Saturation Bounds on Inertially Unstable D-states in Anelastic Equatorial β-plane System

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Group Meeting

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OUTLINE

1. Anelastic Equations

2. Stable and Unstable Equilibria

3. Saturation Bounds

1. (DEEP) ANELASTIC EQUATIONS

- Nonhydrostatic equations used to model convection
- Assume potential temperature varies only small amount from mean value (i.e. weak stratification)
  $\Rightarrow$ allow for vertical motion (convection)
- Filter oscillations faster than gravity waves
  - timescale $\sim \frac{l}{h} N^{-1}$
  - "Elastic" (acoustic) term neglected in energetics
  - $**$ density and pressure variations restricted by anelastic continuity equation

- equatorial β-plane; assume x-symmetry, adiabatic
- Momentum equations:

$$\begin{aligned} u_t &= -vu_y - wu_z + \beta_\delta yv - \gamma_\alpha w \\ v_t &= -vv_y - wv_z - \beta_\delta yu - \frac{1}{B}(\pi_1)_y \\ w_t &= -vw_y - ww_z + \alpha^2 \left[-\gamma_\alpha u + \frac{1}{B}(\pi_1)_z - \theta_1 \right] \end{aligned}$$

- Thermodynamics:

$$\theta_t = -v\theta_y - w\theta_z$$

- Continuity equation:

$$\frac{\partial}{\partial y}(\rho_0 v)\frac{\partial}{\partial z}(\rho_0 w) = 0$$

- Dimensionless parameter $B = \dfrac{hg}{c_p \Theta_0}$, $(0 < B < 1)$

- Reference pressure satisfies $\pi_0(z) = 1 - Bz$

  Reference density satisfies $\rho_0(z) = (1 - Bz)^{\frac{c_p}{c_v}}$

- System is Hamiltonian in variable $\mathbf{x} \equiv (m, \theta, \zeta)$, where

$$m \equiv u - \tfrac{1}{2}\beta_\delta y^2 + \gamma_\alpha z$$

  (m is a material invariant), and

$$\zeta \equiv \frac{1}{\alpha^2} w_y - v_z$$

- System conserves **ENERGY:**

$$\mathcal{H} = \iint_{\mathcal{D}} \Big[\rho_0 \left(\tfrac{1}{2}\beta_\delta y^2 - \gamma_\alpha z\right) m + \frac{1}{2\rho_0}\left(\psi_z^2 + \frac{1}{\alpha^2}\psi_y^2\right) + \frac{1}{\epsilon B}\rho_0\pi_0\theta\Big] \,\mathrm{d}y\,\mathrm{d}z \tag{1}$$

- **and CASIMIRS:**

$$\mathcal{C} = \iint_{\mathcal{D}} \rho_0 C(m,\theta)\,\mathrm{d}y\,\mathrm{d}z$$

2. STABLE AND UNSTABLE EQUILIBRIA

- Consider steady solution:

  $m = M(y, z)$,

  $\theta = \Theta(y, z)$,

  $\zeta = 0$

  (i.e. purely zonal flow)

- M and Θ satisfy thermal wind balance

$$\frac{1}{\epsilon}\Theta_y = -\gamma_\alpha M_y - \beta_\delta y M_z$$

- **STABILITY DEFINITION:**

  Equilibrium $\mathbf{X} \equiv (M, 0, \Theta)$ is stable with respect to the norm $||\mathbf{x} - \mathbf{X}||$ if for every ε, there is a δ such that if $||\mathbf{x}(t = 0) - \mathbf{X}|| < \delta$, then $||\mathbf{x}(t) - \mathbf{X}|| < \varepsilon$ for all times t.

- **NORM DEFINITION:**

$$
\begin{aligned}
||\mathbf{x} - \mathbf{X}||_{\lambda}^{2} \;=\; & \iint_{\mathcal{D}} \left\{ \frac{1}{2\rho_0} \left[\frac{1}{\alpha^2} (\psi_y)^2 + (\psi_z)^2 \right] \right. \\
& \left. + \lambda \frac{\rho_0}{2} \left[(m - M)^2 + (\theta - \Theta)^2 \right] \right\} \mathrm{d}y \, \mathrm{d}z .
\end{aligned}
$$

- **PSEUDOENERGY DEFINITION:**

$$\mathcal{A}(\mathbf{x};\mathbf{X}) \equiv (\mathcal{H}+\mathcal{C})(\mathbf{x}) - (\mathcal{H}+\mathcal{C})(\mathbf{X})$$

where $\mathcal{C}$ **is chosen such that** $\delta\mathcal{A}|_{\mathbf{x}=\mathbf{X}} = 0$
$\Longleftrightarrow$ **Condition on Casimir density** $C(m,\theta)$**:**

$$\begin{aligned} C_m(M,\Theta) &= -\tfrac{1}{2}\beta_\delta Y^2(M,\Theta) + \gamma_\alpha Z(M,\Theta) \\ C_\theta(M,\Theta) &= -\frac{1}{\epsilon B}\left[1 - B\,Z(M,\Theta)\right] \end{aligned}$$

- **Assumes existence of** $[Y^2(M,\Theta), Z(M,\Theta)]$**,** **the inverse of** $[M(y^2,z),\Theta(y^2,z)$**.**

RESULT:

- **A state $\mathbf{X}$ is stable if**

$$0 < \mathcal{A}(\mathbf{x}; \mathbf{X}) < \infty$$

for all $\mathbf{x} \neq \mathbf{X}$.

- **Condition on $\mathcal{A}$ is equivalent to conditions on $C(m, \theta)$:**

$$\begin{aligned} C_{mm} &> 0 \\ C_{\theta\theta} &> 0 \\ C_{mm}C_{\theta\theta} - C_{m\theta}^2 &> 0, \end{aligned}$$

for all m and θ.

EXAMPLE:

- **Consider equilibrium with**

$$
\begin{aligned}
M(y,z) &= M_0 - \tfrac{1}{2} b y^2 \\
\Theta(y,z) &= \Theta_0 + (\epsilon\gamma)(\tfrac{1}{2} b y^2) + \epsilon \Gamma z
\end{aligned}
$$

- **Inverse map:**

$$
\begin{aligned}
Y^2(M,\Theta) &= -\frac{2}{b}(M - M_0) \\
Z(M,\Theta) &= \frac{1}{\epsilon\Gamma}\left[(\Theta - \Theta_0) + \epsilon\gamma(M - M_0)\right]
\end{aligned}
$$

- $C(m,\theta)$ satisfies:

$$C_m(M,\Theta) = \left(\frac{\gamma^2}{\Gamma}+\frac{\beta}{b}\right)(M-M_0)+\frac{\gamma}{\epsilon\Gamma}(\Theta-\Theta_0)$$

$$C_\theta(M,\Theta) = -\frac{1}{\epsilon B}+\frac{1}{\epsilon^2\Gamma}(\Theta-\Theta_0)+\frac{\gamma}{\epsilon\Gamma}(M-M_0)$$

- Therefore,

$$C_{mm}=\frac{\gamma^2}{\Gamma}+\frac{\beta}{b}, \qquad C_{\theta\theta}=\frac{1}{\epsilon^2\Gamma}, \qquad C_{m\theta}=\frac{\gamma}{\epsilon\Gamma}$$

$$C_{mm}C_{\theta\theta}-C_{m\theta}^2=\frac{\beta}{\epsilon^2\Gamma b}$$

$\Rightarrow$ Stable if: $\Gamma > 0$ (static stability), and
$b > 0$ (inertial stability).

- An example of an UNSTABLE equilibrium:

$$M(y,z) = M_0 - \tfrac{1}{2}by^2 + \lambda y$$

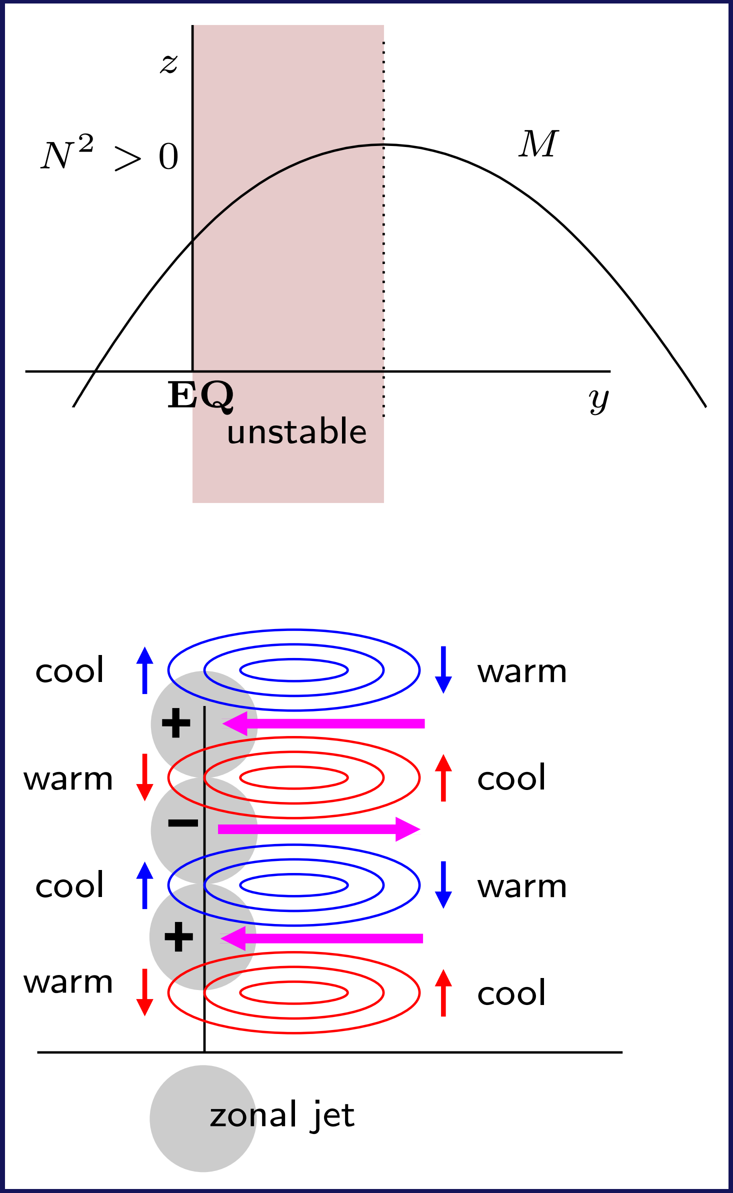
$$\Theta(y,z) = \Theta_0 + (\epsilon\gamma)(\tfrac{1}{2}by^2 + \lambda y) + \epsilon\Gamma z$$

- State is relevant to middle atmosphere solstice dynamics

- Problem considered by Dunkerton (1981)
- Can solve anelastic equations linearized about this state
  - get something like Dunkerton result . . .

Dunkerton problem

- Meridional velocity shear $U = \lambda y$ at the equator violates Rayleigh stability condition in interval $0 < y < \lambda/\beta$
- Dunkerton (1981) solved linearized, hydrostatic equations on β-plane
- Solution exhibits
  - "Taylor Vortices" in unstable region
  - zonal jets over equator
  - pancake structures in temperature perturbation field



2. SATURATION BOUNDS

- Recall: for a stable state $\mathbf{X}$, $\mathcal{A}(\mathbf{x};\mathbf{X}) > 0$ for all $\mathbf{x} \neq \mathbf{X}$.
- $\mathcal{A}$ is conserved, and can be written as

$$\mathcal{A} = \mathcal{K}_\perp + \mathcal{APE}$$

  where $\mathcal{K}_\perp$ is the kinetic energy in the (v, w) components and $\mathcal{APE}$ is available potential energy.

- Consider an unstable equilibrium zonal flow $\mathbf{X}_U$.
  - An infinitessimal disturbance will lead to the conversion of $\mathcal{APE}$ into $\mathcal{K}_\perp$.
  - $\mathcal{K}_\perp$ is always bounded by $\mathcal{A}$
  - Hence $\mathcal{A}(\mathbf{X}_U;\mathbf{X})$, where $\mathbf{X}$ is any stable state is a rigorous upper bound on $\mathcal{K}_\perp$

- We seek a stable state which gives as tight a "saturation bound" on $\mathcal{K}_\perp$ as possible.
- Consider again the Dunkerton state:

$$\begin{aligned} M_U(y,z) &= -\tfrac{1}{2}b'y^2 + \lambda' y \\ \Theta_U(y,z) &= (\epsilon\gamma)(\tfrac{1}{2}b'y^2 + \lambda' y) + \epsilon\Gamma' z \end{aligned}$$

and the class of stable states

$$\begin{aligned} M(y,z) &= M_0 - \tfrac{1}{2}by^2 \\ \Theta(y,z) &= \Theta_0 + (\epsilon\gamma)(\tfrac{1}{2}by^2) + \epsilon\Gamma z \end{aligned}$$

- Seek M_0, Θ_0, b and Γ which minimizes $\mathcal{A}(\mathbf{X}_U;\mathbf{X})$.

- One can write

$$\mathcal{A}(\mathbf{X}_U;\mathbf{X}) = \int_0^1\int_{-1}^1 \rho_0(z)\left[\tfrac{1}{2}C_{mm}(M_U - M)^2 + C_{m\theta}(M_U - M)(\Theta_U - \Theta) + \tfrac{1}{2}C_{\theta\theta}(\Theta_U - \Theta)^2\right] \mathrm{d}y\,\mathrm{d}z$$

- Using earlier results, this is

$$\mathcal{A} = \frac{2\beta I_0}{b}\left\{\frac{1}{40}(b' - b)^2 + \frac{1}{6}\left[\lambda'^2 + M_0(b' - b)\right] + \frac{1}{2}M_0^2\right\} + \frac{I_0}{\Gamma}\left(\gamma M_0 + \frac{1}{\epsilon}\Theta_0\right)^2 - 2I_1\left(\gamma M_0 + \frac{1}{\epsilon}\Theta_0\right)\left(\frac{\Gamma' - \Gamma}{\Gamma}\right) + I_2\frac{(\Gamma' - \Gamma)^2}{\Gamma}$$

…where $I_k \equiv \int_0^1 z^k \rho_0(z) \mathrm{d}z$

- Minimize $\mathcal{A}$ with respect to:

$$M_0,\ b,\ \Gamma,\ \text{and}\ T_0 \equiv \gamma M_0 + \frac{1}{\epsilon}\Theta_0$$

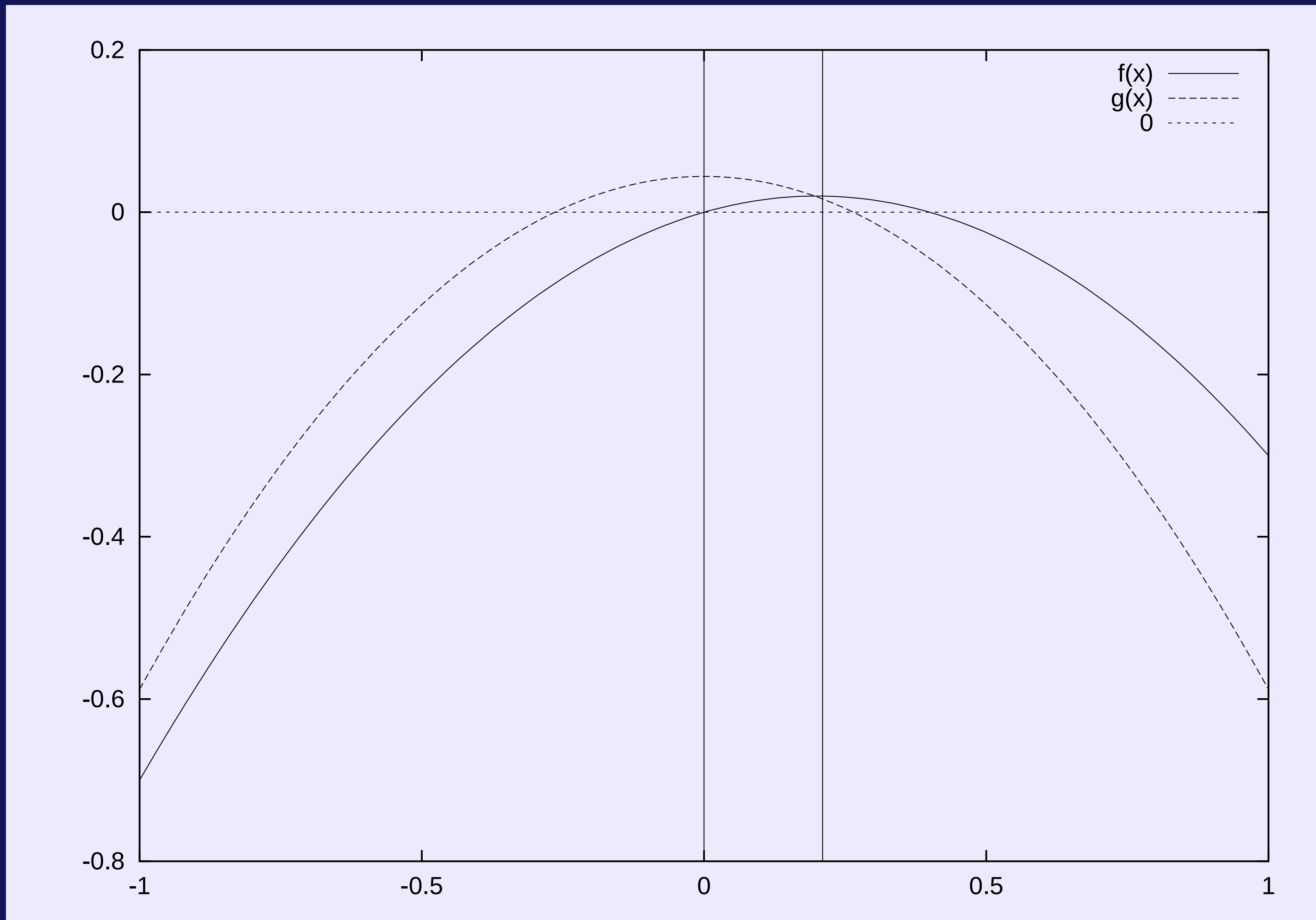
- Solve system: $\dfrac{\partial \mathcal{A}}{\partial M_0} = \dfrac{\partial \mathcal{A}}{\partial b} = \dfrac{\partial \mathcal{A}}{\partial \Gamma} = \dfrac{\partial \mathcal{A}}{\partial T_0} = 0$

  (and test if solution is a minimum of $\mathcal{A}$).

- Solution is …

$$
\begin{aligned}
\Gamma_{\min} &= |\Gamma'| \\
b_{\min} &= |b'|\sqrt{1 + 15\left(\frac{\lambda'}{b'}\right)^2} \\
(M_0)_{\min} &= \frac{1}{6}(b_{\min} - b') \\
(T_0)_{\min} &= -\frac{I_1}{I_0}(\Gamma_{\min} - \Gamma')
\end{aligned}
$$

- $\mathbf{X}_U$ can be made inertially unstable ($\lambda \neq 0$ and/or $b' < 0$) or convectively unstable ($\Gamma' < 0$), or both.
- As $\mathbf{X}_U$ approaches a stable state ($\lambda \to 0$ or $\Gamma' \to 0_-$), $\mathcal{A}_{\min}$ approaches 0 from above.


f(x)
g(x)
0
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1