Equatorial Symmetric Stability in the Middle Atmosphere

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Mercredi, 19 Octobre, 2005

OUTLINE

1. Symmetric instability in the middle atmosphere
2. The inviscid adiabatic anelastic equations
3. Sufficient conditions for symmetric stability
4. Saturation bounds on instability

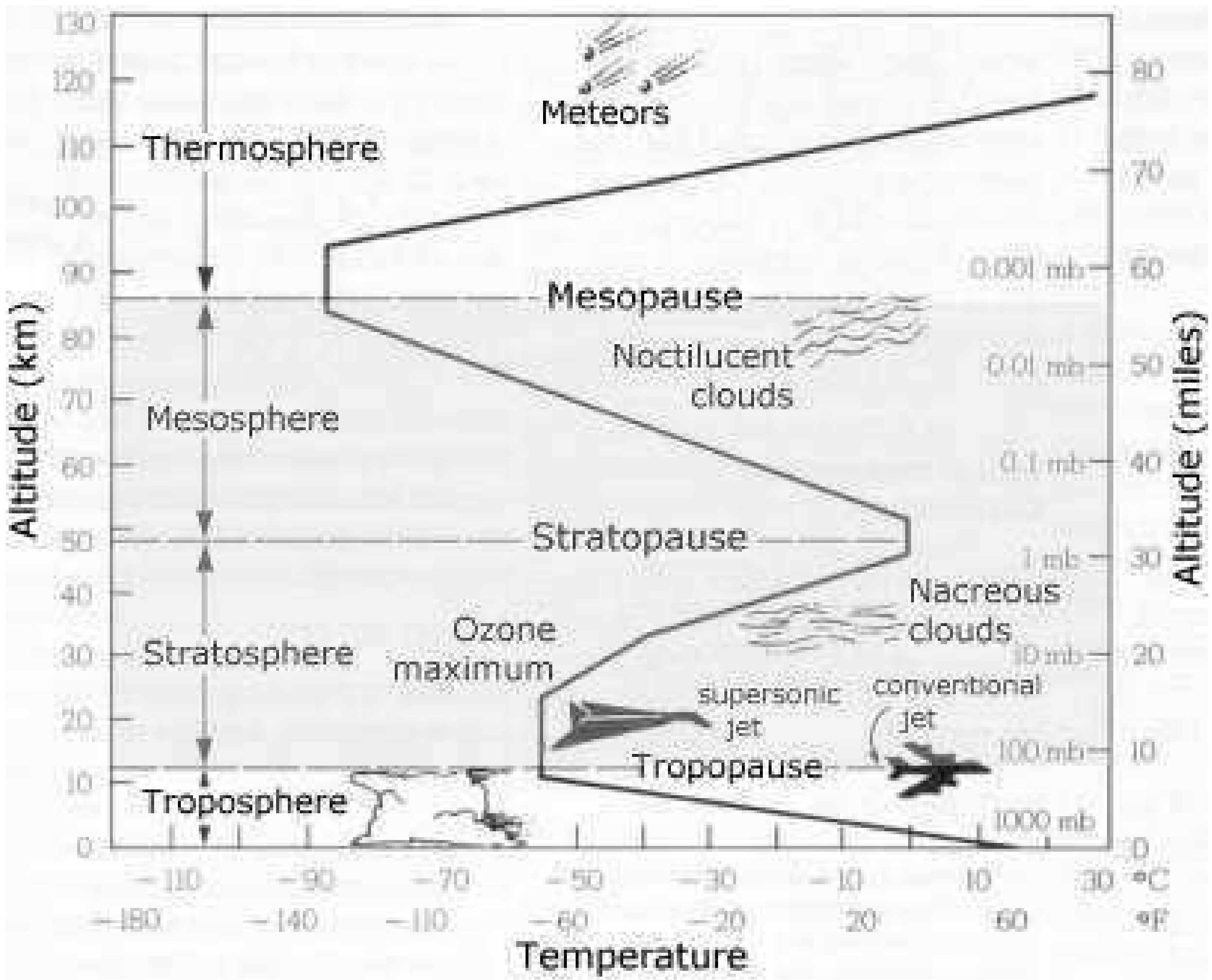
1. Symmetric instability in the middle atmosphere

- Equatorial *symmetric* stability refers to the stability of an axisymmetric zonal flow to axisymmetric perturbations considering both inertial and convective force balances.
- Stability depends on distributions of

  potential temperature $\theta \equiv T(p_{00}/p)^{R/c_p}$ (ideal gas),

  and angular momentum $m \equiv u_{\text{abs}} r \cos\phi$.
- Without rotation: stable if θ increases with decreasing pressure (Helmholtz, 1880's).
- Without stratification: stable if m increases in magnitude with increasing distance from axis of symmetry/rotation (Rayleigh, 1910's).

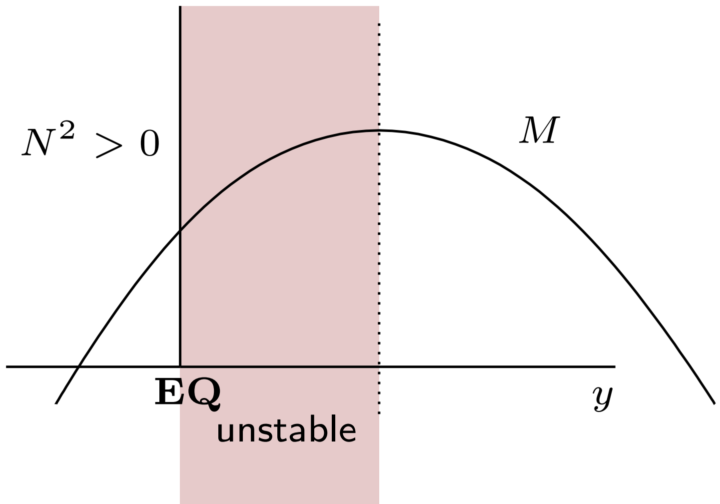
- Temperature (horizontal) versus altitude in atmosphere: (from *What is Chemical Engineering?* website (Dandy, 2003))

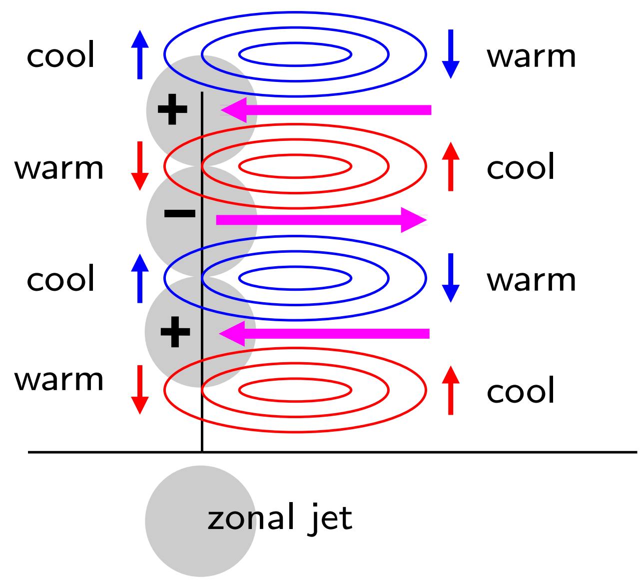


- Middle atmosphere is stably stratified (hence "stratosphere"), but is prone to inertial instability during solstice seasons (winter and summer).
- Characteristic pattern of equatorial inertial instability in atmosphere is due to Dunkerton (1981).
- Signatures of inertial instability activity observed in satellite temperature data of the equatorial stratopause region.
- But Inertial adjustment is thought to take place constantly, smoothing temperature, angular momentum and chemical concentration gradients, and to be part of the underlying physics of the middle atmosphere Hadley circulation and the semi-annual oscillation.

Dunkerton problem

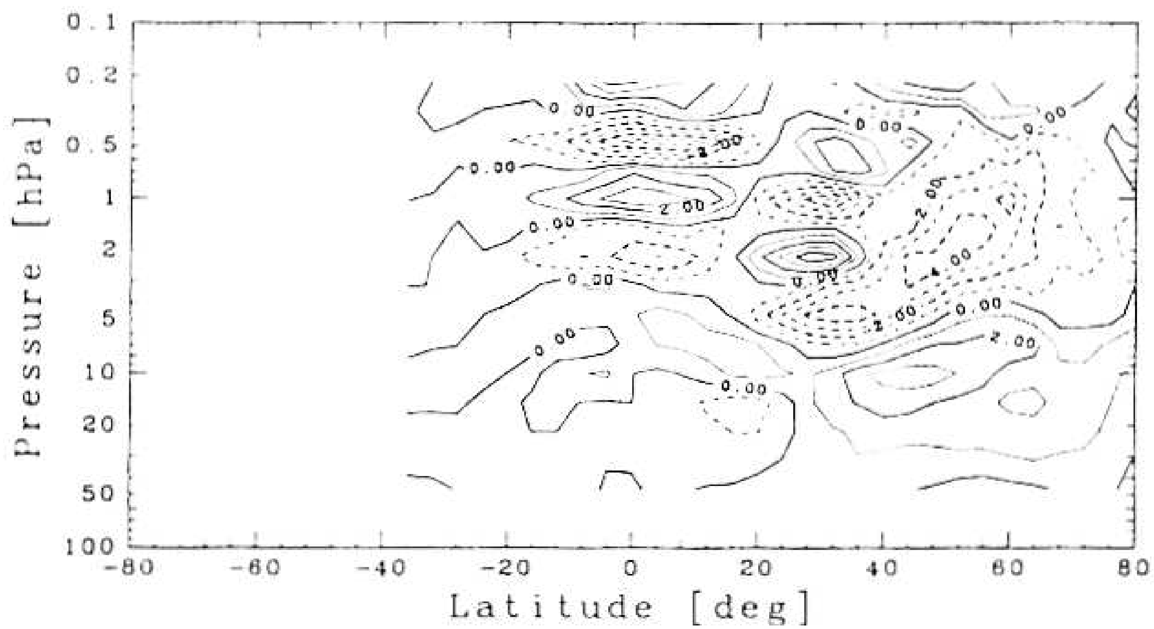
- Uses linearized "traditional" hydrostatic equations on a β-plane.
- Meridional velocity shear $U = \lambda y$ at the equator violates Rayleigh stability condition since $M \equiv U - \frac{1}{2}\beta y^2$ increases with y in the interval $0 < y < \lambda/\beta$.
- Most unstable solution exhibits
  - "Taylor Vortices" in the unstable region,
  - zonal jets over equator,
  - and crêpe structures in the temperature perturbation field.



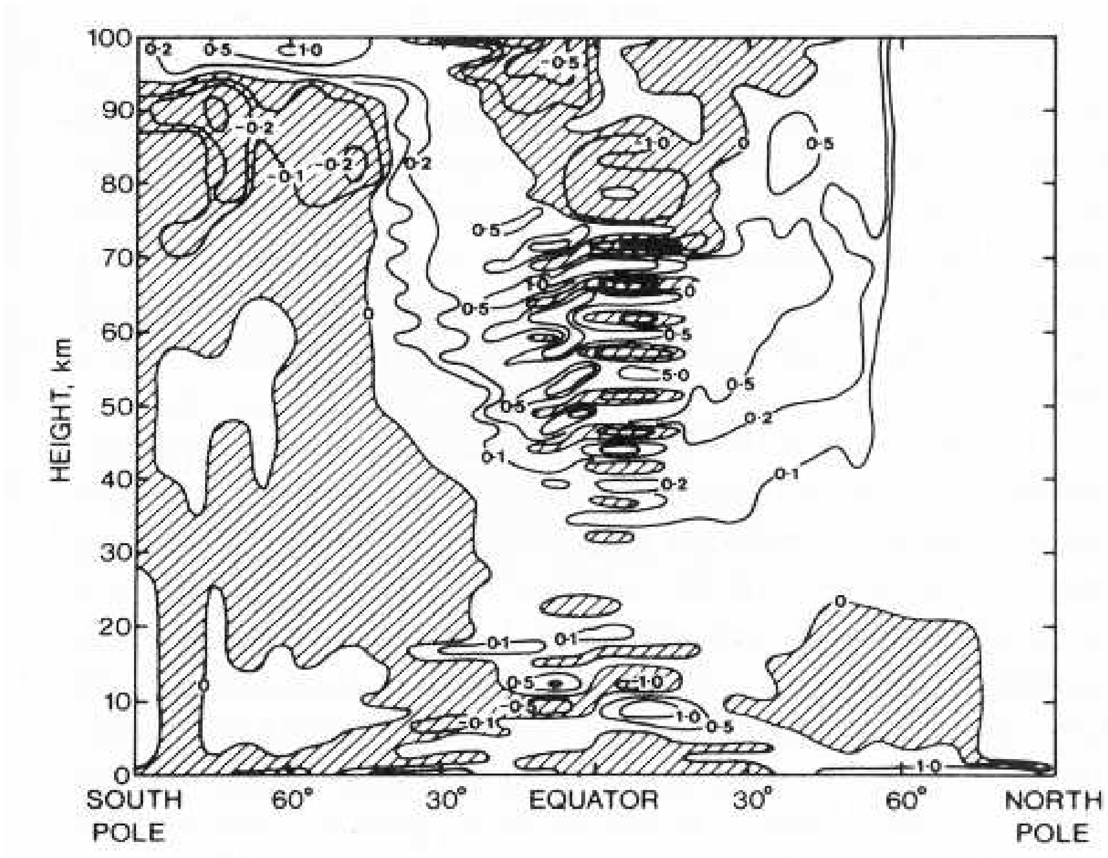


Explicit Inertial Instability

- Hayashi et al. (1998): Temperature anomaly from satellite data, averaged over 7 days starting Dec. 7, 1992 and over 60° longitude centred on 225° East (cells are about 10 km in height, with amplitude 2 Kelvin):



- Hunt (1981): Time averaged zonal-mean meridional wind (v) for January as computed by a GCM:



Inertial Adjustment

- Wehrbein and Leovy (1982): Radiative equilibrium temperature at solstice versus latitude. Note horizontal temperature gradient at stratopause (∼50 km altitude):

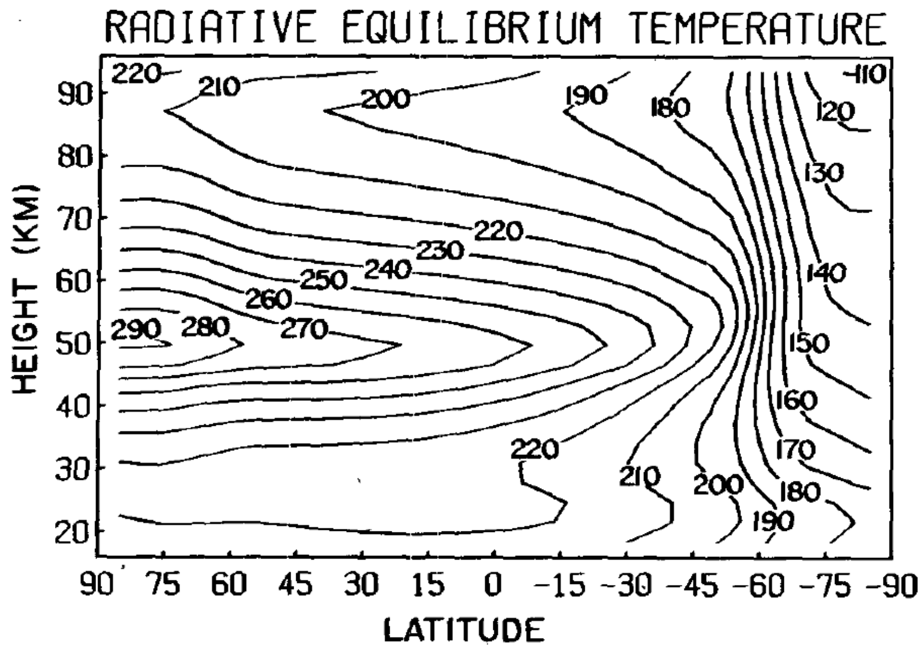
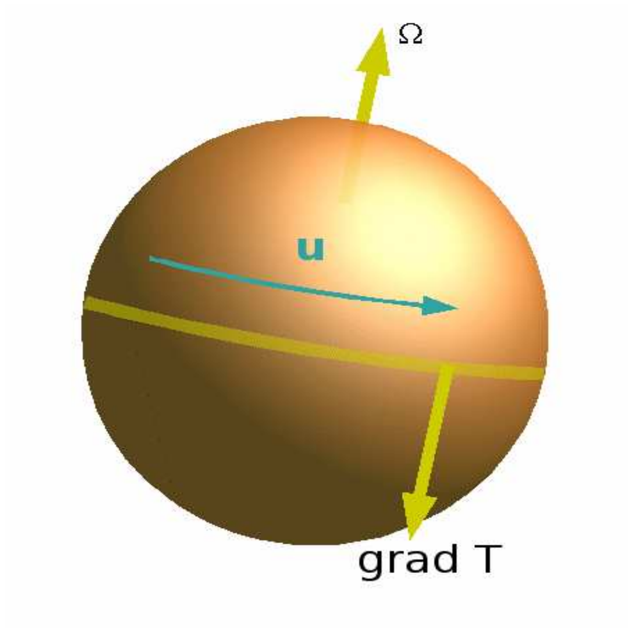


FIG. 3. Derived radiative equilibrium temperature distribution at solstice. Winter hemisphere on right.

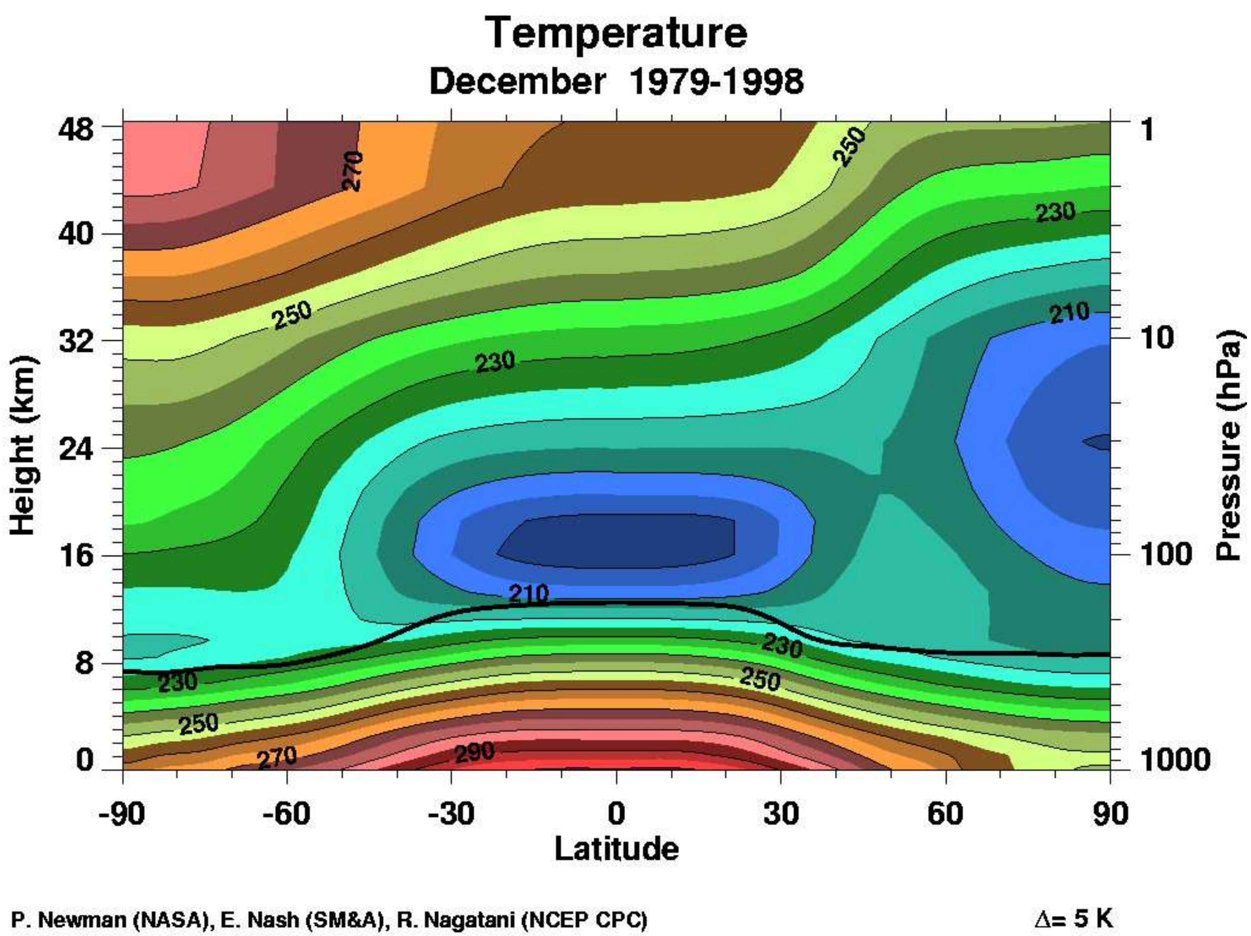


- North-south temperature (and pressure) gradient cannot be balanced by Coriolis force on a zonal flow u: $-2\boldsymbol{\Omega} \times (u\,\hat{\boldsymbol{e}}_\lambda)$, which is in the radial direction.

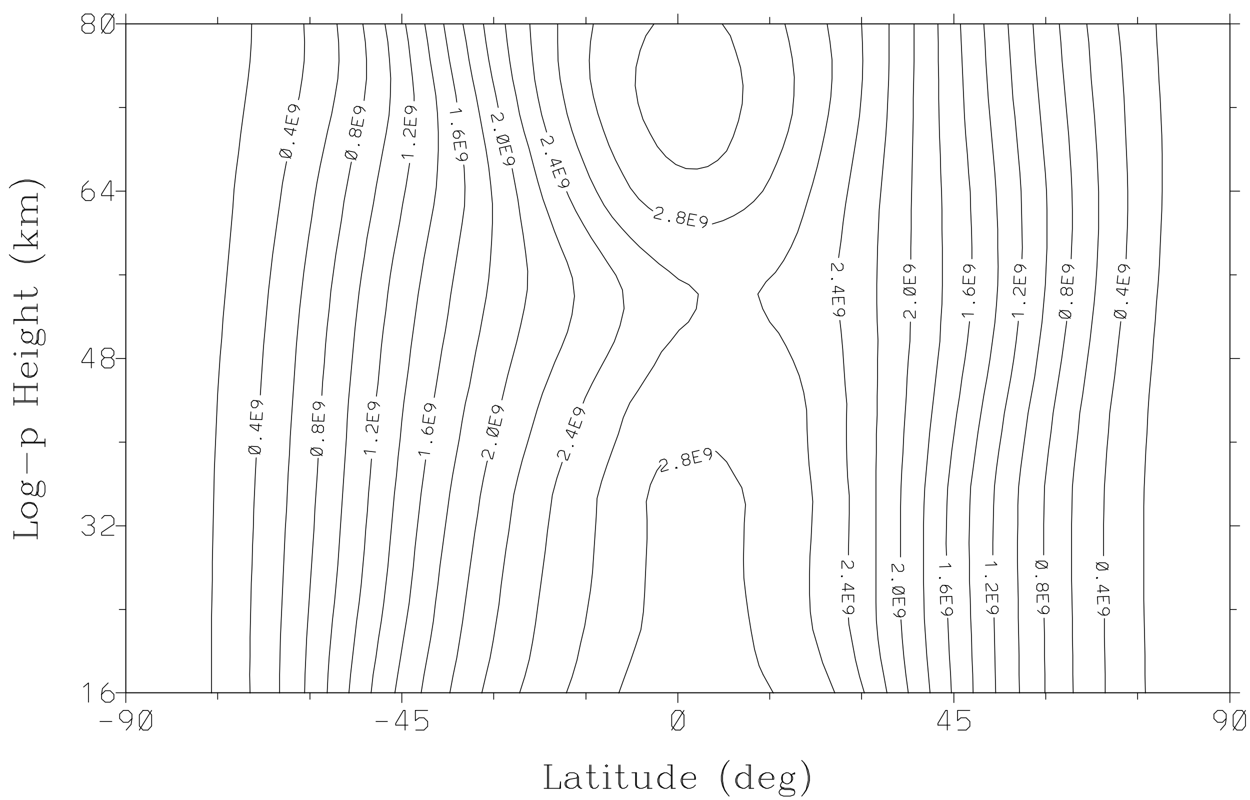
⇒ In the solstice radiative equilibrium state, there must be a north-south acceleration, forcing angular momentum distribution away from inertial stability.

⇒ Inertial instability cells are thought to form constantly (inertial *adjustment*) in response, eventually establishing a radiative-dynamic equilibrium state.

- *Observed* temperature in atmosphere versus latitude for December (NCEP):



- Simulated January monthly- and zonal-mean angular momentum versus latitude (from Canadian Middle Atmosphere Model, courtesy of K. Semeniuk):



2. The inviscid adiabatic anelastic equations

- Invented by Ogura and Phillips (1962).
- (Linearized) system does not admit sound wave solutions (associated with the elastic oscillations of the gas) while still allowing non-hydrostatic effects.
- Based on the assumptions that:
  - potential temperature θ does not vary much from a constant value over the whole domain,
  - the time scale of the motion is that of gravity waves and is slow.
- Similar to Boussinesq equations (in which density approximately constant) but valid for deeper domains.

- The symmetric anelastic equations on an equatorial β-plane, with representation of full planetary rotation vector are

$$
\begin{aligned}
u_t &= -vu_y - wu_z + \beta_\delta yv - \gamma_\alpha w, \\
v_t &= -vv_y - wv_z - \beta_\delta yu - \frac{1}{B}\theta_0(\pi_1)_y, \\
w_t &= -vw_y - ww_z \\
&\qquad + \alpha^2\left\{-\gamma_\alpha u + \frac{1}{B}\left[(\theta_0\pi_1)_z + \frac{\mathrm{d}\pi_0}{\mathrm{d}z}\theta_1\right]\right\}, \\
(\theta_1)_t &= -v(\theta_1)_y - w(\theta_1)_z - \frac{w}{\epsilon}\frac{\mathrm{d}\theta_0}{\mathrm{d}z}, \\
(\rho_0 v)_y + (\rho_0 w)_z &= 0,
\end{aligned}
$$

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where: $2L$ is domain width, H is domain height,

$\epsilon \equiv \Delta\theta/\bar{\Theta}$, $B \equiv H/(c_p\bar{\Theta}/g)$, $\alpha \equiv L/H$,

$\theta_0(z)$ is a specified θ profile,

$\pi_0(z),\ \rho_0(z)$ are pressure and density in hydrostatic balance with $\theta_0(z)$,

$\boldsymbol{\Omega} \equiv (0, \gamma_\alpha, \beta_\delta\, y) \equiv \left(\frac{L}{H}\right)\left(\frac{N}{2\Omega}\right)\left(0, \frac{H}{L}, \frac{L}{a}\, y\right)$.

- Les mêmes, in Hamiltonian form:

$$
\begin{aligned}
m_t &= \frac{1}{\rho_0}\partial(\psi, m) \equiv \frac{1}{\rho_0}\left(\psi_y m_z - \psi_z m_y\right), \\
\zeta_t &= \partial\left(\psi, \frac{1}{\rho_0}\zeta\right) + \partial\left[\frac{1}{\rho_0}\left(\frac{1}{\epsilon B}\rho_0\pi_0\right), \theta\right] \\
&\qquad + \partial\left\{\frac{1}{\rho_0}\left[\rho_0\left(\tfrac{1}{2}\beta_\delta y^2 - \gamma_\alpha z\right)\right], m\right\}, \\
\theta_t &= \frac{1}{\rho_0}\partial(\psi, \theta),
\end{aligned}
$$

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where: $m \equiv u - \frac{1}{2}\beta_\delta y^2 + \gamma_\alpha z$

$$\zeta \equiv \frac{1}{\alpha^2}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \equiv -\left[\frac{1}{\alpha^2}\frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial\psi}{\partial z}\right)\right],$$

$$\theta \equiv \theta_0(z) + \epsilon\theta_1,$$

- Equations conserve the Hamiltonian functional

$$\mathcal{H}(m,\zeta,\theta) \equiv \iint \Bigg\{ \rho_0 \left(\tfrac{1}{2}\beta_\delta y^2 - \gamma_\alpha z\right) m + \frac{1}{2\rho_0}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{1}{\alpha^2}\left(\frac{\partial\psi}{\partial y}\right)^2\right] + \frac{1}{\epsilon B}\rho_0\pi_0\theta \Bigg\}\,\mathrm{d}y\,\mathrm{d}z$$

- They also conserve Casimir-like functionals of the form

$$\mathcal{C}_1(m,\theta) = \iint \rho_0 C_1(m,\theta,q)\,\mathrm{d}y\,\mathrm{d}z,$$

where $q \equiv \dfrac{1}{\rho_0}\partial(\theta,m)$ is the potential vorticity,

and C_1 is an arbitrary function.

3. Sufficient conditions for symmetric stability

- We seek sufficient conditions for the stability of a steady solution $\mathbf{X} \equiv (M, 0, \Theta)$, which satisfies thermal wind balance:

$$\left(\frac{1}{\epsilon B}\frac{\mathrm{d}\pi_0}{\mathrm{d}z}\right)\frac{\partial\Theta}{\partial y} - \partial\left(\tfrac{1}{2}\beta_\delta y^2 - \gamma_\alpha z, M\right) = 0.$$

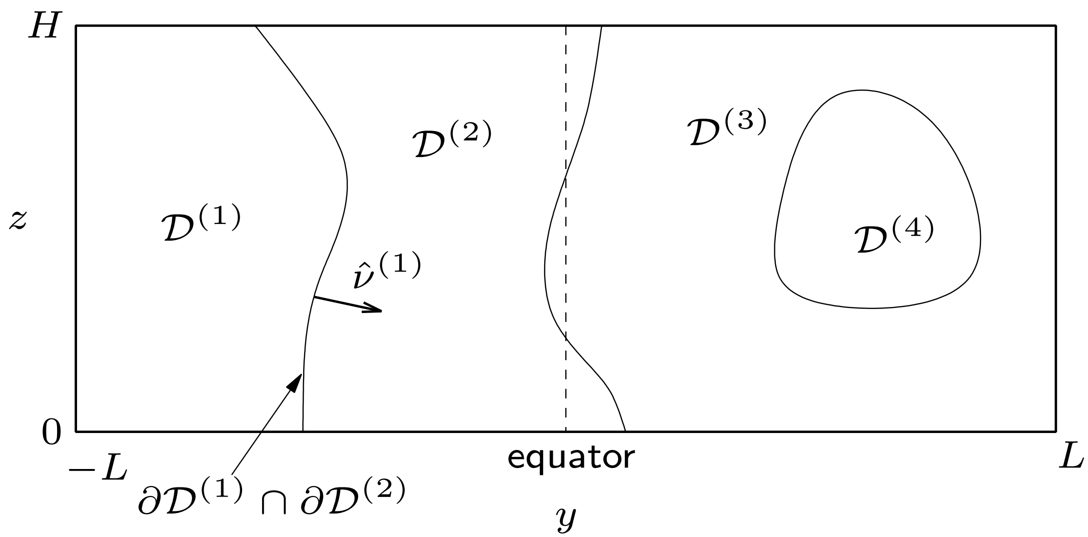
- Stability in the sense of Lyapunov is defined in terms of a norm on phase space displacements $||\mathbf{x} - \mathbf{X}||$, where $\mathbf{x} \equiv (m, \zeta, \theta)$:

  $\mathbf{X}$ is stable if for every ε, there exists a δ such that if $||\mathbf{x}(t=0) - \mathbf{X}|| < \delta$, then $||\mathbf{x}(t) - \mathbf{X}|| < \varepsilon$ for all t.

Linear Stability

- A steady state $\mathbf{X}$ is linearly stable if the zero solution to the equations linearized about $\mathbf{X}$ is stable.
- Linear stability implies stability with respect to infinitesimal perturbations $\mathbf{x}' \equiv \mathbf{x} - \mathbf{X}$.
- We find conditions for linear stability by finding a functional based on $\mathcal{H}$ and $\mathcal{C}_1$ that is conserved by the linearized equations and has a critical point at $\mathbf{x}' = 0$.
- Stability of $\mathbf{X}$ follows if $\mathbf{x}' = 0$ is a global minimum of the conserved functional.

- Assume that $Q = \frac{1}{\rho_0}\partial(\Theta, M)$ is nonzero everywhere in the domain except on a finite number of curves:



- Define the functional

$$\mathcal{C}_L = \sum_{i=1}^{n} \iint\limits_{\mathcal{D}^{(i)}} \rho_0 C^{(i)}(m, \theta)\,\mathrm{d}y\,\mathrm{d}z,$$

where the functions $C^{(i)}$ are chosen such that $\mathcal{H} + \mathcal{C}_L$ has a critical point at $\mathbf{x} = \mathbf{X}$ (i.e. its first variation vanishes at $\mathbf{X}$).

- The second order Taylor expansion of $\mathcal{H} + \mathcal{C}_L$ about $\mathbf{X}$ is conserved by the linearized equations. Hence

$$\mathcal{H}_L(m', \zeta', \theta'; M, \Theta)$$

$$= \iint\limits_{\mathcal{D}} \frac{1}{\rho_0} \left[\frac{1}{\alpha^2} (\psi'_y)^2 + (\psi'_z)^2 \right] \mathrm{d}y\, \mathrm{d}z$$

$$+ \sum_{i=1}^{n} \iint\limits_{\mathcal{D}^{(i)}} \rho_0 \left[C^{(i)}_{mm}(M,\Theta) m'^2 + 2C^{(i)}_{\theta m}(M,\Theta)\theta' m' + C^{(i)}_{\theta\theta}(M,\Theta)\theta'^2 \right] \mathrm{d}y\, \mathrm{d}z$$

is conserved by the linearized equations.

- $\mathcal{H}_L > 0$ for all $\mathbf{x}' \neq 0$ if the matrices

$$\Lambda^{(i)}(M,\Theta) \equiv \begin{bmatrix} C^{(i)}_{mm}(M,\Theta) & C^{(i)}_{m\theta}(M,\Theta) \\ C^{(i)}_{\theta m}(M,\Theta) & C^{(i)}_{\theta\theta}(M,\Theta) \end{bmatrix}$$

are all positive definite.

- Define the norm

$$||\mathbf{x}'||_L^2 \equiv \iint_{\mathcal{D}} \frac{1}{\rho_0}\left[\frac{1}{\alpha^2}(\psi_y')^2 + (\psi_z')^2\right] \mathrm{d}y\,\mathrm{d}z + \sum_{i=1}^{n} \iint_{\mathcal{D}(i)} \rho_0 \lambda_- \left[m'^2 + \theta'^2\right] \mathrm{d}y\,\mathrm{d}z$$

where λ_- is the *smallest* of the eigenvalues of all of the $\Lambda^{(i)}$.

- If the $\Lambda^{(i)}$ are positive definite, then $\lambda_- > 0$.
- Let λ_+ be the *largest* of the eigenvalues. Then, since $\mathcal{H}_L$ is conserved,

$$||\Delta\mathbf{x}'(t)||_L^2 \leq \mathcal{H}_L(\mathbf{x}'(t)) = \mathcal{H}_L(\mathbf{x}'(0)) \leq \frac{\lambda_+}{\lambda_-}||\Delta\mathbf{x}'(0)||_L^2$$

$\Rightarrow$ Therefore, $\mathcal{H}_L > 0$ implies that $\mathbf{X}$ is linearly stable.

- The $\Lambda^{(i)}$ are positive definite if

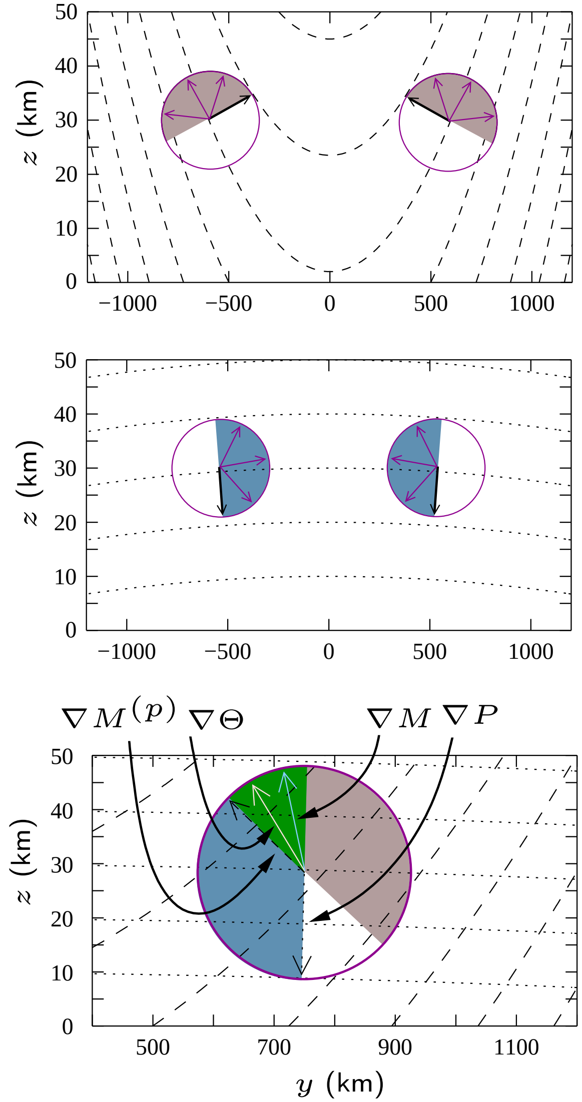
$$C_{mm}^{(i)}(M,\Theta) > 0,$$

$$C_{\theta\theta}^{(i)}(M,\Theta) > 0,$$

$$C_{mm}^{(i)}(M,\Theta)C_{\theta\theta}^{(i)}(M,\Theta) - C_{m\theta}^{(i)}(M,\Theta)C_{\theta m}^{(i)}(M,\Theta) > 0.$$

- From the requirement that $\mathcal{H} + \mathcal{C}_L$ have a critical point at $\mathbf{X}$, we calculate $C_{mm}^{(i)}(M,\Theta)$, $C_{\theta\theta}^{(i)}(M,\Theta)$, and $C_{m\theta}^{(i)}(M,\Theta)$ in terms of $M(y,z)$ and $\Theta(y,z)$, and find ...

Linear Stability Conditions:



$$\frac{1}{Q}\partial(\Theta, -\tfrac{1}{2}\beta_\delta y^2 + \gamma_\alpha z) > 0$$

STATIC STABILITY

$$\frac{1}{Q}\frac{\partial M}{\partial y} < 0$$

INERTIAL STABILITY

$$\frac{y}{Q} > 0$$

SYMMETRIC STABILITY

Nonlinear Stability

- A steady state is nonlinearly stable if it is stable (in the sense of Lyapunov) with respect to arbitrarily large perturbations.
- To prove nonlinear stability, we need a functional conserved by the full nonlinear equations ($\mathcal{H}_L$ is *not*).
- Since linear stability is obviously a prerequisite for nonlinear stability, we need only consider states $\mathbf{X}$ that satisfy the linear stability conditions. In particular, that have $Q = 0$ at the equator, and $yQ > 0$ everywhere else.

- Conserved functional for the nonlinear anelastic equations is ($\mathcal{K}_\perp$ is kinetic energy in (u, v) components):

$$\begin{aligned}\mathcal{A}(\mathbf{x};\mathbf{X}) &= \mathcal{K}_\perp + \iint \rho_0 \left[(\tfrac{1}{2}\beta_\delta y^2 - \gamma_\alpha z)(m - M) + \tfrac{1}{\epsilon B}\pi_0(\theta - \Theta) \right] \mathrm{d}y\,\mathrm{d}z \\ &+ \iint_{q<0} \rho_0 \left[C^-(m,\theta) - C^-(M,\Theta) \right] \mathrm{d}y\,\mathrm{d}z \\ &+ \iint_{q>0} \rho_0 \left[C^+(m,\theta) - C^+(M,\Theta) \right] \mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

- Notice that the domains of the last two integrals change with time as the sign of potential vorticity q changes.

- Define the norm

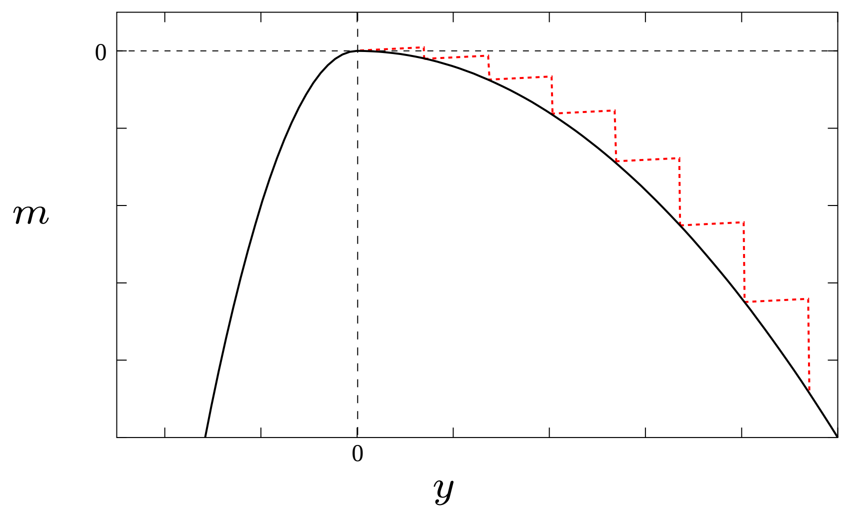
$$||\mathbf{x} - \mathbf{X}||_\lambda^2 = \mathcal{K}_\perp + \iint \left\{ \lambda \frac{\rho_0}{2} \left[(m - M)^2 + (\theta - \Theta)^2 \right] \right\} \mathrm{d}y\,\mathrm{d}z$$

- Steady states that are even functions of y (so that $C^- = C^+ \equiv C$) and satisfy the linear conditions are candidates for nonlinear stability.
- We must test that $C(m, \theta)$ and $C^+(m, \theta)$ functions can be constructed such that $\mathcal{A}$ has a global minimum at $\mathbf{X}$.
- Simplest example of a stable state is:

$$
\begin{aligned}
M(y, z) &= M_0 - \tfrac{1}{2} b y^2 \\
\Theta(y, z) &= \Theta_0 + (\epsilon \gamma_\alpha)(\tfrac{1}{2} b y^2) + \epsilon \Gamma z
\end{aligned}
$$

- The required $C(m, \theta)$ is a quadratic function of m and θ.

- Steady states that are not even functions of latitude cannot be Lyapunov stable (in our norm).
- Consider the perturbation below:
  - red curve is $m(y)$ and black curve is $M(y)$
  - red curve has $q < 0$ everywhere
  - as steps $\to 0$, dominant term in $\mathcal{A}$ is $C^{-}(m, \theta) - C^{+}(M, \Theta) < 0$



- But isn't that small amplitude? No, look at $\dfrac{\partial m}{\partial y} - \dfrac{\partial M}{\partial y}$ but it does satisfy $||\mathbf{x} - \mathbf{X}|| \to 0$

4. Saturation bounds on instability

- Recall that when $\mathbf{X}$ is nonlinearly stable, $\mathcal{A}(\mathbf{x};\mathbf{X})$ is the sum of two positive terms: the kinetic energy term $\mathcal{K}_\perp(\mathbf{x})$ and what we might call the available potential energy term $\mathcal{APE}(\mathbf{x};\mathbf{X})$.

- Since $\mathcal{A}$ is conserved, its initial value is a rigorous upper bound on $\mathcal{K}_\perp(\mathbf{x}(t))$.

- Given $\mathbf{x}(0)$ close to an unstable equilibrium $\mathbf{X}_U$, seek the smallest $\mathcal{A}(\mathbf{X}_U;\mathbf{X})$ among all nonlinearly stable $\mathbf{X}$.

- This is a measure of how large the instability can grow before it saturates. $\mathcal{A}$ is called a saturation bound.

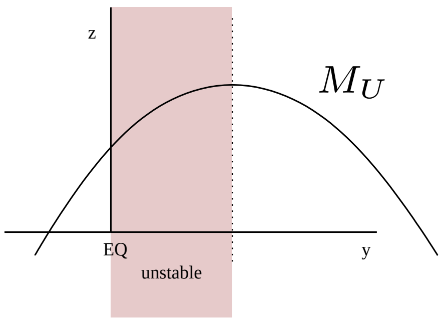
- Can be used as part of a parameterization scheme for subgridscale adjustment in numerical models.

Anelastic Dunkerton Problem

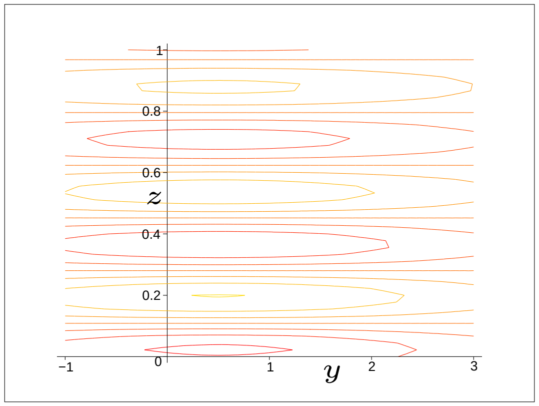
- Consider the unstable equilibrium $\mathbf{X}_U(y,z)$:

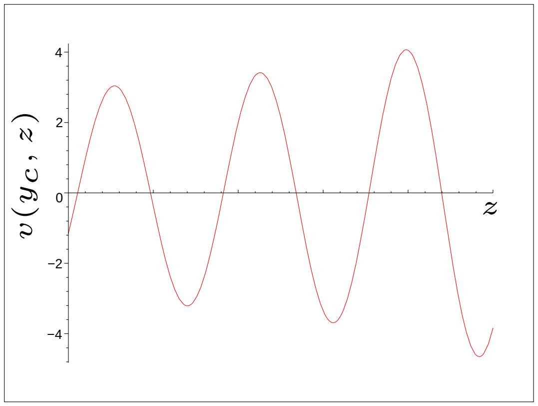
$$M_U(y,z) = -\tfrac{1}{2}b_U y^2 + \lambda_U y$$

$$\Theta_U(y,z) = (\epsilon\gamma_\alpha)(\tfrac{1}{2}b_U y^2 - \lambda_U y) + \epsilon\Gamma_U z$$



- Can solve anelastic equations linearized about $\mathbf{X}_U(y,z)$:





- Taylor Vortices
- v increases with z depending on slope of $\rho_0(z)$

Streamfunction $\qquad$ $v(y_c, z)$

- Again, consider Dunkerton state $\mathbf{X}_U$:

$$\begin{aligned} M_U(y,z) &= -\tfrac{1}{2}b_U y^2 + \lambda_U y \\ \Theta_U(y,z) &= (\epsilon\gamma_\alpha)(\tfrac{1}{2}b_U y^2 - \lambda_U y) + \epsilon\Gamma_U z \end{aligned}$$

- Minimize $\mathcal{A}(\mathbf{X}_U;\mathbf{X})$ over the class of nonlinearly stable states of the form

$$\begin{aligned} M(y,z) &= M_0 - \tfrac{1}{2}by^2 \\ \Theta(y,z) &= \Theta_0 + (\epsilon\gamma_\alpha)(\tfrac{1}{2}by^2) + \epsilon\Gamma z \end{aligned}$$

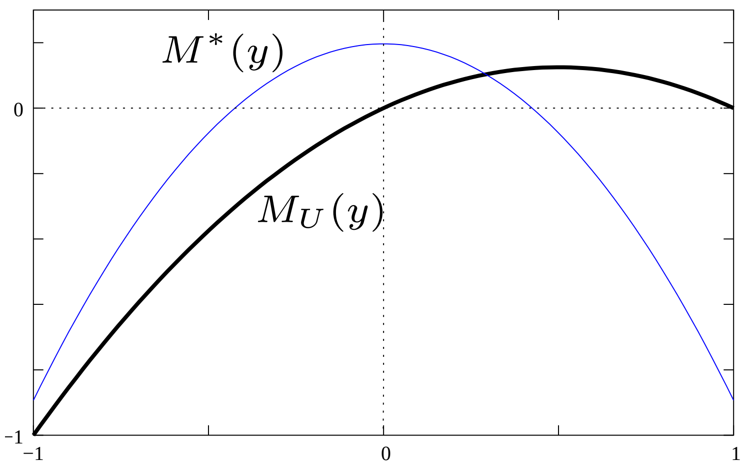
- i.e. find M_0, b, Θ_0, Γ which minimize $\mathcal{A}$

Example 1 - Inertial Instability

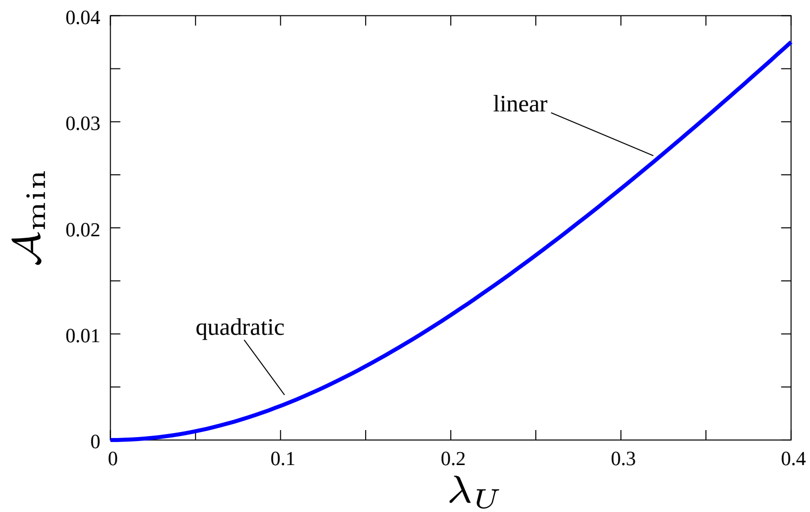
- Consider statically stable ($\Gamma_U > 0$), inertially unstable ($\lambda_U > 0$)
- minimizing $\mathbf{X}$ has:

$$\Gamma_{\min} = \Gamma_U \,, \qquad b_{\min} = |b_U| \sqrt{1 + 15 \left(\frac{\lambda_U}{b_U} \right)^2}$$

$$(M_0)_{\min} = \tfrac{1}{6}(b_{\min} - b_U), \qquad (\Theta_0)_{\min} = \tfrac{1}{6}(\epsilon \gamma_\alpha)(b_{\min} - b_U)$$



Stable $M(y)$ (blue) which minimizes $\mathcal{A}$



Minimum $\mathcal{A}$ as function of λ_U

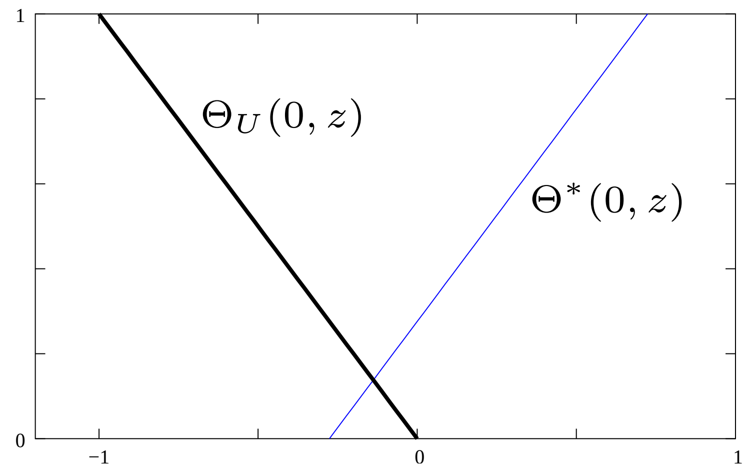
Notice $\mathcal{A}_{\min} \to 0$ as $\lambda_U \to 0$

Example 2 - Static Instability

- Consider statically unstable ($\Gamma_U < 0$), inertially stable ($\lambda_U = 0$)
- minimizing $\mathbf{X}$ has:

$$\Gamma_{\min} = -|\Gamma_U|, \quad b_{\min} = b_U$$

$$(M_0)_{\min} = 0, \quad (\Theta_0)_{\min} = -2\frac{I_1}{I_0}\epsilon|\Gamma_U|$$



Minimizing $\Theta(0, z)$ (blue) (z is vertical axis)

- Saturation bound is $\mathcal{A}_{\min} = \frac{4}{I_0}(I_0 I_2 - I_1^2)|\Gamma_U|$

Summary

- Symmetric (inertial) instability is fundamental to equatorial middle atmosphere dynamics and is not adequately resolved in observations or simulations.
- We calculated sufficient conditions for linear and nonlinear symmetric stability in the anelastic equations on an equatorial β-plane with representation of full planetary rotation vector.
- Results generalize conditions calculated in "traditional" hydrostatic calculations.
- Families of nonlinearly stable states can be used to calculate saturation bounds on the growth of instabilities out of unstable steady states.