## INTRODUCTION

The "traditional" hydrostatic approximation to the equations of fluid motion on a sphere includes certain geometrical approximations necessary for retaining conservation of energy and angular momentum. In particular, the radial coordinate is replaced by the earth's mean radius, and the horizontal component of the earth's rotation vector is neglected. The neglected terms are most significant near the equator. Studies by White and Bromley (1995) and de Verdière and Schopp (1994) have suggested that neglecting these terms near the equator is not justified.

We consider the importance of making the hydrostatic approximation in the context of equatorial inertial instability. Inertial instability refers to a flow becoming unstable due to its distribution of angular momentum. The simplest case, axisymmetric circular flow, is unstable if the angular momentum decreases with distance from the axis of rotation. Adjustment to a stable state involves the formation of vertical rolls superposed on the circular flow. These are known as *Taylor vortices* after the Taylor-Couette experiment on flow between co-rotating cylinders (Fig. 1a). In the equatorial middle atmosphere, the approximately zonal mean flow can become inertially unstable if the angular momentum is maximum away from the equator, leading to the formation of vortices in the vertical-meridional plane (Fig. 1b).

**PREVIOUS RESULTS**

*Dunkerton (1981)* - considered a zonal flow on an equatorial $\beta$-plane with no vertical velocity shear and linear horizontal shear. The linearized system can be solved exactly. The most unstable mode consists of rolls in the $yz$ plane, the formation of vertically stacked zonal jets of alternating direction, and vertically stacked cells of relatively warm and relatively cold air (see Fig. 2). The zonal jets are a signature of inertial instability in observations of the equatorial deep ocean (Hua et al 1997) and the temperature cells in observations of the equatorial middle atmosphere (Hitchman et al 1987).

*Stevens (1983)* - derived linear stability criteria for symmetric zonal flow on a sphere in hydrostatic system.

*Cho et al (1993)* and *Mu et al (1996)* - applied energy-Casimir method to symmetric stability problem on an $f$-plane. Found linear and nonlinear stability criteria.

*Bowman and Shepherd (1995)* - found linear and nonlinear stability criteria for flow on a sphere in hydrostatic system using energy-Casimir method.

*White and Bromley (1995)* - reexamined traditional hydrostatic approximation for purposes of numerical weather prediction. Presented scaling arguments that showed neglected Coriolis terms can be as large as 10
*Hua, Moore and Le Gentil (1997)* - motivated by observations of equatorial ocean, performed numerical experiments on inertial adjustment keeping the nonhydrostatic Coriolis terms. They derived the criterion for linear stability in nonhydrostatic system.

## GOVERNING EQUATIONS

Fluid parcels undergoing frictionless, zonal motion in a spherical shell conserve absolute zonal angular momentum

$$L \equiv ur\cos\phi + \Omega r^2\cos^2\phi,$$

where $u$ is zonal velocity, $r$ is distance from the centre of the earth, and $\phi$ is latitude.

For flow near the equator, we consider only the largest non-constant terms in the Taylor expansion of $L$

$$\frac{L}{a} = (\Omega a) + (u - \frac{\Omega}{a}y^2 + 2\Omega z) + \mathcal{O}(\frac{\Omega}{a}z^2),$$

where $\Omega$ is the earth's angular velocity, $a$ is the mean radius of the earth, $y \equiv r\phi$ is arclength away from the equator, and $z$ is distance above the earth's surface, and we approximate the spherical geometry with that of a plane. In this study, we consider flow in a rectangular $(x, y, z)$ box with periodic boundary conditions in $x$ that materially conserves the "angular momentum"

$$m \equiv u - \tfrac{1}{2}\beta y^2 + \gamma z,$$

where $\beta \equiv 2\Omega/a$ and $\gamma \equiv 2\Omega$. The appropriate set of equations is

$$\left.\begin{aligned} u_t &= -vu_y - wu_z + \beta yv - \gamma w \\ v_t &= -vv_y - wv_z - \beta yu - \tfrac{1}{\rho}p_y \end{aligned}\right\} \quad \text{horizontal momentum}$$

$$\gamma u - g - \tfrac{1}{\rho}p_z = 0 \qquad \text{"hydrostatic" condition}$$

$$\rho_t = -(\rho v)_y - (\rho w)_z \qquad \text{continuity equation}$$

$$s_t = -vs_y - ws_z \qquad \text{adiabatic motion}$$

In the above, $\rho$ and $p$ are density and pressure, $u$, $v$, and $w$ are velocities in the $x$, $y$ and $z$ directions, $g$ is the acceleration due to gravity, and $s$ is the entropy.

Setting $\gamma = 0$ reduces the system to the usual *hydrostatic equatorial $\beta$-plane equations* (see Andrews et al 1987). The system as it stands incorporates the more realistic angular momentum information. It might be called the *quasi-hydrostatic equatorial $\beta$-plane equations* after White and Bromley (1995).

## THE ENERGY-CASIMIR METHOD OF STABILITY ANALYSIS

We now introduce the *energy-Casimir* method of stability analysis. The idea is to exploit the connection between the stability of an equilibrium and the phase space geometry of a conserved function of the dynamics in the neighbourhood of the corresponding fixed point. As an introduction, we outline the method for finite dimensional systems (governed by sets of ordinary differential equations in canonical form) before describing and applying the method to an infinite dimensional fluid problem (governed by a set of partial differential equations in noncanonical form).

### Ordinary Differential Equations

The state of a system governed by a set of ordinary differential equations $\mathbf{x}_t = \mathbf{F}(\mathbf{x})$ can be represented by a point in a finite dimensional *phase space*. If the system is conservative, then there is a conserved function $H(\mathbf{x})$ called the *Hamiltonian*, and the evolution of the system in phase space is constrained to curves of constant $H$.

If the system is in *canonical form*, then equilibrium solutions $\mathbf{X}$ of the equations (satisfying $\mathbf{F}(\mathbf{X}) = 0$) are *critical points* of the Hamiltonian. That is

$$\mathbf{F}(\mathbf{X}) = 0 \quad \Rightarrow \quad \left.\frac{\partial H}{\partial x_i}\right|_{\mathbf{X}} = 0, \;\; i = 1..n,$$

where $n$ is the dimension of the phase space.

$\mathbf{X}$ is a *linearly stable* solution if the equations linearized about $\mathbf{X}$ have no exponentially growing solutions. Linear stability follows if $\mathbf{X}$ is an extreme point of the Hamiltonian, or equivalently, if the matrix

$$\left.\frac{\partial^2 H}{\partial x_i \partial x_j}\right|_{\mathbf{X}}, \;\; i, j = 1..n$$

is positive definite (has strictly positive eigenvalues). This is intuitive in two-dimensions: in the neighbourhood of an extremum, H is shaped like a bowl, and level curves of H are closed; in the neighbourhood of an unstable equilibrium, H is shaped like a saddle, and level curves are open (so the system is not constrained to stay near the equilibrium).

**Partial Differential Equations and Noncanonical Representation**

The same analysis can be applied to systems of partial differential equations. In that case, the state of the system is represented by a point $x$ in an infinite dimensional phase space, and stability is assessed based on the geometry of a conserved *functional* of the dynamical fields.

Hamiltonian representations of fluid systems are typically cast in *noncanonical form.* The *Hamiltonian functional* $\mathcal{H}$ is conserved as well as families of *Casimir invariants* $\mathcal{C}$. A fixed point $X$ of the system is a critical point of a combined invariant $\mathcal{H}+\mathcal{C}$. We denote this condition by

$$\left.\frac{\delta(\mathcal{H}+\mathcal{C})}{\delta x}\right|_X = 0.$$

*Linear stability* of the solution $X$ follows if it can be shown that the second variation,

$$\delta^2(\mathcal{H}+\mathcal{C})|_X$$

is of definite sign. The Casimirs are typically arbitrary functions of materially conserved quantities. In practice, we use the first criterion above to determine the appropriate Casimir and the second to determine sufficient conditions for stability of the equilibrium flow.

## STABILITY OF ZONAL EQUATORIAL FLOW

### Linear stability

The quasi-hydrostatic $\beta$-plane system conserves the total energy

$$\mathcal{H}(m, v, \rho, s) = \iint \rho \left( \tfrac{1}{2}(u^2 + v^2) + gz + E(\rho, s) \right) dydz$$

where $E(\rho, s)$ is the internal energy, obeying the thermodynamic identity

$$dE = \frac{p}{\rho^2} d\rho + T ds,$$

where $T$ is temperature. It also conserves the family of Casimirs

$$\mathcal{C}(m, \rho, s) = \iint \rho C(m, s) dydz,$$

where $C(m, s)$ is an arbitrary function.

We investigate the stability of a steady, purely zonal flow, with $m = M$ $(u = U)$, $v = 0$ $(w = 0)$, $\rho = R$, $s = S$ (with corresponding $p = P(R, S)$ and steady state temperature $T(R, S)$ determined by a thermodynamic equation of state). The flow must satisfy the *thermal wind equation*

$$\beta y M_z + \gamma M_y = \left( \frac{\partial T}{\partial R} \right)_S (R_z S_y - R_y S_z).$$

We determine $C(m, s)$ by calculating the first variation of $\mathcal{H}+\mathcal{C}$ and requiring that it vanish for the equilibrium flow. We obtain

$$C = -\tfrac{1}{2} U^2 - gz(M, S) - E(R, S) - R \left( \tfrac{\partial E}{\partial R} \right)_S,$$

$$C_m = -U, \qquad C_s = -T,$$

which is meaningful and consistent only if the steady state fields $M(y, z)$ and $S(y, z)$ uniquely define inverse maps $y(M, S)$ and $z(M, S)$. That is equivalent to assuming that the steady state potential vorticity

$$Q(y, z) \equiv M_z S_y - M_y S_z$$

is single-signed in the domain of interest.

The next step is to determine conditions on $C$ such that the basic flow is an extremum of $\mathcal{H}+\mathcal{C}$. To that end, we calculate the second variation of $\mathcal{H}+\mathcal{C}$, and require it to be positive definite (positive for arbitrary variations of the dynamical variables $m$, $v$, $\rho$, $s$ from their basic states). We obtain the conditions

$$(\rho E_{\rho\rho}+2E_\rho)|_{(R,S)}>0$$

$$(1+C_{mm})|_{(M,S)}>0$$

$$\left[(\rho E_{\rho\rho}+2E_\rho)(E_{ss}+C_{ss})-\rho E_{\rho s}^2\right]\Big|_{(R,M,S)}>0$$

$$\left\{\left[(\rho E_{\rho\rho}+2E_\rho)(E_{ss}+C_{ss})-\rho E_{\rho s}^2\right](1+C_{mm})-(\rho E_{\rho\rho}+2E_\rho)C_{ms}^2\right\}\Big|_{(R,M,S)}>0.$$

We can rewrite the conditions in the more instructive forms

| | |
|---|---|
| $c^2\equiv\left(\dfrac{\partial P}{\partial R}\right)_S>0$ | c is the speed of sound, so this is trivially true |
| $\dfrac{1}{Q}(\beta y S_z+\gamma S_y)>0$ | static stability |
| $\dfrac{1}{Q}\left(\dfrac{\partial T}{\partial R}\right)_S[\beta y U M_z+(\gamma U-g)M_y]>0$ | inertial stability |
| $g\left(\dfrac{\partial T}{\partial R}\right)_S\dfrac{\beta y}{Q}>0$ | $fQ>0$ (result of Hua et al 1997) |

Observe that for an ideal gas, for example, $(\partial T/\partial R)_S$ is positive, so the inertial stability condition implies that $M$ decreases away from the equator ($g\gg\gamma U\approx 2\times10^{-4}$ m/s$^2$ - the dominant balance is $-gM_y/Q>0$).

The final condition partly validates our assumption that $Q$ be single-signed. Since $f$ changes sign only at the equator, $Q$ must be positive in the northern hemisphere, negative in the southern, and (for continuity) must vanish at the equator itself. Under our stability conditions, the map $(y,z)\rightarrow(M,S)$ is invertible in each hemisphere. $Q$ changing sign away from the equator can be shown to imply instability (Hua et al 1997).

**Nonlinear stability**

We have found conditions under which a zonal flow near the equator is *linearly* stable. This ensures that solutions to the equations *linearized* about that basic state have no exponentially growing modes.

True stability, however, requires that a solution that starts "near" the equilibrium, remains "near" for all subsequent time. We define nearness in terms of a *disturbance norm*. *Nonlinear* (or *finite amplitude*) stability, then, means

$$||x'(t=0)|| < \epsilon \Rightarrow ||x'(t)|| < \delta(\epsilon)\,\forall t > 0,$$

where $x'(t) = (m'(t), v'(t), \rho'(t), s'(t))$ is the departure of the state of the system from equilibrium $X$ at time $t$, and $||\cdot||$ is a norm.

We can demonstrate nonlinear stability of our basic flow by defining the *pseudoenergy* of the perturbed state

$$\mathcal{A} \equiv (\mathcal{H}+\mathcal{C})(x) - (\mathcal{H}+\mathcal{C})(X)$$

and showing that it can be bounded from above and below for all time by two comparable disturbance norms $||\cdot||_-$ and $||\cdot||_+$. Since $\mathcal{A}$ is an exact invariant, the following logical chain establishes nonlinear stability

$$||x'(t)||_- < \mathcal{A}(t) = \mathcal{A}(0) < ||x'(0)||_+ < k||x'(0)||_-$$

so that

$$||x'(0)|| < \epsilon \Rightarrow ||x'(t)|| < k\epsilon.$$

It can be shown that for some $\tilde{\rho} \in [R, R+\rho']$, $\tilde{m} \in [M, M+m']$ and $\tilde{s} \in [S, S+s']$,

$$\begin{aligned}\mathcal{A} \geq \iint \tfrac{1}{2}\rho \Bigg\{ v'^2 \; &+ \; \left[(\tilde{C}_{mm}+1) - \frac{s_0}{m_0}|\tilde{C}_{ms}|\right] m'^2 \\ &+ \; \left[(\tilde{E}_{\rho\rho} + \frac{2}{\tilde{\rho}}\tilde{E}_\rho) - \frac{s_0}{\rho_0}|\tilde{E}_{\rho s}|\right]\rho'^2 \\ &+ \; \left[(\tilde{C}_{ss}+\tilde{E}_{ss}) - \frac{\rho_0}{s_0}|\tilde{E}_{\rho s}| - \frac{m_0}{s_0}|\tilde{C}_{ms}|\right] s'^2 \Bigg\}\, dydz\end{aligned}$$

and

$$\mathcal{A} \leq \iint \tfrac{1}{2}\,\rho \left\{ v'^2 \;+\; \left[ (\tilde{C}_{mm}+1) + \frac{s_0}{m_0}|\tilde{C}_{ms}| \right] m'^2 \right.$$
$$+ \;\left[ (\tilde{E}_{\rho\rho} + \frac{2}{\tilde{\rho}}\tilde{E}_\rho) + \frac{s_0}{\rho_0}|\tilde{E}_{\rho s}| \right] \rho'^2$$
$$\left. + \;\left[ (\tilde{C}_{ss} + \tilde{E}_{ss}) + \frac{\rho_0}{s_0}|\tilde{E}_{\rho s}| + \frac{m_0}{s_0}|\tilde{C}_{ms}| \right] s'^2 \right\} dydz$$

where $\tilde{}$ above a symbol indicates that it is evaluated at $(\tilde{\rho}, \tilde{m}, \tilde{s})$ and $\rho_0$, $m_0$ and $s_0$ are arbitrary, positive dimensional constants.

We define a family of norms by

$$||x'||_k^2 \equiv \iint \rho \left\{ \tfrac{1}{2}v^2 + k(\tfrac{m_0^2}{\rho_0^2}\rho'^2 + m'^2 + \tfrac{m_0^2}{s_0^2}s'^2) \right\} dydz,$$

and observe that if

$$\min \left\{ \; (\tilde{C}_{mm}+1) - \frac{s_0}{m_0}|\tilde{C}_{ms}|, \right.$$
$$\frac{\rho_0^2}{m_0^2}\left[ (\tilde{E}_{\rho\rho} + \frac{2}{\tilde{\rho}}\tilde{E}_\rho) - \frac{s_0}{\rho_0}|\tilde{E}_{\rho s}| \right],$$
$$\left. \frac{s_0^2}{m_0^2}\left[ (\tilde{C}_{ss} + \tilde{E}_{ss}) - \frac{\rho_0}{s_0}|\tilde{E}_{\rho s}| - \frac{m_0}{s_0}|\tilde{C}_{ms}| \right] \right\} \geq c^- > 0$$

and

$$\max \left\{ \; (\tilde{C}_{mm}+1) + \frac{s_0}{m_0}|\tilde{C}_{ms}|, \right.$$
$$\frac{\rho_0^2}{m_0^2}\left[ (\tilde{E}_{\rho\rho} + \frac{2}{\tilde{\rho}}\tilde{E}_\rho) + \frac{s_0}{\rho_0}|\tilde{E}_{\rho s}| \right],$$
$$\left. \frac{s_0^2}{m_0^2}\left[ (\tilde{C}_{ss} + \tilde{E}_{ss}) + \frac{\rho_0}{s_0}|\tilde{E}_{\rho s}| + \frac{m_0}{s_0}|\tilde{C}_{ms}| \right] \right\} \leq c^+ < \infty$$

(where the minimum and maximum are taken over the whole domain and all time), then for all $t$,

$$||x'||_{c^-}^2 \leq \mathcal{A} \leq ||x'||_{c^+}^2.$$

and nonlinear stability follows as outlined above.

There is a fine point that should be addressed now. The linear stability criteria imply nonlinear stability only if the interpolated phase point $\tilde{x}$ at which the functions are evaluated in the norms is part of the domain at the initial time. This is true of $\tilde{m}$ and $\tilde{s}$ for a natural perturbation because $m$ and $s$ are Lagrangian invariants of symmetric adiabatic flow. The evolution of the system does not introduce any new values of $m$ and $s$. It is not necessarily true for $\tilde{\rho}$, but if we assume that $E_{\rho s}$ and $E_{ss}$ are positive and bounded for all time (reasonable for an ideal gas in the normal atmosphere), then nonlinear stability can be claimed. See Bowman and Shepherd (1995) for a discussion of this issue.

The nonlinear stability result has an interesting application. Given an unstable equilibrium, it is natural to ask how large the instability can grow before it *saturates*. To answer this question, we consider the unstable equilibrium as a finite amplitude perturbation to a nonlinearly stable equilibrium. We can thereby compute an upper bound to the saturation of the instability. The upper bound will depend on $c^-$, $c^+$, and the particular stable state that we choose. The problem then becomes one of minimizing the upper bound over these variables. See Mu et al (1996) for an application of this method to a related problem.

## CONCLUSION

We have considered the effects of relaxing the traditional hydrostatic approximation on the classic problem of symmetric stability. It is reassuring that the basic results concerning the stability criteria generalize from the hydrostatic $\beta$-plane case. In that sense, the fundamental structure of the system is unchanged.

Future work will focus on deriving saturation bounds on various unstable basic states and will look for quantitative differences from the hydrostatic case.

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