Symmetric stability of equilibrium zonal flow in generalized equatorial β-plane anelastic system

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Part I

ANELASTIC EQUATIONS ON EQUATORIAL β-PLANE

Rotating earth and β-plane

$$\mathbf{\Omega_s} = \Omega \left(\cos\phi\, \hat{\mathbf{e}}_\phi + \sin\phi\, \hat{\mathbf{e}}_r\right)$$

$$\Omega \equiv 2\pi\, \mathrm{day}^{-1}$$

$$\mathbf{\Omega} = \tfrac{1}{2}(\gamma\, \hat{\mathbf{e}}_y + \beta y\, \hat{\mathbf{e}}_z)$$

Euler equations on β-plane

$$
\begin{aligned}
u_t + uu_x + vu_y + wu_z &= \beta y v - \gamma w - \tfrac{1}{\rho}\, p_x \\
v_t + uv_x + vv_y + wv_z &= -\beta y u - \tfrac{1}{\rho}\, p_y \\
w_t + uw_x + vw_y + ww_z &= \gamma u - g - \tfrac{1}{\rho}\, p_z \\
\rho_t + u\rho_x + v\rho_y + w\rho_z &= -\rho(u_x + v_y + w_z) \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= 0
\end{aligned}
$$

$$p = \rho R T$$

$$\theta = \left(\frac{p_{00}}{p}\right)^{\kappa} T \qquad \left(\kappa \equiv \frac{R}{c_p}\right)$$

Nondimensionalization

$$\pi = \left(\frac{p}{p_{00}}\right)^{\kappa}$$

$$\rho = \left(\frac{p_{00}}{R}\right)\frac{\pi^{\frac{1}{\kappa}-1}}{\theta}$$

$$T = \pi\theta$$

$$\begin{aligned}
(x, y) &= l(x^*, y^*) \\
z &= hz^* \\
t &= \tau t^* \\
\theta &= \Theta\theta^* \\
\beta &= (2\Omega/a)\beta^* \\
\gamma &= (2\Omega)\gamma^*
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
(u, v) &= (l/\tau)(u^*, v^*) \\
w &= (h/\tau)w^* \\
\rho &= (p_{00}/R\Theta)\rho^* \\
T &= \Theta T^*
\end{aligned}$$

Nondimensional Euler equations

$$\frac{l^2}{\tau^2}\frac{Du}{Dt} = \frac{2\Omega l^2}{\tau}(\beta y v - \frac{h}{l}\gamma w) - (c_p\Theta)\theta\frac{\partial \pi}{\partial x}$$

$$\frac{l^2}{\tau^2}\frac{Dv}{Dt} = \frac{2\Omega l^2}{\tau}(-\beta y u) - (c_p\Theta)\theta\frac{\partial \pi}{\partial y}$$

$$\frac{h^2}{\tau^2}\frac{Dw}{Dt} = \frac{2\Omega h l}{\tau}\gamma u - hg - (c_p\Theta)\theta\frac{\partial \pi}{\partial z}$$

$$\left(\frac{1}{\kappa} - 1\right)\frac{D}{Dt}(\ln \pi) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{D\theta}{Dt} = 0$$

Classical anelastic approximation

$$\epsilon \equiv \frac{\Delta\theta}{\Theta} = \Delta\theta^*$$

$$\epsilon \ll 1$$

$$N_0^2 \equiv \frac{g}{\bar{\Theta}} \frac{d\bar{\Theta}}{dz}$$

$$N^2 \equiv \frac{g}{\Theta} \frac{\Delta\theta}{h} = \frac{g}{h}\epsilon$$

$$\tau = \frac{l}{h} N^{-1} = \sqrt{\frac{l^2}{gh\epsilon}}$$

Nondimensional momentum equations (again)

$$
\begin{aligned}
\left(\frac{hg}{c_p\Theta}\right)\epsilon\left[\frac{Du}{Dt}-\left(\frac{2\Omega}{N}\right)\left(\frac{l}{h}\right)\left(\beta yv-\frac{h}{l}\gamma w\right)\right]+\theta\frac{\partial\pi}{\partial x} &= 0\\
\left(\frac{hg}{c_p\Theta}\right)\epsilon\left[\frac{Dv}{Dt}+\left(\frac{2\Omega}{N}\right)\left(\frac{l}{h}\right)(\beta yu)\right]+\theta\frac{\partial\pi}{\partial y} &= 0\\
\left(\frac{h}{l}\right)^2\left(\frac{hg}{c_p\Theta}\right)\epsilon\left[\frac{Dw}{Dt}-\left(\frac{2\Omega}{N}\right)\left(\frac{l}{h}\right)^2(\gamma u)\right]+\frac{hg}{c_p\Theta}+\theta\frac{\partial\pi}{\partial z} &= 0
\end{aligned}
$$

Nondimensional parameters

$$
\begin{aligned}
B &= \frac{gh}{c_p \Theta} \\
\alpha &= \frac{l}{d} \\
\mathrm{Ro} &= \frac{N}{2\Omega}
\end{aligned}
$$

$$B + \theta \frac{\partial \pi}{\partial z} = 0$$

$$\pi_{\mathrm{hydrostatic}}(z) = 1 - Bz$$

$$B = \frac{h}{d}$$

Momentum equations (again ... sigh)

$$\epsilon\left[\frac{Du}{Dt}-\alpha\frac{1}{\mathrm{Ro}}\left(\beta y v-\frac{1}{\alpha}\gamma w\right)\right]+\frac{1}{B}\theta\frac{\partial\pi}{\partial x} = 0$$

$$\epsilon\left[\frac{Dv}{Dt}+\alpha\frac{1}{\mathrm{Ro}}(\beta y u)\right]+\frac{1}{B}\theta\frac{\partial\pi}{\partial y} = 0$$

$$\frac{1}{\alpha^2}\epsilon\left[\frac{Dw}{Dt}-\alpha^2\frac{1}{\mathrm{Ro}}(\gamma u)\right]+1+\frac{1}{B}\theta\frac{\partial\pi}{\partial z} = 0$$

Perturbation expansions

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \ldots$$

$$\pi = \pi_0 + \epsilon \pi_1 + \epsilon^2 \pi_2 + \ldots$$

$$\theta = 1 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \ldots$$

$\mathcal{O}(1)$ equations

$$\frac{\partial \pi_0}{\partial x} = \frac{\partial \pi_0}{\partial y} = 0$$

$$\frac{\partial \pi_0}{\partial z} = -B$$

$$\frac{\partial \rho_0}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho_0 u_0) + \frac{\partial}{\partial y}(\rho_0 v_0) + \frac{\partial}{\partial z}(\rho_0 w_0)\right]$$

$$\rho_0 = \pi_0^{\frac{1}{\kappa}-1}$$

$\mathcal{O}(1)$ solutions

$$\pi_0(z,t) = \pi_0(0,t) - Bz$$

$$\iiint \frac{\partial \rho_0}{\partial t} dxdydz = -\iiint \nabla \cdot (\rho_0 \mathbf{u_0}) dxdydz = 0$$

$$\left[\frac{\partial}{\partial t}\pi_0(0,t)\right] \int \pi_0^{\frac{1}{\kappa}-2} dz = 0$$

$$\frac{\partial \rho_0}{\partial t} = 0$$

$$\frac{\partial}{\partial x}(\rho_0 u_0) + \frac{\partial}{\partial y}(\rho_0 v_0) + \frac{\partial}{\partial z}(\rho_0 w_0) = 0$$

$\mathcal{O}(\epsilon)$ equations

$$\begin{aligned}
\frac{D_0 u_0}{Dt} - \frac{1}{\mathrm{Ro}}\alpha\left(\beta y v_0 - \frac{1}{\alpha}\gamma w_0\right) + \frac{1}{B}\frac{\partial \pi_1}{\partial x} &= 0 \\
\frac{D_0 v_0}{Dt} + \frac{1}{\mathrm{Ro}}\alpha\left(\beta y u_0\right) + \frac{1}{B}\frac{\partial \pi_1}{\partial y} &= 0 \\
\frac{D_0 w_0}{Dt} + \alpha^2\left[-\frac{1}{\mathrm{Ro}}\gamma u_0 + \frac{1}{B}\frac{\partial \pi_1}{\partial z} - \theta_1\right] &= 0 \\
\frac{D_0 \theta_1}{Dt} &= 0
\end{aligned}$$

Diagnostic equation for pressure perturbation π_1

$$\frac{1}{B}\left[\frac{\partial}{\partial x}\left(\rho_0\frac{\partial \pi_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho_0\frac{\partial \pi_1}{\partial y}\right)+\alpha^2\frac{\partial}{\partial z}\left(\rho_0\frac{\partial \pi_1}{\partial z}\right)\right]$$

$$= \quad \alpha^2\frac{\partial}{\partial z}(\rho_0\theta_1)-\nabla\cdot[\rho_0(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0]$$

$$+\,\alpha^2\frac{1}{\mathrm{Ro}}\left\{\frac{\partial}{\partial x}\left[\rho_0\left(\beta y v_0-\frac{1}{\alpha}\gamma w_0\right)\right]\right.$$

$$\left.+\,\frac{\partial}{\partial y}\left[\rho_0(-\beta y u_0)\right]+\frac{\partial}{\partial z}[\rho_0(\alpha\gamma u_0)]\right\}$$

Exact total energy

$$
\begin{aligned}
E_{\text{comp}} &= \iiint \rho \left(\tfrac{1}{2}|\mathbf{u}|^2 + gz + c_v T\right) dxdydz \\
&= \left(\frac{p_{00}}{\kappa}\right) l^2 hB \iiint \rho^* \left\{ \frac{\epsilon}{2}\left[u^{*2} + v^{*2} + \frac{1}{\alpha^2} w^{*2}\right]\right. \\
&\qquad \left. + z^* + \frac{1}{B}(1-\kappa)T^* \right\} dx^* dy^* dz^*
\end{aligned}
$$

$$
E_{\text{comp}} = \iiint \rho \left\{ \frac{\epsilon}{2}\left[u^2 + v^2 + \frac{1}{\alpha^2} w^2\right] + z + \frac{1}{B}(1-\kappa)T \right\} dxdydz
$$

Expansion of energy in power series in ϵ

$$
\begin{aligned}
E_{\text{comp}} &= \iiint \rho_0 \left[\left(\frac{1-\kappa}{B} \right) + \kappa z \right] dxdydz \\
&\quad + \epsilon \iiint \rho_0 \left\{ \tfrac{1}{2} \left[u_0^2 + v_0^2 + \frac{1}{\alpha^2} w_0^2 \right] - \theta_1 z \right\} dxdydz \\
&\quad + \epsilon \iiint \left(\frac{1-\kappa}{B} \right) \left(\rho_0 \frac{p_1}{p_0} \right) dxdydz \\
&\quad + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

Anelastic energy

$$E_{\text{elastic}} = \epsilon \iiint \left(\frac{1-\kappa}{B} \right) \left(\rho_0 \frac{p_1}{p_0} \right) dxdydz$$

$$E = \iiint \rho_0 \left\{ \tfrac{1}{2} \left[u_0^2 + v_0^2 + \frac{1}{\alpha^2} w_0^2 \right] - \theta_1 z \right\} dxdydz$$

Modified anelastic equations

$$\begin{aligned}\theta &= 1 + \epsilon(\bar{\theta_1}(z) + \theta_1) + \epsilon^2\theta_2 + \ldots \\ &\equiv \theta_0(z) + \epsilon\theta_1 + \epsilon^2\theta_2 + \ldots\end{aligned}$$

$$B + \theta_0(z)\frac{d\pi_0}{dz} = 0$$

$$\frac{D_0 w_0}{Dt} + \alpha^2\left\{-\frac{1}{\mathrm{Ro}}\gamma u_0 + \frac{1}{B}\left[\frac{\partial}{\partial z}(\theta_0\pi_1) + \frac{d\pi_0}{dz}\theta_1\right]\right\} = 0$$

Modified anelastic equations summary

$$\frac{D_0 u_0}{Dt} - \frac{1}{\mathrm{Ro}}\alpha\left(\beta y v_0 - \frac{1}{\alpha}\gamma w_0\right) + \frac{1}{B}\theta_0\frac{\partial \pi_1}{\partial x} \quad = \quad 0$$

$$\frac{D_0 v_0}{Dt} + \frac{1}{\mathrm{Ro}}\alpha\left(\beta y u_0\right) + \frac{1}{B}\theta_0\frac{\partial \pi_1}{\partial y} \quad = \quad 0$$

$$\frac{D_0 w_0}{Dt} + \alpha^2\left\{-\frac{1}{\mathrm{Ro}}\gamma u_0 + \frac{1}{B}\left[\frac{\partial}{\partial z}(\theta_0\pi_1) + \frac{d\pi_0}{dz}\theta_1\right]\right\} \quad = \quad 0$$

$$\frac{D_0 \theta_1}{Dt} + \frac{w_0}{\epsilon}\frac{d\theta_0}{dz} \quad = \quad 0$$

$$\left(\ \frac{\partial}{\partial t}\left[\nabla\cdot(\rho_0\mathbf{u}_0)\right] \ = \ 0\ \right)\ , \qquad \nabla\cdot(\rho_0\mathbf{u}_0) \quad = \quad 0$$

Energy in modified anelastic system

$$E_{\text{comp}} = \iiint \rho_0 \left[z + \left(\frac{1-\kappa}{B} \right) \theta_0 \pi_0 \right] dxdydz$$

$$+ \epsilon \iiint \rho_0 \left\{ \tfrac{1}{2} \left[u_0^2 + v_0^2 + \frac{1}{\alpha^2} w_0^2 \right] - \frac{\theta_1}{\theta_0} z \right\} dxdydz$$

$$+ \epsilon \iiint \left(\frac{1-\kappa}{B} \right) \left(\rho_0 \frac{p_1}{p_0} \right) (\pi_0 \theta_0 + Bz) dxdydz$$

$$+ \mathcal{O}(\epsilon^2)$$

$$E' = \iiint \rho_0 \left\{ \tfrac{1}{2} \left[u_0^2 + v_0^2 + \frac{1}{\alpha^2} w_0^2 \right] + \frac{1}{B} \pi_0 \theta_1 \right\} dxdydz$$

Part II

SYMMETRIC STABILITY OF EQUILIBRIUM ZONAL FLOW

Symmetric equations $\left(\frac{\partial}{\partial x} \equiv 0 \right)$

$$
\begin{aligned}
(u_0)_t &= -v_0(u_0)_y - w_0(u_0)_z + \frac{1}{\mathrm{Ro}}\alpha\left(\beta y v_0 - \frac{1}{\alpha}\gamma w_0\right) \\
(v_0)_t &= -v_0(v_0)_y - w_0(v_0)_z - \frac{1}{\mathrm{Ro}}\alpha\left(\beta y u_0\right) - \frac{1}{B}\theta_0(\pi_1)_y \\
(w_0)_t &= -v_0(w_0)_y - w_0(w_0)_z + \alpha^2\left\{-\frac{1}{\mathrm{Ro}}\gamma u_0 + \frac{1}{B}\left[(\theta_0\pi_1)_z + \frac{d\pi_0}{dz}\theta_1\right]\right\} \\
(\theta_1)_t &= -v_0(\theta_1)_y - w_0(\theta_1)_z - \frac{w_0}{\epsilon}\frac{d\theta_0}{dz} \\
0 &= (\rho_0 v_0)_y + (\rho_0 w_0)
\end{aligned}
$$

Hamiltonian variables

Define the variables

$$\begin{aligned} m &= u_0 + \frac{1}{\mathrm{Ro}}\left(-\tfrac{1}{2}\beta y^2 + \tfrac{1}{\alpha}\gamma z\right) \\ \zeta &= -\left(\frac{\partial v_0}{\partial z} - \frac{1}{\alpha^2}\frac{\partial w_0}{\partial y}\right) \\ \theta &= \theta_0 + \epsilon\theta_1 \end{aligned}$$

and the stream function $\psi(y, z, t)$, satisfying

$$v_0 = \frac{1}{\rho_0}\frac{\partial \psi}{\partial z}, \qquad w_0 = -\frac{1}{\rho_0}\frac{\partial \psi}{\partial y}$$

Hamiltonian equations

$$m_t = \frac{1}{\rho_0}\partial(\psi, m) = \frac{1}{\rho_0}\partial\left(\frac{\delta\mathcal{H}}{\delta\zeta}, m\right)$$

$$\zeta_t = \partial(\psi, \frac{1}{\rho_0}\zeta) + \partial\left[\frac{1}{\rho_0}\left(\frac{1}{\epsilon B}\rho_0\pi_0\right), \theta\right] + \partial\left[\frac{1}{\rho_0}\left(\frac{1}{\mathrm{Ro}}\alpha(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z)\right), m\right]$$

$$= \partial\left(\frac{\delta\mathcal{H}}{\delta\zeta}, \frac{1}{\rho_0}\zeta\right) + \partial\left(\frac{1}{\rho_0}\frac{\delta\mathcal{H}}{\delta\theta}, \theta\right) + \partial\left(\frac{1}{\rho_0}\frac{\delta\mathcal{H}}{\delta m}, m\right)$$

$$\theta_t = \frac{1}{\rho_0}\partial(\psi, \theta) = \frac{1}{\rho_0}\partial\left(\frac{\delta\mathcal{H}}{\delta\zeta}, \theta\right)$$

The Hamiltonian

where the Hamiltonian is

$$
\begin{aligned}
\mathcal{H} &= E' + \text{constant} \\
&= \iint \left\{ \rho_0 \frac{1}{\text{Ro}} \alpha \left(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z \right) m \right. \\
&\qquad \left. + \frac{1}{2\rho_0} \left[\left(\frac{\partial \psi}{\partial z} \right)^2 + \frac{1}{\alpha^2} \left(\frac{\partial \psi}{\partial y} \right)^2 \right] + \frac{1}{\epsilon B} \rho_0 \pi_0 \theta \right\} dy dz
\end{aligned}
$$

$$
\frac{\delta \mathcal{H}}{\delta \zeta} = \psi, \qquad \frac{\delta \mathcal{H}}{\delta \theta} = \frac{1}{\epsilon B} \rho_0 \pi_0
$$

$$
\frac{\delta \mathcal{H}}{\delta m} = \rho_0 \frac{1}{\text{Ro}} \alpha \left(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z \right)
$$

Symplectic representation

$$\frac{\partial x}{\partial t} = \underline{\underline{J}} \frac{\delta \mathcal{H}}{\delta x}$$

$$x \equiv (\zeta, m, \theta)^T, \qquad \underline{\underline{J}} \equiv \begin{pmatrix} \partial(\cdot, \frac{1}{\rho}\zeta) & \partial(\frac{1}{\rho}\cdot, m) & \partial(\frac{1}{\rho}\cdot, \theta) \\ \frac{1}{\rho}\partial(\cdot, m) & 0 & 0 \\ \frac{1}{\rho}\partial(\cdot, \theta) & 0 & 0 \end{pmatrix}$$

The associated Poisson Bracket

$$\{\mathcal{F}, \mathcal{G}\} = \iint \frac{\delta \mathcal{F}}{\delta x} \underline{\underline{J}} \frac{\delta \mathcal{G}}{\delta x} dydz$$

is skew-symmetric, etc.

Casimirs

In general, the time derivative of a functional $\mathcal{F}(x)$ is $\mathcal{F}_t = \{\mathcal{F}, \mathcal{H}\}$, so invariants commute with $\mathcal{H}$ in the sense of Poisson brackets. Functionals of a special class, called Casimirs, commute with all functionals (including $\mathcal{H}$, so they are invariant). In this case, the Casimirs are all of the form

$$\mathcal{C} = \int\int \rho_0 C(m, \theta, q) dydz$$

for arbitrary function $C(m, \theta, q)$, where

$$q \equiv \frac{1}{\rho_0} \partial(\theta, m)$$

is the potential vorticity.

Derivatives of Casimirs

$$\frac{\delta \mathcal{C}}{\delta \zeta} = 0$$

$$\frac{\delta \mathcal{C}}{\delta m} = \rho_0 C_m + \partial(C_q, \theta)$$

$$\frac{\delta \mathcal{C}}{\delta \theta} = \rho_0 C_\theta - \partial(C_q, m)$$

Can verify that $\underline{\underline{J}}\frac{\delta \mathcal{C}}{\delta x} = 0$ for all $C(m, \theta, q)$.

Steady zonal flow

A steady solution to the equations of motion $x = X$, with $\zeta = 0$ (ψ =constant), $m = M$ and $\theta = \Theta$, satisfies

$$\partial\left(\Theta, \frac{\pi_0}{\epsilon B}\right) + \partial\left[M, \frac{1}{\mathrm{Ro}}\alpha\left(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z\right)\right] = 0$$

Choosing $C(m, \theta, q)$

Choose $C(m, \theta, q)$ in order that

$$\left.\frac{\delta \mathcal{C}}{\delta x}\right|_{x=X} = - \left.\frac{\delta \mathcal{H}}{\delta x}\right|_{x=X}$$

That is,

$$\begin{aligned} \rho_0 C_m|_X + \partial(C_q|_X, \Theta) &= -\rho_0 \frac{1}{\mathrm{Ro}} \alpha(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z) \\ \rho_0 C_\theta|_X - \partial(C_q|_X, M) &= -\frac{1}{\epsilon B} \rho_0 \pi_0 \end{aligned}$$

Pseudoenergy

Construct pseudoenergy

$$
\begin{aligned}
\mathcal{A}(\zeta, m, \theta; M, \Theta) \quad &\equiv \quad (\mathcal{H}+\mathcal{C})(\zeta, m, \theta) - (\mathcal{H}+\mathcal{C})(0, M, \Theta) \\
&= \iint \left\{ \rho_0 \frac{1}{\mathrm{Ro}} \alpha \left(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z\right) m \right. \\
&\quad + \frac{1}{2\rho_0}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{1}{\alpha^2}\left(\frac{\partial\psi}{\partial y}\right)^2\right] + \frac{1}{\epsilon B}\rho_0\pi_0\theta + \rho_0 C(m, \theta, q) \\
&\quad - \rho_0 \frac{1}{\mathrm{Ro}} \alpha \left(\tfrac{1}{2}\beta y^2 - \tfrac{1}{\alpha}\gamma z\right) M - \frac{1}{\epsilon B}\rho_0\pi_0\Theta \\
&\quad \left. - \rho_0 C(M, \Theta, Q) \right\} dydz
\end{aligned}
$$

Properties of $\mathcal{A}$

The pseudoenergy is

- conserved
- vanishes at $x = X$
- is of quadratic or higher order in $(x - X)$

Apply Taylor's Theorem

Rewrite $\mathcal{A}$ using

$$
\begin{aligned}
& C(m,\theta,q) - C(M,\Theta,Q) \\
&= (m-M)C_m(M,\Theta,Q) + (\theta-\Theta)C_\theta(M,\Theta,Q) + (q-Q)C_q(M,\Theta,Q) \\
&\quad + \frac{1}{2}\Big[(m-M)^2 C_{mm}(\tilde{m},\tilde{\theta},\tilde{q}) + (\theta-\Theta)^2 C_{\theta\theta}(\tilde{m},\tilde{\theta},\tilde{q}) + (q-Q)^2 C_{qq}(\tilde{m},\tilde{\theta},\tilde{q}) \\
&\quad + 2(m-M)(\theta-\Theta)C_{m\theta}(\tilde{m},\tilde{\theta},\tilde{q}) + 2(m-M)(q-Q)C_{mq}(\tilde{m},\tilde{\theta},\tilde{q}) \\
&\quad + 2(\theta-\Theta)(q-Q)C_{\theta q}(\tilde{m},\tilde{\theta},\tilde{q})\Big]
\end{aligned}
$$

where $\tilde{x}(y,z,t)$ is on the line segment connecting $X(y,z)$ and $x(y,z,t)$.

... inserting into $\mathcal{A}$

$$\mathcal{A} = \iint \Bigg\{ -(\Delta m)\partial(C_q, \Theta) + (\Delta\theta)\partial(C_q, M) + (\Delta q)C_q$$

$$+ \frac{1}{2\rho_0}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{1}{\alpha^2}\left(\frac{\partial\psi}{\partial y}\right)^2\right]$$

$$+ \tfrac{1}{2}\big[(\Delta m)^2\tilde{C}_{mm} + (\Delta\theta)^2\tilde{C}_{\theta\theta} + (\Delta q)^2\tilde{C}_{qq}$$

$$+ 2(\Delta m)(\Delta\theta)\tilde{C}_{m\theta} + 2(\Delta m)(\Delta q)\tilde{C}_{mq} + 2(\Delta\theta)(\Delta q)\tilde{C}_{\theta q}\big]\Bigg\} dydz$$

... continuing ...

We want conditions on X to exist for which $\mathcal{A}$ is positive for all x. We cannot have terms linear in components of Δx, so we require that

$$C_q(m, \theta, q) = 0$$

for all $(m, \theta, q) \in \mathrm{range}\,((M, \Theta, Q)(y, z))$. Let $C_q = 0$ for all x.

Inspection of $\mathcal{A}$ now reveals that it is positive definite if the matrix

$$\Lambda(m, \theta) = \left[\begin{array}{cc} C_{mm} & C_{m\theta} \\ C_{\theta m} & C_{\theta\theta} \end{array} \right]$$

is positive definite for all $m \in \mathrm{range}\,(m(t = 0))$ and $\theta \in \mathrm{range}\,(\theta(t = 0))$.

Comment

In general, need to extend definitions of the second derivatives of $C(m, \theta, q)$ outside of ranges of $M(y, z)$, $\Theta(y, z)$, and $Q(y, z)$, but it is arbitrary how we are to do this.

(by which is meant that the relevant properties of $\mathcal{A}$, namely conservation of $\mathcal{A}$ and $\mathcal{A}(X, X) = 0$, are independent of C outside the ranges of M, Θ, and Q).

Second derivatives of C

If we assume that $\rho_0 Q = \partial(\Theta, M) \neq 0$ in the domain, then there is an invertible map between (y, z) and (M, Θ).

$$
\begin{aligned}
C_{mm} &= -\frac{1}{\mathrm{Ro}}\alpha\left[\beta y\left(\frac{\partial y}{\partial M}\right)_{\Theta} - \frac{1}{\alpha}\gamma\left(\frac{\partial z}{\partial M}\right)_{\Theta}\right] \\
C_{\theta\theta} &= -\frac{1}{\epsilon B}\frac{d\pi_0}{dz}\left(\frac{\partial z}{\partial \Theta}\right)_M \\
C_{m\theta} &= -\frac{1}{\mathrm{Ro}}\alpha\left[\beta y\left(\frac{\partial y}{\partial \Theta}\right)_M - \frac{1}{\alpha}\gamma\left(\frac{\partial z}{\partial \Theta}\right)_M\right] \\
&= -\frac{1}{\epsilon B}\frac{d\pi_0}{dz}\left(\frac{\partial z}{\partial M}\right)_{\Theta}
\end{aligned}
$$

Second derivatives of C (again)

Equivalently,

$$C_{mm} = \frac{1}{\mathrm{Ro}}\alpha\left(\frac{1}{\rho_0 Q}\right)\left[\beta y\frac{\partial\Theta}{\partial z} + \frac{1}{\alpha}\gamma\frac{\partial\Theta}{\partial y}\right]$$

$$C_{\theta\theta} = \left(\frac{1}{\epsilon B}\frac{d\pi_0}{dz}\right)\left(\frac{1}{\rho_0 Q}\frac{\partial M}{\partial y}\right)$$

$$C_{m\theta} = -\frac{1}{\mathrm{Ro}}\alpha\left(\frac{1}{\rho_0 Q}\right)\left[\beta y\frac{\partial M}{\partial z} + \frac{1}{\alpha}\gamma\frac{\partial M}{\partial y}\right]$$

$$= -\left(\frac{1}{\epsilon B}\frac{d\pi_0}{dz}\right)\left(\frac{1}{\rho_0 Q}\frac{\partial\Theta}{\partial y}\right)$$

Equality of two forms of $C_{m\theta}$ follows from thermal wind.

Stability conditions

Λ is positive definite if

$$C_{mm} > 0, \qquad C_{\theta\theta} > 0$$

$$C_{mm}C_{\theta\theta} - (C_{m\theta})^2 > 0$$

which translates to

$$\frac{1}{Q}\left[\beta y \frac{\partial \Theta}{\partial z} + \frac{1}{\alpha}\gamma \frac{\partial \Theta}{\partial y}\right] > 0 \qquad \text{Static stability}$$

$$-\frac{1}{Q}\frac{\partial M}{\partial y} > 0 \qquad \text{Inertial stability}$$

$$\frac{\beta y}{Q} > 0$$

Remarks

- Conditions agree with our intuition about a stable atmosphere:
  - Q positive (negative) in the northern (southern) hemisphere
  - $|M|$ decreases away from the equator
  - Θ increases with height

- Only contribution by γ to stability conditions is that stability is aided if temperature increases away from the equator.

  (?!) ... centrifugally stable to have more mass (colder air) at widest circle?

To do next

- Assess impact of anelastic approximations to results
- Compare with hydrostatic result (this covers “Euler” and “Phillips” cases)
- Linear solution to simple case (compare to Dunkerton, 1982)
- Quantitative calculation of saturation amplitudes for unstable basic states (must decide on domain characteristics, etc.)

FINAL SCORE: 4.3/7