Nonlinear inertial stability of zonal flows near the equator

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OUTLINE

- Inertial instability, examples
- Hamiltonian form of hydrostatic β-plane equations
- Stability of zonal solution
- Solution to equations linearized about zonal solution
- Saturation of instability

Inertial instability

- Steadily rotating flows have balance between centrifugal and pressure gradient forces

- Flow is *inertially unstable* if radial perturbations are amplified by imbalance of forces

- Rayleigh criterion for stability:

  absolute angular momentum everywhere increases with distance from axis of rotation

- Examples:
  - Couette-Taylor experiment
  - Equatorial ocean and middle atmosphere

Equatorial β-plane

- Model equatorial dynamics with β-plane instead of sphere

- Since inertial stability depends on angular momentum distribution, we must retain as much information about angular momentum in spherical as we can.

- Zonal flows in rotating, spherical shell conserve

$$m_0 = ur\cos\phi + \Omega r^2 \cos^2\phi$$

- Construct symmetric β-plane equations to conserve largest terms in m_0. Hydrostatic system conserves

$$m = u - \tfrac{1}{2}\beta y^2$$

where $\beta \equiv \dfrac{2\Omega}{a}$.

Hydrostatic β-plane equations

(for adiabatic, zonally symmetric flow)

$$
\begin{aligned}
u_t &= -vu_y - wu_z + \beta yv \\
v_t &= -vv_y - wv_z - \beta yu - \frac{1}{\rho} p_y \\
g - \frac{1}{\rho} p_z &= 0 \\
\rho_t &= -(\rho v)_y - (\rho w)_z \\
\theta_t &= -v\theta_y - w\theta_z
\end{aligned}
$$

- Conserve total energy

$$
\begin{aligned}
\mathcal{H}(m, v, \rho, \theta) &= \int\int \rho\{\tfrac{1}{2}(u^2 + v^2) \\
&+ gz + \mathcal{E}(\rho, \theta)\} dydz
\end{aligned}
$$

- and integrals of the form

$$
\mathcal{C}(m, \theta) = \int\int \rho C(m, \theta) dydz
$$

Hamiltonian form

- By switching to pressure coordinates we can put the system in the Hamiltonian form $\frac{\partial x_i}{\partial t} = \mathsf{J}_{ij}\frac{\delta \mathcal{H}}{\delta x_j}$

$$\frac{\partial}{\partial t}\begin{bmatrix} \zeta \\ m \\ \theta \end{bmatrix} = \begin{bmatrix} \partial(\zeta,\cdot) & \partial(m,\cdot) & \partial(\theta,\cdot) \\ \partial(m,\cdot) & 0 & 0 \\ \partial(\theta,\cdot) & 0 & 0 \end{bmatrix}\begin{bmatrix} \psi \\ \frac{1}{2}\beta y^2 \\ c_p\frac{T}{\theta} \end{bmatrix}$$

  where ψ is a streamfunction defined by

$$v = -\frac{\partial\psi}{\partial p} \quad \omega = \frac{\partial\psi}{\partial y}$$

  and $\zeta \equiv \frac{\partial v}{\partial p}$.

- The vector on the right side is the functional derivatives of

$$\mathcal{H}(\zeta,m,\theta) = \int\int\{\tfrac{1}{2}v^2 + \tfrac{1}{2}\beta y^2 m + \mathcal{E}(\rho,\theta) + p/\rho^2\}dydp$$

Noncanonical structure

- System is generalization of Hamilton's equations of particle dynamics:

$$\begin{bmatrix} \frac{\partial p_i}{\partial t} \\ \frac{\partial q_i}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q_i} \end{bmatrix}$$

- Difference is that fluid systems are noncanonical.
  - J is not invertible
  - $\mathbf{x}_t = 0$ does not imply that $\frac{\delta \mathcal{H}}{\delta \mathbf{x}} = 0$ (equilibria are not critical points of $\mathcal{H}$)
  - There exist Casimir invariants satisying $\mathsf{J}\dfrac{\delta \mathcal{C}}{\delta \mathbf{x}} = 0 \quad \forall \mathbf{x}$. Namely,

$$\mathcal{C}(m, \theta) = \iint C(m, \theta) dy dp$$

Stability of equilibrium

- An equilibrium $\mathbf{X}_0$ is stable with respect to the norm $||\cdot||$ if

  $\forall \epsilon \; \exists \delta$ such that

  $$||\mathbf{x}(0) - \mathbf{X}_0|| < \delta \Rightarrow ||\mathbf{x}(t) - \mathbf{X}_0|| < \epsilon \; \forall t$$

  (the solution $\mathbf{x}(t)$ can be bounded arbitrarily close to $\mathbf{X}_0$ if it starts close enough)

- Energy-Casimir method

  – Choose $C(m, \theta)$ such that $\frac{\delta}{\delta x}(\mathcal{H} + \mathcal{C})\Big|_{X_0} = 0$

  – Define pseudoenergy

  $$\mathcal{A}(\mathbf{x}) \equiv (\mathcal{H} + \mathcal{C})(\mathbf{x}) - (\mathcal{H} + \mathcal{C})(\mathbf{X}_0)$$

  – Find a norm $||\cdot||$ such that for all t

  $$||\mathbf{x}(t) - \mathbf{X}_0|| \leq \mathcal{A}(t) \leq c||\mathbf{x}(t) - \mathbf{X}_0|| < \infty$$

  – Then $\mathbf{X}_0$ is stable with respect to $||\cdot||$.

Zonal equilibrium

- Consider a purely zonal equilibrium solution $\mathbf{X}_0$ with

$$m = M(y,p), \ \ \zeta = 0, \ \ \theta = \Theta(y,p)$$

with associated temperature $T(p,\Theta)$ and density $R(p,\Theta)$

- Must satisfy thermal wind equation

$$\beta y \left(\frac{\partial M}{\partial p}\right)_y = \frac{1}{R\Theta}\left(\frac{\partial \Theta}{\partial y}\right)_p$$

- $\left.\dfrac{\delta}{\delta x}(\mathcal{H}+\mathcal{C})\right|_{\mathbf{X}_0} = 0$ if

$$C_m = -\tfrac{1}{2}\beta\left[y(M,\Theta)\right]^2 \ \ \text{and} \ \ C_\theta = -c_p\frac{T}{\Theta}$$

Aside: Linear stability

- Necessary condition for stability is that the equations linearized about $\mathbf{X}_0$ be stable.

- Follows if quadratic approximation to $\mathcal{A}$ is positive definite

bowl shaped

versus

saddle shaped

$$\mathcal{A}(m', v', \theta') \approx \iint \left\{ \tfrac{1}{2} v'^2 + \tfrac{1}{2} \mathbf{x}'^{\mathsf{T}} \Lambda \mathbf{x}' \right\} dy\, dp$$

where

$$\mathbf{x}' \equiv \begin{bmatrix} \frac{m'}{m_0} \\ \frac{\theta'}{\theta_0} \end{bmatrix}, \Lambda \equiv \begin{bmatrix} m_0^2 C_{mm} & m_0 \theta_0 C_{m\theta} \\ m_0 \theta_0 C_{m\theta} & \Theta_0^2 C_{\theta\theta} \end{bmatrix}\Bigg|_{(M,\Theta)}$$

Linear stability $\iff$ Λ positive definite

Linear stability (cont'd)

- Λ is positive definite if and only if its eigenvalues $\lambda_1(M, \Theta)$ and $\lambda_2(M, \Theta)$ are positive almost everywhere.

  Equivalently, if

$$C_{mm} > 0, \qquad C_{\theta\theta} > 0$$

$$C_{mm}C_{\theta\theta} - C_{m\theta}^2 > 0$$

- Evaluating the derivatives, we find conditions for linear stability

$$\frac{-\beta y}{Q}\left(\frac{\partial \Theta}{\partial p}\right)_y > 0 \quad \text{static stability}$$

$$\frac{-1}{R\Theta Q}\left(\frac{\partial M}{\partial y}\right)_p > 0 \quad \text{inertial stability}$$

$$\frac{\beta y}{R\Theta Q} > 0 \quad \text{symmetric stability}$$

Normed stability

- Linear stability criteria are in fact nonlinear stability criteria with slight modification

- Using Taylor's Remainder Theorem, write
$$\mathcal{A}(m', v', \theta') = \iint \left\{ \tfrac{1}{2} v'^2 + \tfrac{1}{2} \mathbf{x}'^{\mathsf{T}} \tilde{\Lambda} \mathbf{x}' \right\} dy dp$$
where $\tilde{\Lambda} \equiv \left. \begin{bmatrix} m_0^2 C_{mm} & m_0 \theta_0 C_{m\theta} \\ m_0 \theta_0 C_{m\theta} & \Theta_0^2 C_{\theta\theta} \end{bmatrix} \right|_{(\tilde{m}, \tilde{\theta})}$
and $\tilde{m} \in [M, M + m']$, $\tilde{\theta} \in [\Theta, \Theta + m']$

- If A is symmetric with eigenvalues $\alpha_1 \leq \alpha_2$,
$$\alpha_1 |\mathbf{x}|^2 \leq \mathbf{x}^{\mathsf{T}} \mathsf{A} \mathbf{x} \leq \alpha_2 |\mathbf{x}|^2 \;\; \forall \mathbf{x}$$
suggesting the family of norms
$$\left\| (m', v', \theta')^{\mathsf{T}} \right\|_{\lambda}^2 = \iint \left\{ \tfrac{1}{2} v'^2 + \tfrac{\lambda}{2} \mathbf{x}'^{\mathsf{T}} \mathbf{x}' \right\} dy dp$$

Normed stability (cont'd)

- Define
  $\lambda_{min} \equiv \min(\lambda(m,\theta))$, $\lambda_{max} \equiv \max(\lambda(m,\theta))$

- Then

$$||\mathbf{x}(t) - \mathbf{X}_0||^2_{\lambda_{min}}$$
$$\leq \mathcal{A}(t) = \mathcal{A}(0) \leq ||\mathbf{x}(0) - \mathbf{X}_0||^2_{\lambda_{max}}$$

  and, finally,

$$||\mathbf{x}(0) - \mathbf{X}_0||^2_{\lambda_{min}} < \delta$$
$$\Rightarrow ||\mathbf{x}(t) - \mathbf{X}_0||^2_{\lambda_{min}} < \frac{\lambda_{max}}{\lambda_{min}}\delta$$

- If the linear stability conditions are satisfied for all values of m and θ accessible by perturbation, the equilibrium is nonlinearly stable.

Saturation of instability

- The nonlinear stability analysis suggests a method for bounding the amount of kinetic energy released when an unstable equilibrium is perturbed

- Consider unstable equilibrium as finite amplitude perturbation to a stable equilibrium, and use

$$\mathcal{K}_y(t) \equiv \iint \tfrac{1}{2} v'^2(t)\, dy dp \le \mathcal{A}(t) = \mathcal{A}(0)$$

- Example: Dunkerton basic state

$$m = U_0 + sy - \tfrac{1}{2}\beta y^2$$
$$\theta = \theta_0 p^{-HN^2/g}$$

as perturbation to

$$M = U_0 - \tfrac{1}{2}By^2$$
$$\Theta = \theta_0 p^{-HN^2/g}$$

yields $\mathcal{K}_y \le \frac{8}{3}\frac{s^5}{\beta^3}$ (c.f. $K_y \approx \frac{27}{20}\frac{s^5}{\beta^3}$)

SUMMARY

- Linearized solution to equatorial inertial instability problem exhibits Taylor vortices as in Couette-Taylor experiment

- Hamiltonian formulation of equatorial β-plane equations for symmetric, adiabatic, hydrostatic flow exists

- Inertial stability conditions analogous to those in Couette-Taylor experiment can be derived for zonal flows on the equatorial β-plane using energy-Casimir method

- Leads to method for bounding saturation of instability