Analyse Hamiltonienne d'instabilité à inertie équatoriale

Mark Fruman
T. G. Shepherd
University of Toronto, Canada

Université de Bretagne Occidentale
Mardi le 11ème Octobre, 2005

OUTLINE

1. L'ENERGIE et LA STABILITE

2. La stabilité de la "RIGID BODY" TOURNANT

3. Critères pour LA STABILITE SYMETRIQUE EQUATORIALE

1. L'ENERGIE ET LA STABILITE

MAIN IDEA:

- For systems that conserve energy, the stability of fixed points (i.e. time independent solutions) is closely related to the behaviour of the energy function(al) near the fixed point.

- If a system is in canonical Hamiltonian form and the Hamiltonian function(al) does not depend *explicitly* on time, then:
  - The Hamiltonian is conserved in time.
  - Steady state solutions are critical points of the Hamiltonian function(al).
- For many systems, the Hamiltonian is the total energy (kinetic energy T plus potential energy U).

- Example: If the state of the system is described by generalized coordinate $\mathbf{q}(t) = (q_1(t), .., q_n(t))$ and its conjugate momentum $\mathbf{p}(t) = (p_1(t), .., p_n(t))$, and the Hamiltonian function is $H(\mathbf{q}, \mathbf{p})$, then Hamilton's Equations are

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}$$

(cf. Newton's equations $m\frac{\mathrm{d}q_i}{\mathrm{d}t} = p_i, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial U(\mathbf{q})}{\partial q_i}$)

- Therefore,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i^n \left(\frac{\partial H}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \right) = 0$$

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\mathrm{d}p_i}{\mathrm{d}t} = 0 \iff \nabla_{(\mathbf{q},\mathbf{p})} H = 0$$

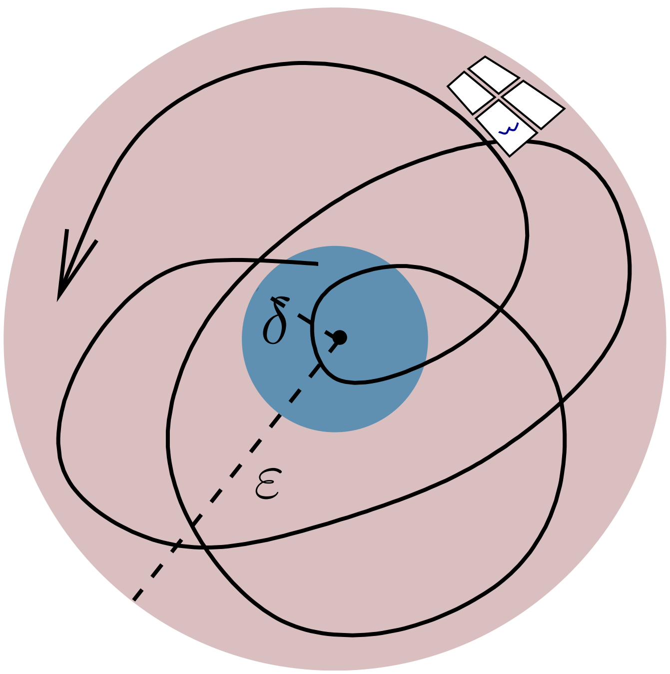
- What does this have to do with stability?
  - Stability of a steady solution means that if the system starts close enough to the steady solution, it will remain close for all time.
  - Mathematically (Lyapunov):

    Steady state $(\mathbf{Q}, \mathbf{P})$ is stable with respect to the norm $||(\mathbf{q} - \mathbf{Q}, \mathbf{p} - \mathbf{P})||$ if:

for every ε, there is a δ such that, if $||(\mathbf{q}(t=0), \mathbf{p}(t=0)) - (\mathbf{Q}, \mathbf{P})|| < \delta$, then $||(\mathbf{q}(t), \mathbf{p}(t)) - (\mathbf{Q}, \mathbf{P})|| < \varepsilon$ for all times t.

- Norm might be Euclidean norm (distance)

$$||(\mathbf{q}-\mathbf{Q},\mathbf{p}-\mathbf{P})|| \equiv \sqrt{\sum_i [(q_i - Q_i)^2 + (p_i - Q_i)^2]}$$

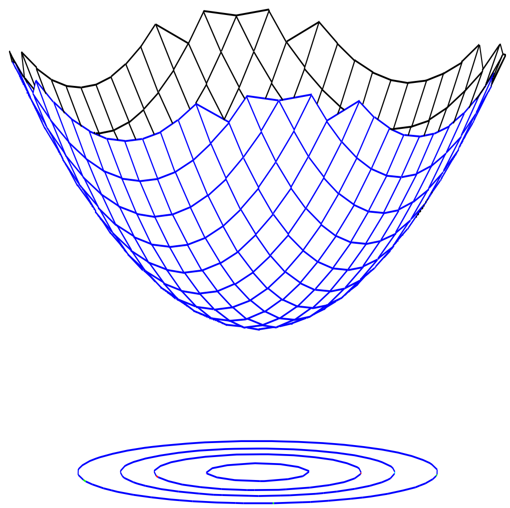


- Black line is trajectory of system through phase space (the space of all states $(\mathbf{q},\mathbf{p})$)

(← these balls are at least 3-dimensional)

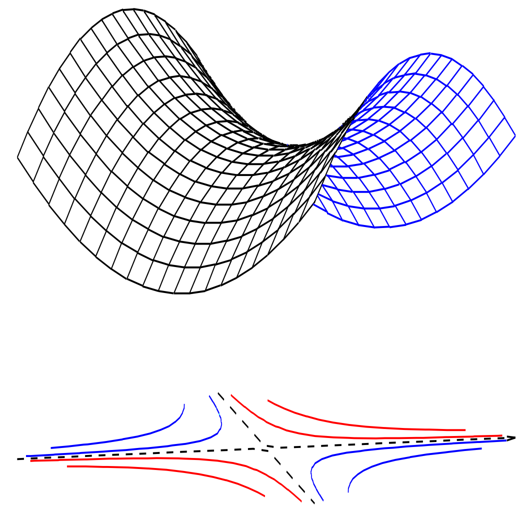
- In the neighbourhood of a critical point, the Hamiltonian can have one of two geometries, corresponding to stability and instability:

Phase space level curves $\longrightarrow$



"bowl"

Point is stable



"saddle"

Point is unstable

- Hamilton's equations can be generalized to non-canonical forms; i.e. systems for which the state of the system $\mathbf{x}$ is not described by conjugate pairs of coordinates and momenta.

- In that case, the equations (still) take the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{J}(\mathbf{x})\nabla_{\mathbf{x}}H(\mathbf{x})$$

but the matrix $\mathbf{J} \neq \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix}$ and is not invertible.

- Therefore, fixed points might not be critical points of the Hamiltonian!

- However . . . in non-canonical systems, there exist other conserved functionals called Casimir invariants.
- They may be thought of as constraints on the dynamics:
  - Fixed points are critical points of the Hamiltonian given the constraints of the Casimir invariants.
  - Equivalently, fixed points are points at which there exists a Casimir invariant $C(\mathbf{x})$ such that the surface of constant Hamiltonian $H(\mathbf{x})$ is tangent to the surface of constant $C(\mathbf{x})$.

    (cf. the method of Lagrange multipliers for finding extrema of constrained systems).
- For stability, we look at the geometry of $H(\mathbf{x}) + C(\mathbf{x})$ near the fixed point.

2. EXEMPLE: LA "RIGID BODY" TOURNANT

(e.g. Arnold, V. I. *Mathematical Methods of Classical Mechanics*)

- The rotational properties of a free rigid body are characterized by its moments of inertia I_1, I_2, and I_3 about its three principal axes.
- The state of the system is given by $\mathbf{m} = (m_1, m_2, m_3)$, the body's angular momentum about each of its principal axes. The Hamiltonian is the kinetic energy

$$H(\mathbf{m}) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right)$$

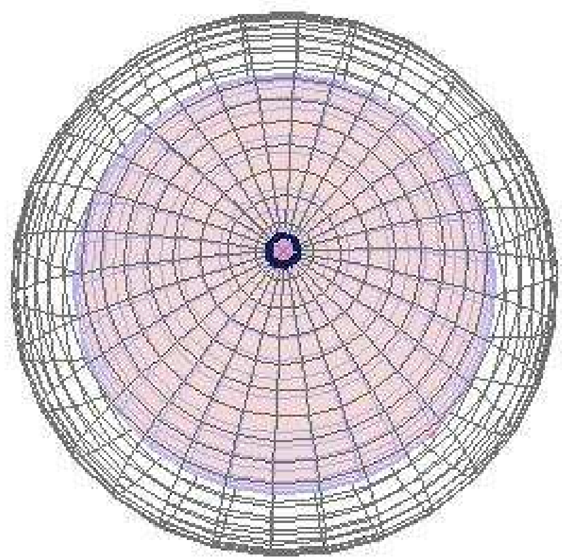
and the Casimir is total angular momentum

$$C(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2$$

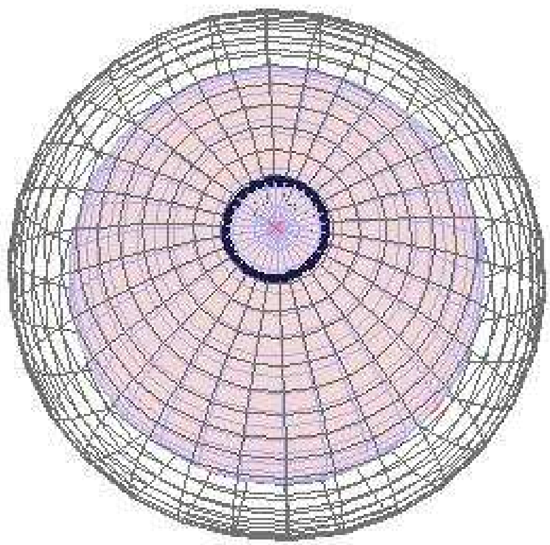
- The trajectories of the system through phase space (i.e. the three dimensional space of angular momentum) are thus the intersections of a sphere (angular momentum), and an ellipsoid (kinetic energy).
- Fixed points are points at which the sphere and the ellipsoid are tangent.
- If the moments of inertia are all different, the only fixed points correspond to rotation strictly about one of the principal axes.
- Are these fixed points STABLE or UNSTABLE?

- Rotation about the **lightest** axis . . .

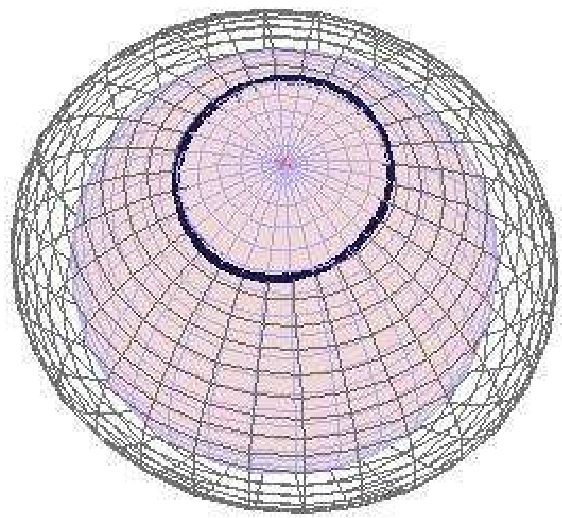
- Rotation about the **lightest** axis . . .



- Rotation about the **lightest** axis . . .



- Rotation about the **lightest** axis . . .

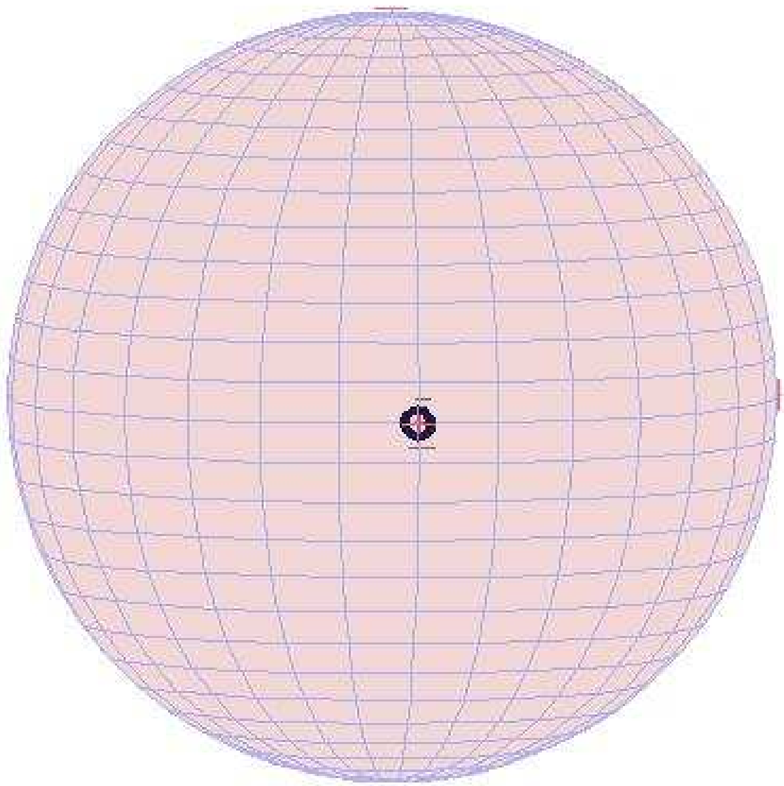


- Rotation about the **lightest** axis is

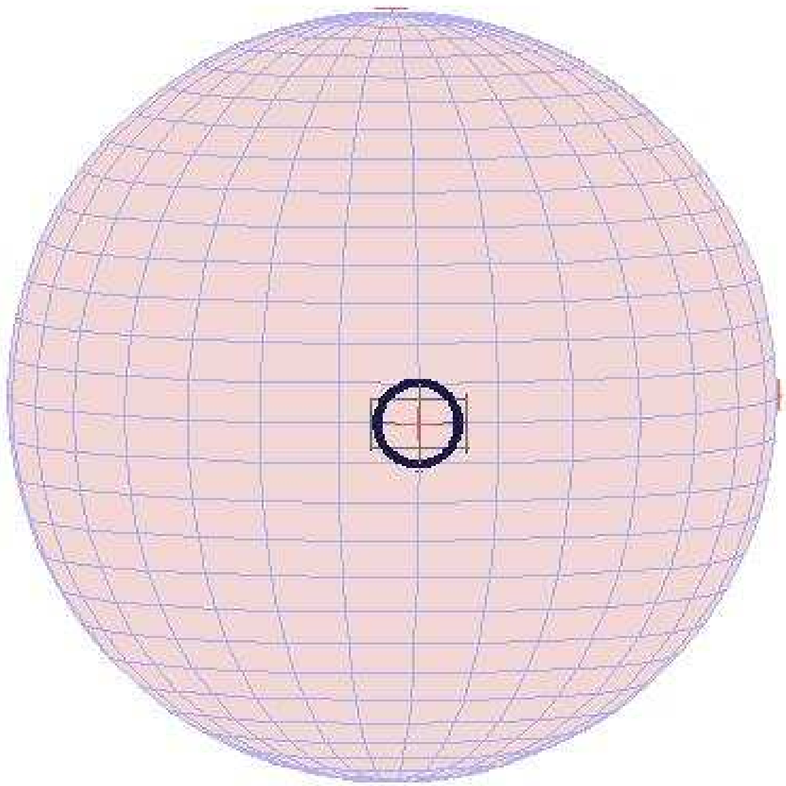
$\Rightarrow$ **STABLE!**

- Rotation about the **heaviest** axis . . .

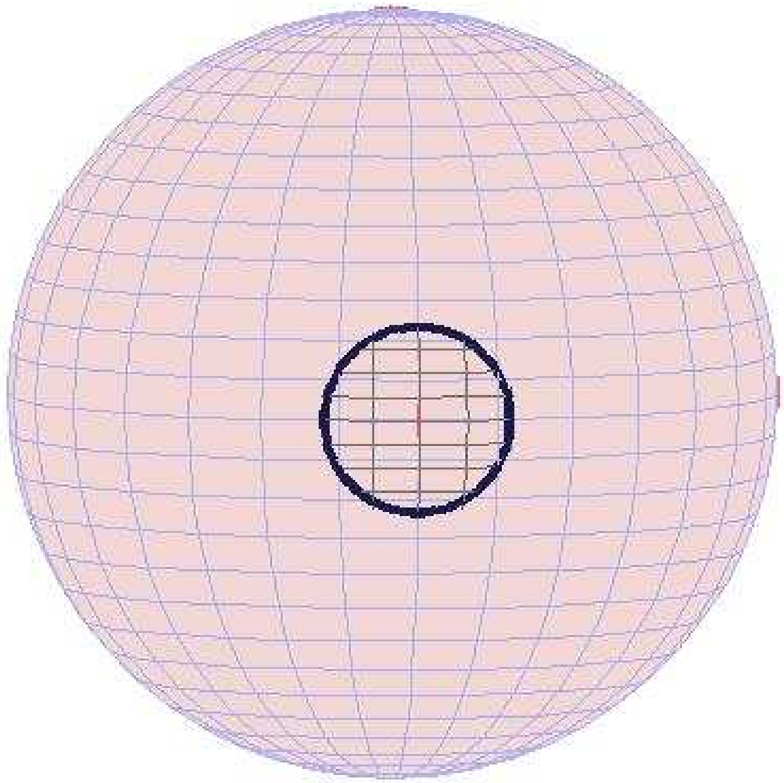
- Rotation about the **heaviest** axis . . .



- Rotation about the **heaviest** axis . . .



- Rotation about the **heaviest** axis . . .

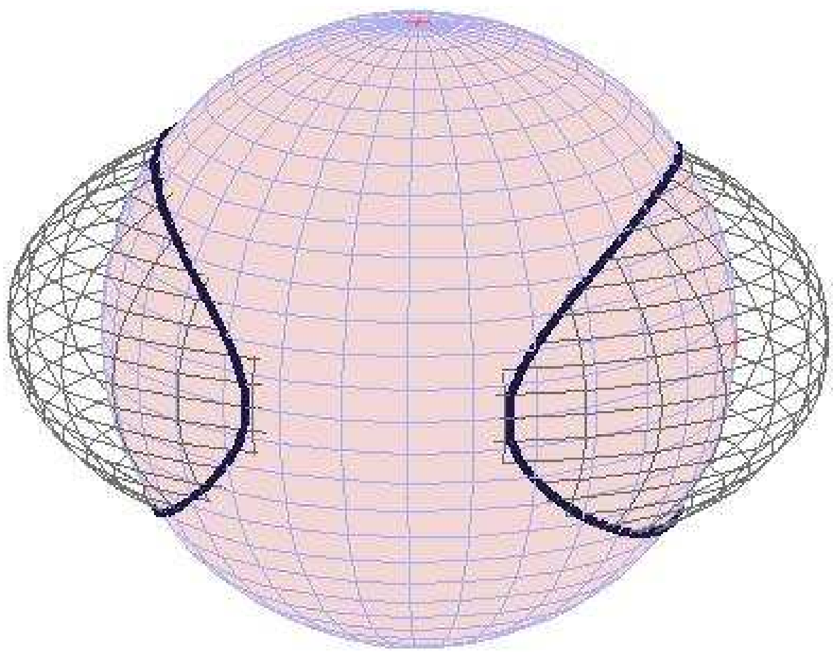


- Rotation about the **heaviest** axis is

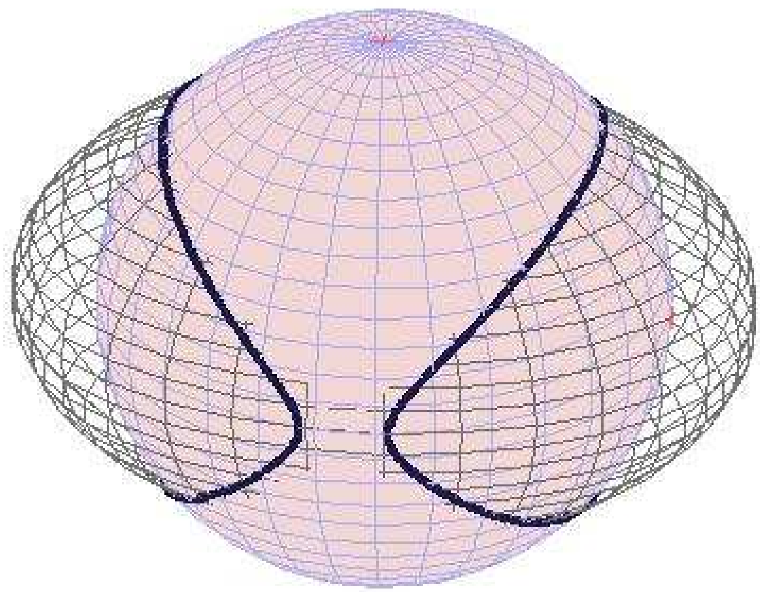
$\Rightarrow$ **STABLE!**

- Rotation about the **middle** axis . . .

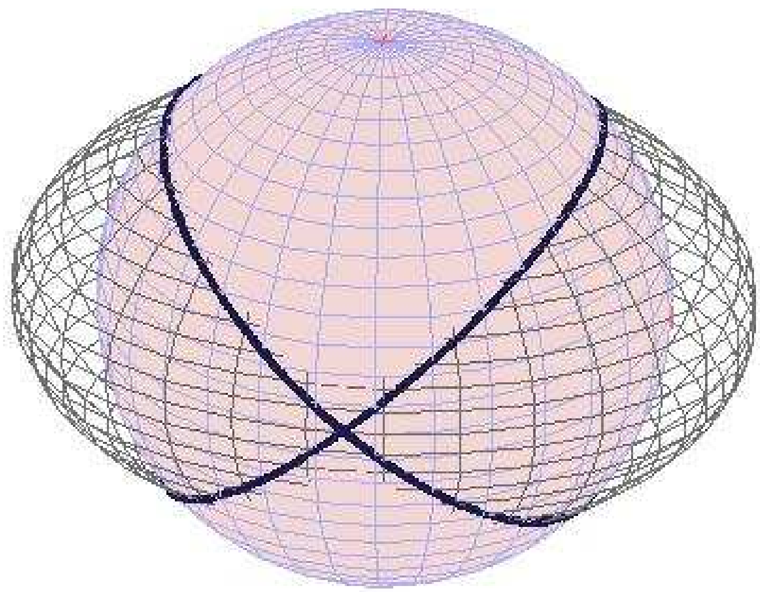
- Rotation about the **middle** axis . . .



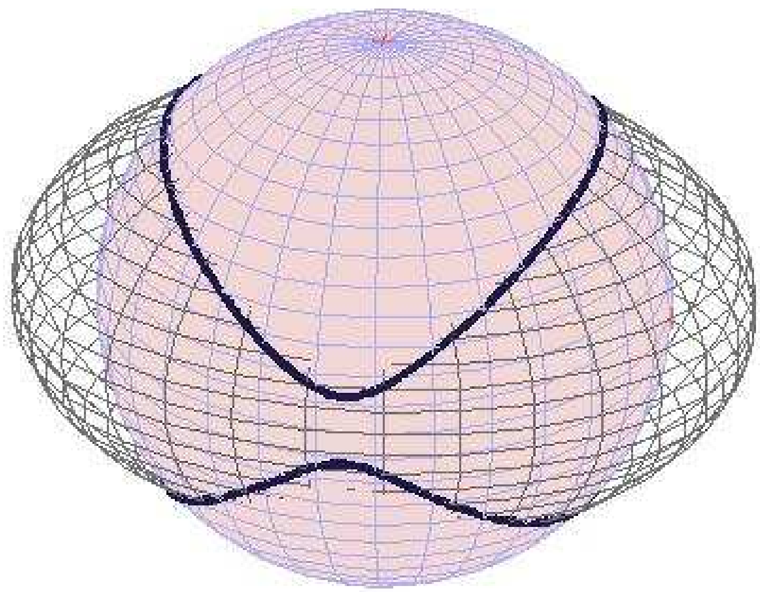
- Rotation about the **middle** axis . . .



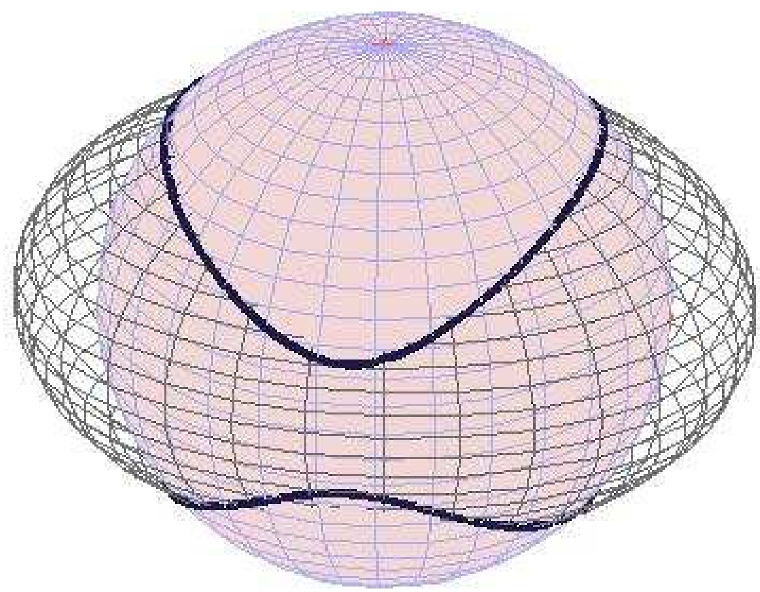
- Rotation about the **middle** axis . . .



- Rotation about the **middle** axis . . .



- Rotation about the **middle** axis . . .



- Rotation about the **middle** axis is

$\Rightarrow$ **UNSTABLE!**

3. LA STABILITE SYMETRIQUE EQUATORIALE

- Hamiltonian methods can also apply to the partial differential equations of fluid mechanics.
- The independent variables are continuous functions of time *and* space, and the Hamiltonian is a functional of the independent variables.
- Hamiltonian fluid systems written in Eulerian variables (velocity, temperature, entropy, etc.) are non-canonical.
- The Casimirs are commonly functionals of Lagrangian invariants like potential vorticity and entropy.

- Consider the problem of the stability of a zonal (east-west) flow in the equatorial atmosphere.
- Instability in equatorial zonal flows is a significant process in shaping the dynamics in the equatorial stratosphere during solstice seasons, and in organizing moist convection in the equatorial troposphere.
- If the flow is adiabatic and inviscid, it can be described by a non-canonical Hamiltonian system of equations.
- If the system is also assumed to be independent of longitude, its stability characteristics can be determined with the "energy-Casimir" method.
- We use the anelastic approximation on the equatorial β-plane.

- The state of the system is described by $\mathbf{x} = (m, \zeta, \theta)$, where m is the component of absolute angular momentum corresponding to zonal motion, ζ is the component of relative vorticity in the zonal direction (corresponding to meridional and vertical motion), and θ is potential temperature.
- The Hamiltonian is:

$$\mathcal{H} = \iint \left\{ \rho_0 \left(\tfrac{1}{2}\beta y^2 - \gamma z \right) m + \frac{1}{2\rho_0} \left[\left(\frac{\partial \psi}{\partial z} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] + \rho_0 \pi_0 \theta \right\} \mathrm{d}y \, \mathrm{d}z$$

  where (y, z) are latitude and altitude, $(0, \gamma, \beta y)$ is the local planetary rotation vector, ψ is a streamfunction for motion in the (y, z) plane, and $\pi_0(z)$ and $\rho_0(z)$ are prescribed pressure and density fields.

- The Casimirs are functionals of the form

$$\mathcal{C} = \iint C(m,\theta)\,\mathrm{d}y\,\mathrm{d}z$$

where $C(m,\theta)$ is an arbitrary twice differentiable function.

- Steady states $\mathbf{X} = (M(y,z), 0, \Theta(y,z))$ satisfy the thermal wind equation

$$\left(\frac{\mathrm{d}\pi_0}{\mathrm{d}z}\right)\frac{\partial\Theta}{\partial y} - \beta y\frac{\partial M}{\partial z} + \gamma\frac{\partial M}{\partial y} = 0$$

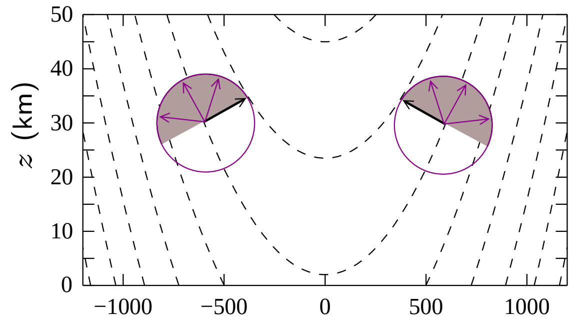
- Step 1: Find the Casimirs which are tangent to the Hamiltonian at the fixed points. These give relations between the derivatives of $C(m, \theta)$ and the steady state functions M and Θ.

- Step 2: Find conditions under which $\mathcal{H} + \mathcal{C}$ has a minimum at the basic state. These are conditions on the second derivatives of $C(m, \theta)$ and in turn on the gradients of M and Θ, and in particular on the potential vorticity $Q = \frac{1}{\rho_0}\partial(\Theta, M)$.

- Step 3: We can then define a norm on the displacements $\mathbf{x} - \mathbf{X}$ with respect to which the conditions found in Step 2 are sufficient for Lyapunov stability.

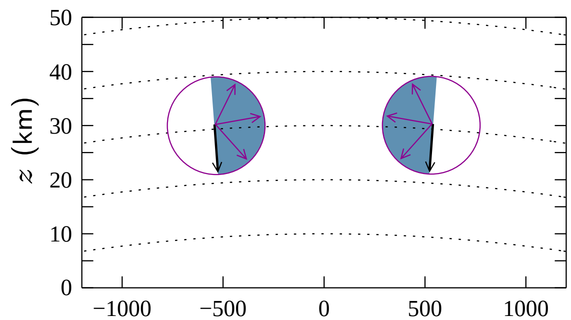
$\Rightarrow$ Stability Conditions

$$\frac{1}{Q}\partial(\Theta, -\tfrac{1}{2}\beta y^2 + \gamma z) > 0$$



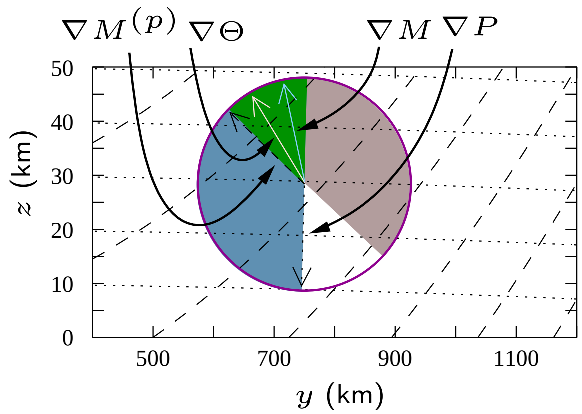
STATIC STABILITY

$$\frac{1}{Q}\frac{\partial M}{\partial y} < 0$$



INERTIAL STABILITY

$$\frac{y}{Q} > 0$$



SYMMETRIC STABILITY

SOMMAIRE

- In Hamiltonian systems, there is a direct connection between the geometry of the energy functional near fixed points and their stability properties.
- Non-canonical Hamiltonian systems have additional invariants (besides energy) called Casimir invariants. Fixed points are critical points of the Hamiltonian given the Casimir invariants as constraints.
- The free rotations of a rigid body are an example of a system described by a non-canonical Hamiltonian system of equations. Rotation about the principal axis of intermediate moment of inertia is an unstable fixed point.

- Inviscid, adiabatic fluid systems in Eulerian variables are also non-canonical Hamiltonian.
- Longitudinally symmetric zonal flows at the equator are stable if the potential vorticity has the sign of latitude, and the absolute zonal angular momentum increases towards the equator.