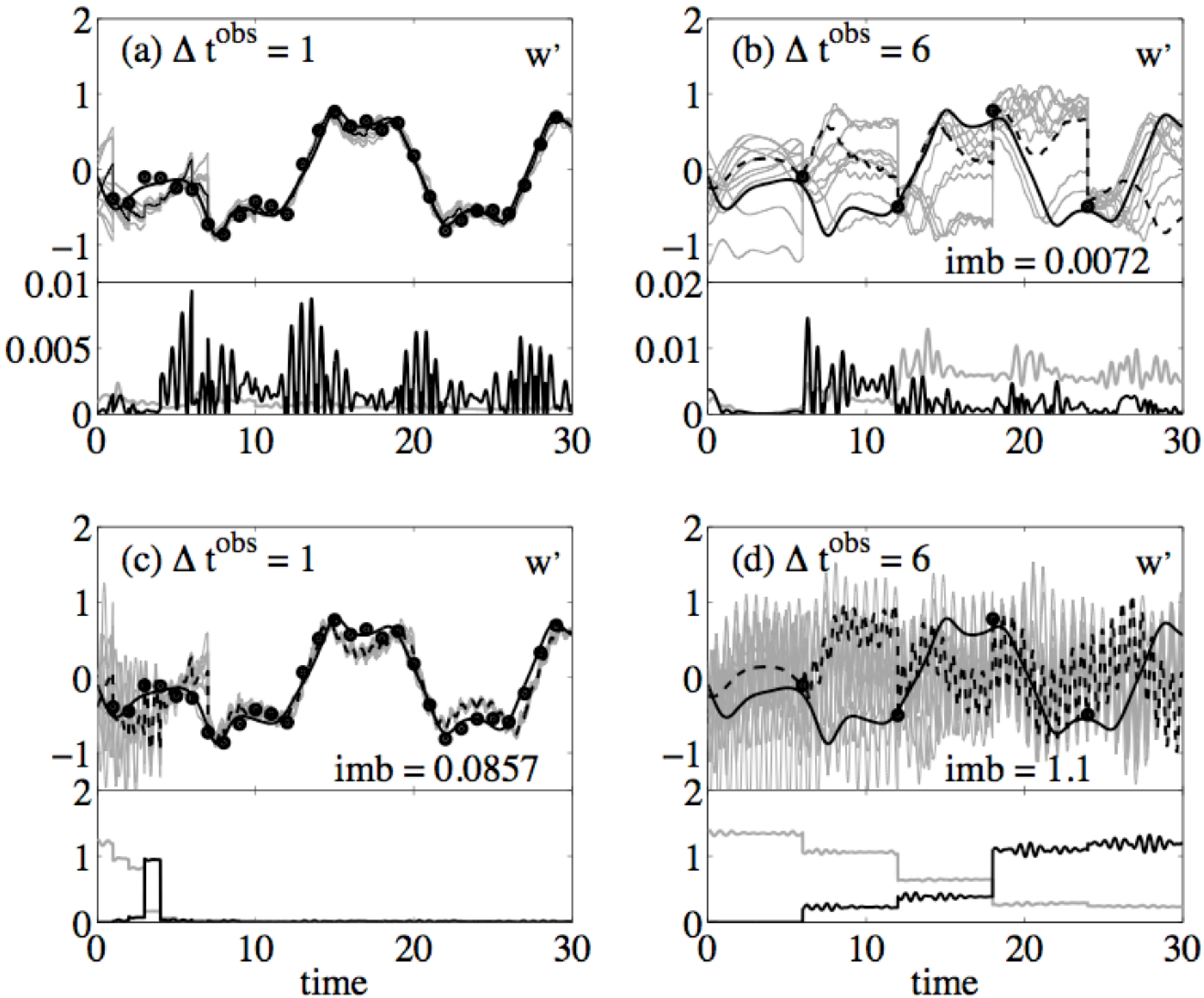
$$\epsilon \equiv \frac{U}{\left(c_{\mathrm{gw}}^2 + c_{\mathrm{i}}^2\right)^{1/2}} = \frac{U}{\left(gH + f^2L^2\right)^{1/2}}$$

$$\epsilon \equiv \frac{Ro\ B}{\sqrt{1+B^2}}, \qquad \tau_1 = f^{-1}$$

$$R \equiv U/fL \qquad \tau_2 = L/U$$

$$B \equiv fL/\sqrt{gH}$$

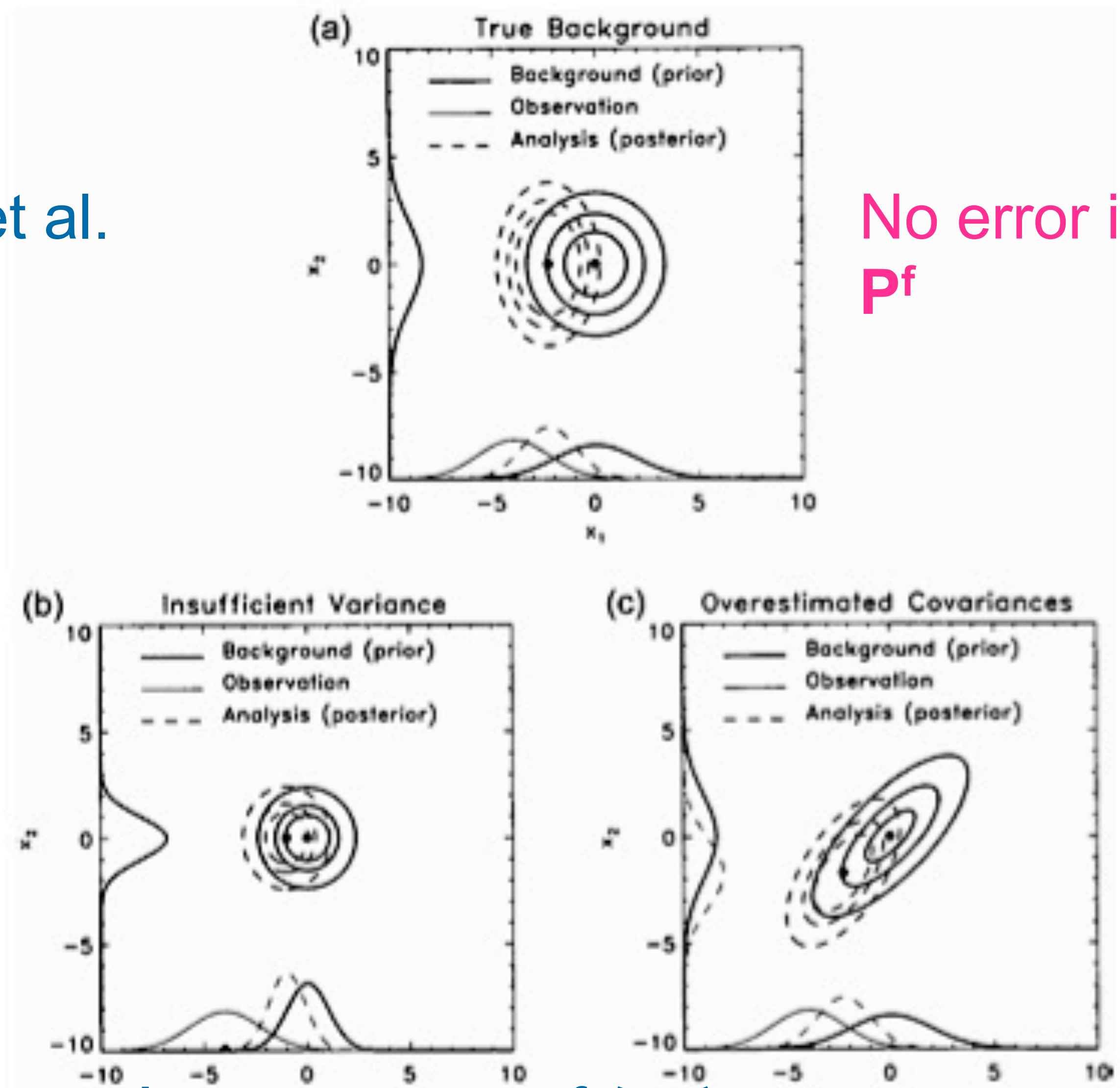
Loss of Balance: EnKF example



Analysis errors vs forecast errors

(Hamill et al. 2001)

No error in $\mathbf{P}^f$



Insufficient variance

Misestimated covariances

If error in the estimation of (co)variances is too great, the assimilation **diverges**, and obs are **rejected**.