Symmetric stability in the equatorial middle atmosphere

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Group meeting

Thursday, July 15th, 2004

I.

INERTIAL INSTABILITY

Definition of inertial instability

- A fluid in equilibrium is said to be inertially unstable when disturbances are amplified by an imbalance between the pressure gradient force and the centrifugal force due to the fluid's rotating.
- Axisymmetric systems conserve angular momentum $m \equiv rv$, where r is the perpendicular distance from the axis of rotation, and v is the component of velocity tangent to the corresponding circle of radius r.
- Rayleigh criterion for inertial stability of an axisymmetric fluid is that the magnitude of the angular momentum increases with distance from the axis of rotation.

Inertial instability vs. convection

- Angular momentum plays the rôle in axisymmetric inertial instability that potential temperature plays in adiabatic convection.
- Recall that potential temperature, defined by

$$\theta = \left(\frac{p_{00}}{p}\right)^{\frac{R}{c_p}} T$$

  is the temperature a fluid parcel would have if its pressure were changed adiabatically to p_{00}.
- Since θ is conserved by fluid parcels, a parcel lifted to a new height will be warmer (more buoyant) than the surrounding fluid if its potential temperature is higher than the ambient potential temperature at its new height. It would then keep rising and we conclude that the initial configuration was therefore unstable.

- The condition for static stability is thus $\frac{\partial \theta}{\partial z} > 0$
- Similarly, the angular momentum is a measure of the speed a ring of fluid would have if displaced symmetrically to a radius of unity.
- The centrifugal force on the fluid in the ring is $F_c = \frac{v^2}{r}$.

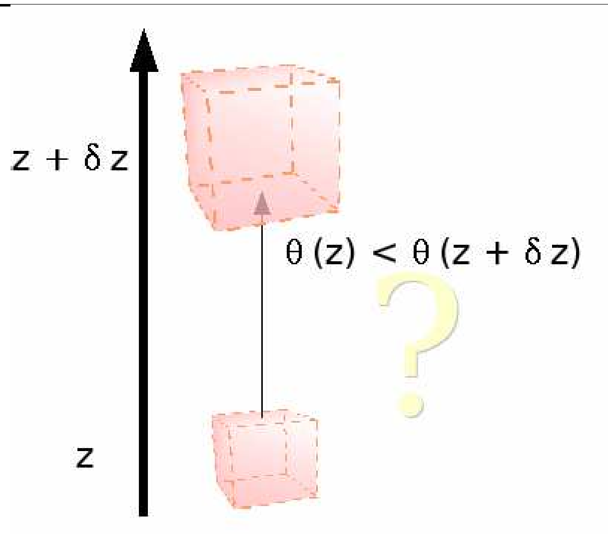
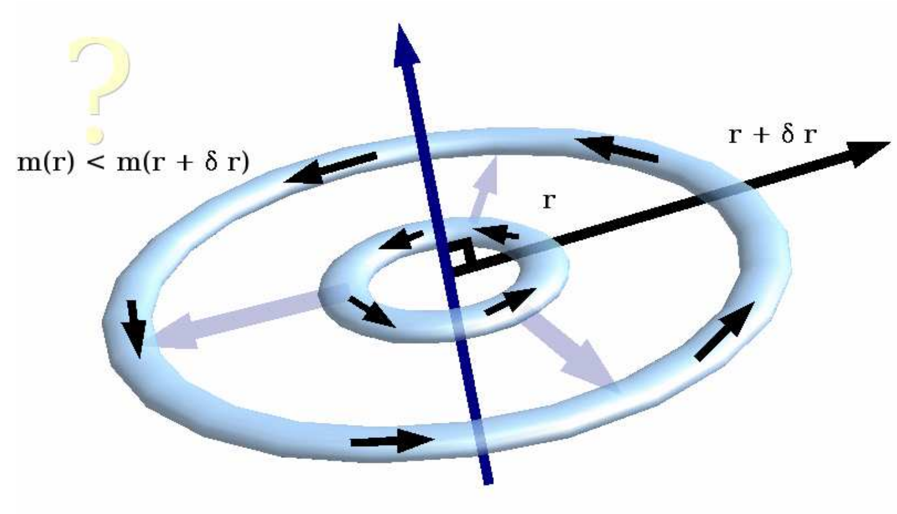
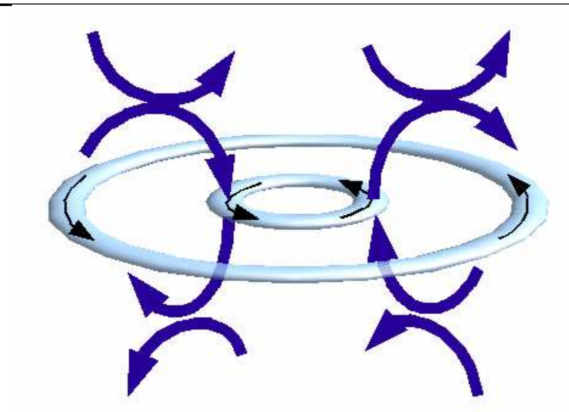
⇒ A ring of fluid displaced outward will have greater absolute velocity than the ambient fluid at its new position, and hence be accelerated outwards, if the magnitude of its angular momentum is greater than that of the ambient fluid.

⇒ Hence the Rayleigh condition for inertial stability $\frac{\partial (m^2)}{\partial r} > 0$

- Similar phenomena observed in laboratory experiments of Rayleigh-Bénard (convection) and Taylor-Couette (flow between rotating cylinders).

Inertial adjustment

- An inertially unstable steady state subject to a small symmetric perturbation will develop Taylor vortex rolls superposed on the tangential flow, thus mixing angular momentum.
- Analogously, a statically unstable steady state subject to a vertical perturbation will lead to cells of rising and descending fluid, mixing potential temperature.
- Evolution of system from unstable basic state towards stable equilibrium called adjustment.
- If forcing which created the basic state is removed (for example by nonlinear interaction between secondary circulation and basic state), adjustment leads to smoothing of offending angular momentum (potential temperature) gradient and removal of instability.

CONVECTIVE STABILITY	INERTIAL STABILITY
	
	

II.

GEOPHYSICAL CONTEXT OF INERTIAL INSTABILITY

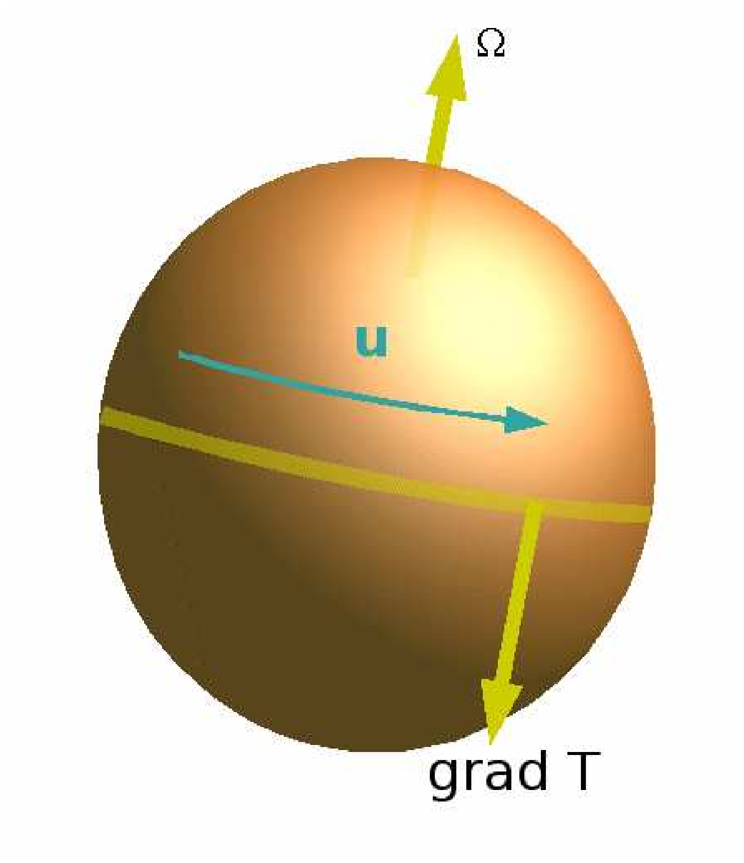
- There is approximately hydrostatic balance in the vertical in the Earth's atmosphere. Motion is predominantly horizontal (at constant distance from the Earth's centre).
- But the Rayleigh criterion refers to distances from the axis of rotation.
- Horizontal motion towards the equator implies motion away from the axis of rotation, and motion away from the equator implies motion towards the axis of rotation.
- Rayleigh criterion in atmosphere becomes: $\phi \frac{\partial m}{\partial \phi}$, where ϕ is latitude and

$$m = r \cos \phi (\Omega r \cos \phi + u)$$

- Notice that the planetary angular momentum is symmetric about the equator, so any latitudinal wind shear $\frac{\partial u}{\partial \phi}$ at equator is inertially unstable.

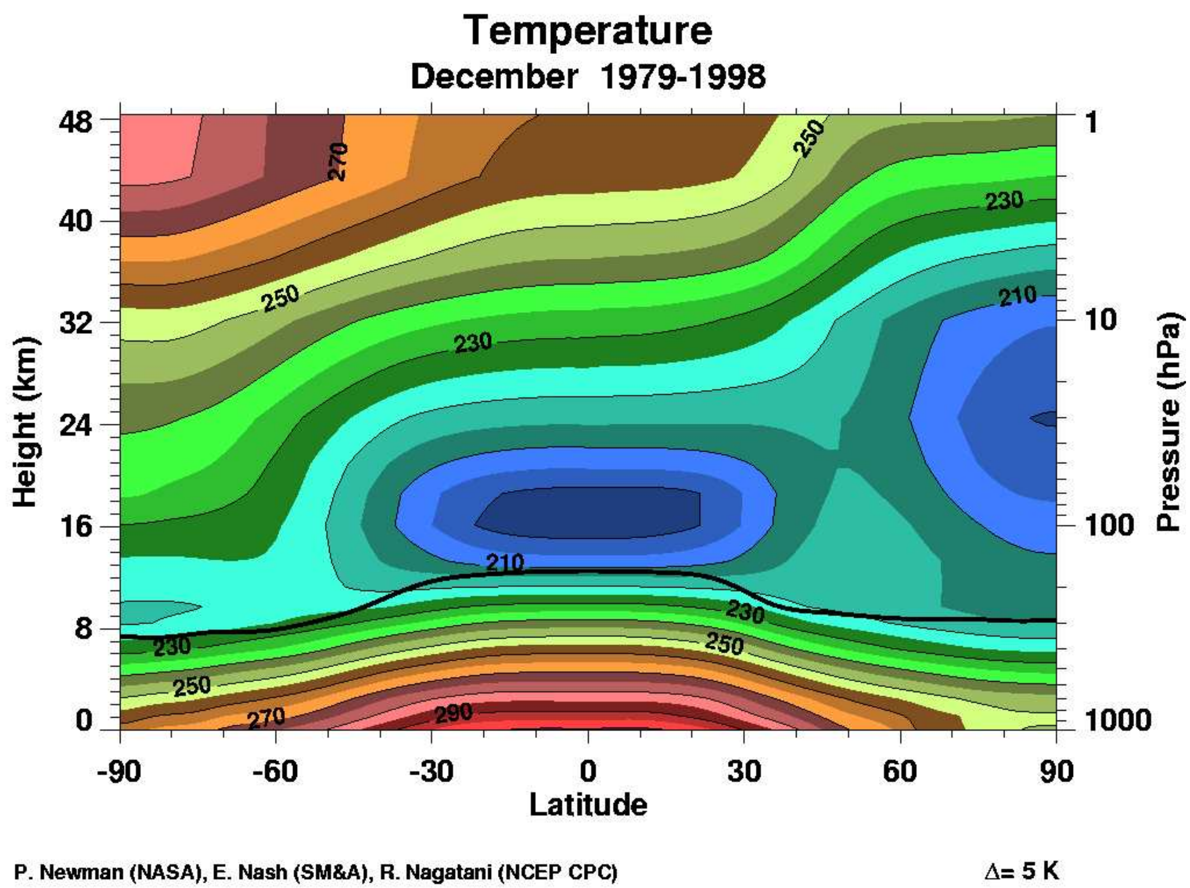
Rôle of inertial instability in solstice dynamics

- The estimated radiative equilibrium temperature distribution in the middle atmosphere (that is the temperature distribution that would obtain due to solar heating, radiatively active chemistry and outgoing radiation - in the absence of dynamics) is symmetric about the equator during the equinox seasons.
- But during the solstices, T_{rad} is decidedly warmer in the summer hemisphere. In particular, the maximum value occurs away from the equator and there is a latitudinal gradient of temperature across the equator.
- The corresponding pressure gradient cannot be balanced by Coriolis forces because it is parallel to the rotation axis.
- Therefore, we don't expect to observe the radiative equilibrium temperature profile at the equator during the solstices.



- Recall $\mathbf{F}_{\text{Coriolis}} = -2\mathbf{\Omega} \times \mathbf{u}$, which is necessarily orthogonal to $\mathbf{\Omega}$ and so cannot possibly balance the temperature (pressure) gradient.

- Away from the equator, the T_{rad} gradient is approximately in geostrophic balance with the zonal wind.
- On the summer side of the equator, the geostrophically balanced winds would be strong enough so that the maximum angular momentum would be off the equator, violating the Rayleigh criterion.
- This condition is therefore not observed. It is believed that continuous inertial adjustment smooths the temperature around the equator, and a Hadley cell develops preventing the temperature from relaxing towards radiative equilibrium. (Cause and effect are a bit confusing, but this is what is observed in models)
- The Hadley circulation pushes air from summer to winter, smoothing the angular momentum gradient in the winter equatorial region.



- Zonal mean temperature for December, averaged over 16 year period (from NCEP)
- Notice temperature gradients flatten over equatorial region.

(from Semeniuk and Shepherd, 2001)

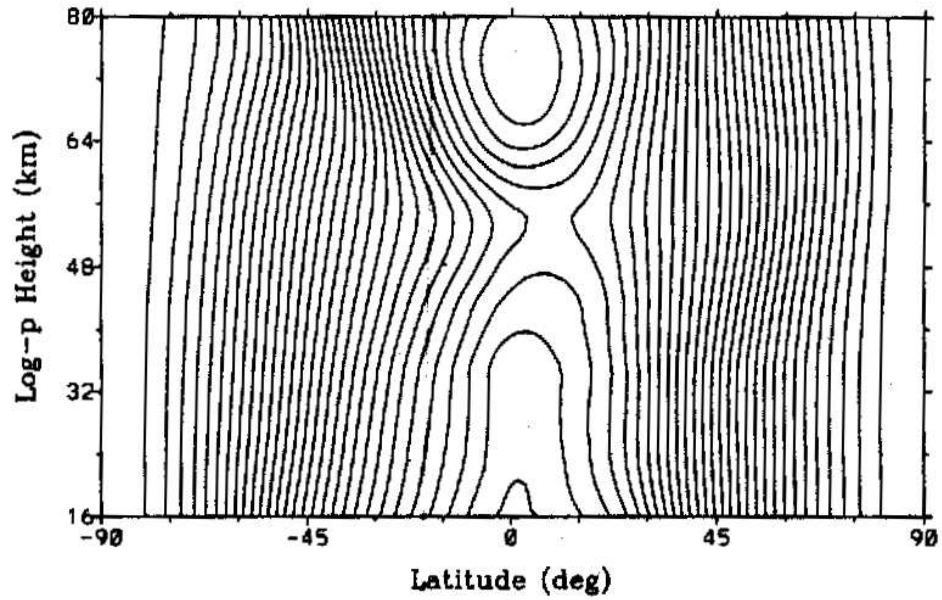


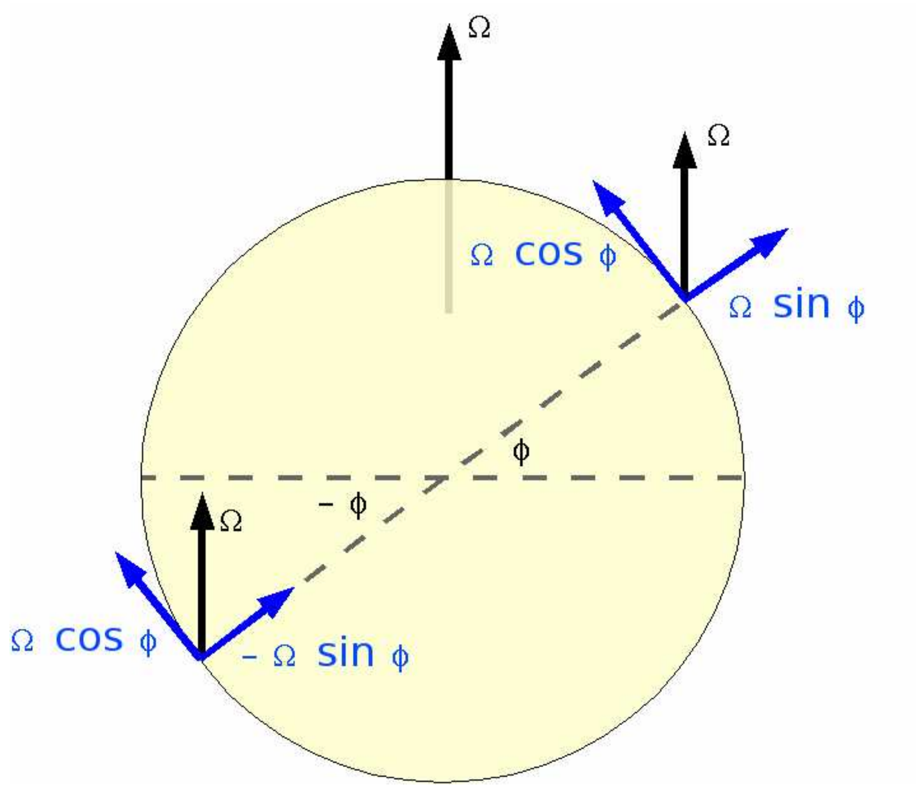
FIG. 19. January mean CMAM absolute angular momentum distribution ($10^8\ m^2s^{-1}$).

- Angular momentum gradient in winter hemisphere weakens due to cross equatorial flow
- Effect most pronounced at stratopause because of maximum ozone heating (and hence maximum gradient in $T_{\rm rad}$) and low density.

III.

SYMMETRIC EQUATORIAL β-PLANE ANELASTIC SYSTEM

Traditional hydrostatic approximation



- Traditional hydrostatic approximation assumes hydrostatic balance in the vertical direction and neglects the $\cos\phi$ Coriolis force terms due to northward component of rotation vector.

Enhanced equatorial β-plane

- Near equator, can approximate rotation vector by its second order Taylor expansion about $\phi = 0$

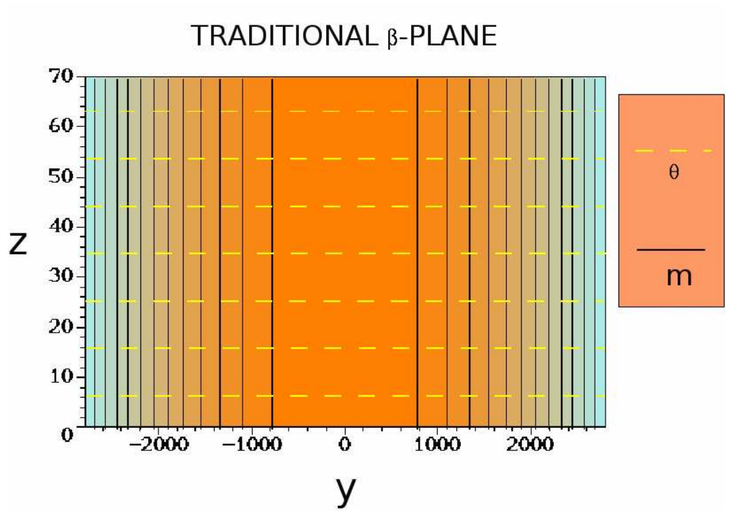
$$\boldsymbol{\Omega} = \tfrac{1}{2}(\gamma\,\hat{\mathbf{e}}_y + \beta y\,\hat{\mathbf{e}}_z),$$

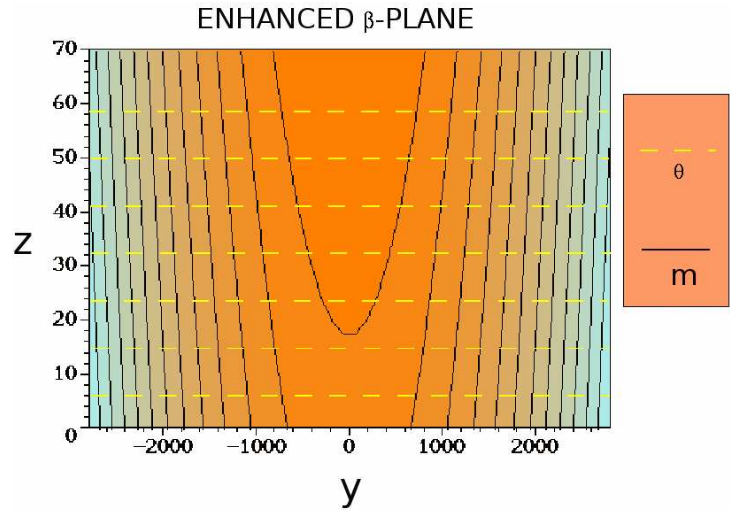
  where $\gamma = 2\Omega$ and $\beta = 2\Omega/a$
  (a being the mean radius of the earth).

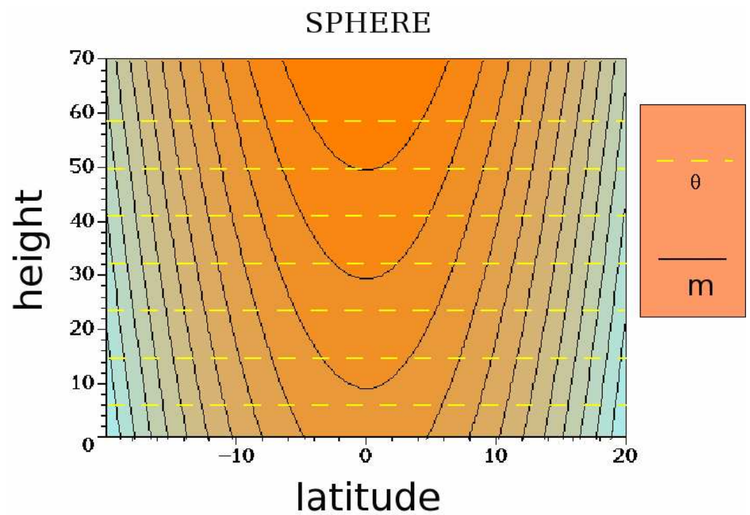
- Latitude is replaced by arclength away from equator $y = a\phi$, and y and z are treated as cartesian coordinates.

- The inclusion of the γz term has an effect on contours of planetary angular momentum ...

Contours of planetary angular momentum







Anelastic equations

- Anelastic system derives name because the energy that is conserved by the equations omits the elastic energy term which involves pressure perturbations.
- Anelastic equations do not admit sound wave solutions but allow for nonhydrostatic motion; used to model deep convection.
- Based on assumptions that potential temperature varies by a small fraction of its mean value over the domain and that time scale of motions is at least N^{-1} (time scale of gravity waves).
- The middle atmosphere does not strictly satisfy the first assumption because of the strong stratification. We use the anelastic model anyway for a technical reason.
- Continuity equation is $\nabla \cdot (\rho_0 \mathbf{u})$ (quasi-incompressible) (c.f. Boussinesq equations)

Modified anelastic equations

$$\frac{Du}{Dt} - \frac{1}{S}\left(-\frac{\beta}{\delta}yv - \frac{\gamma}{\alpha}w\right) + \frac{1}{B}\theta_0\frac{\partial \pi_1}{\partial x} = 0$$

$$\frac{Dv}{Dt} + \frac{1}{S}\left(\frac{\beta}{\delta}yu\right) + \frac{1}{B}\theta_0\frac{\partial \pi_1}{\partial y} = 0$$

$$\frac{1}{\alpha^2}\frac{Dw}{Dt} - \frac{1}{S}\frac{\gamma}{\alpha}u + \frac{1}{B}\left[\frac{\partial}{\partial z}(\theta_0\pi_1) + \frac{d\pi_0}{dz}\theta_1\right] = 0$$

$$\frac{D\theta_1}{Dt} + \frac{w}{\epsilon}\frac{d\theta_0}{dz} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla\cdot(\rho_0\mathbf{u})\right] = 0 \ , \qquad \nabla\cdot(\rho_0\mathbf{u}) = 0$$

Symmetric equations

- If $\dfrac{\partial}{\partial x} \equiv 0$, the resulting equations materially conserve m and θ.
- Symmetric equations have a noncanonical Hamiltonian structure.
- Conserve an energy functional (the Hamiltonian, $\mathcal{H}$) and Casimir invariants $\mathcal{C}$ which depend on m, θ and potential vorticity

$$q = \frac{1}{\rho_0}\left(\frac{\partial\theta}{\partial y}\frac{\partial m}{\partial z} - \frac{\partial\theta}{\partial z}\frac{\partial m}{\partial y}\right),$$

  related to particle relabelling symmetry.
- Can use conserved functionals to calculate stability criteria for an equilibrium.
- Recall that functionals depend on entire functions; they are functions of an infinite number of "independent variables".

Symmetric equations have Hamiltonian form

$$
\begin{aligned}
m_t &= \frac{1}{\rho_0}\,\partial(\psi, m) \\
\zeta_t &= \partial(\frac{1}{S}(\tfrac{1}{2}\tfrac{\beta}{\delta}y^2 - \tfrac{\gamma}{\alpha}z, m) + \partial(\psi, \tfrac{1}{\rho_0}\,\zeta) - \partial(\tfrac{1}{\epsilon B}\pi_0), \theta) \\
\theta_t &= \frac{1}{\rho_0}\,\partial(\psi, \theta)
\end{aligned}
$$

where

$$
\begin{aligned}
m &= u - \tfrac{1}{2}\tfrac{\beta}{\delta}y^2 + \tfrac{\gamma}{\alpha}z \\
\zeta &= v_z - w_y \\
\theta &= \theta_0(z) + \epsilon\theta_1
\end{aligned}
$$

and

$$
v = \frac{1}{\rho_0}\,\psi_z, \quad w = -\frac{1}{\rho_0}\,\psi_y
$$

with Hamiltonian

$$\begin{aligned}\mathcal{H} &= \iint \left\{ \rho_0 \frac{1}{S} \left(\tfrac{1}{2} \tfrac{\beta}{\delta} y^2 - \tfrac{\gamma}{\alpha} z \right) m \right. \\ &\qquad \left. + \frac{1}{2\rho_0} \left[\left(\frac{\partial \psi}{\partial z} \right)^2 + \frac{1}{\alpha^2} \left(\frac{\partial \psi}{\partial y} \right)^2 \right] + \frac{1}{\epsilon B} \rho_0 \pi_0 \theta \right\} dydz\end{aligned}$$

and Casimirs of the form

$$\mathcal{C}_1 = \iint \rho_0 C_1(m, \theta) dydz$$

More generally, the equations conserve

$$\mathcal{C} = \iint \rho_0 C(m, \theta, q) dydz$$

IV.

CONDITIONS FOR SYMMETRIC STABILITY: EQUATORIAL β-PLANE ANELASTIC SYSTEM

Consider rather the cleaner system

$$\begin{aligned} m_t &= \tfrac{1}{\rho_0}\,\partial(\psi, m) \\ \zeta_t &= \partial(\tfrac{1}{2}y^2 - z, m) + \partial(\psi, \tfrac{1}{\rho_0}\,\zeta) - \partial(z, \theta) \\ \theta_t &= \tfrac{1}{\rho_0}\,\partial(\psi, \theta), \end{aligned}$$

with reference state for the anelastic system

$$\begin{aligned} \theta_0(z) &= \Theta_0 \\ \rho_0(z) &= R_0(1 - B(\Theta_0)z)^{c_v/c_p}, \end{aligned}$$

in the domain

$$\mathcal{D} = \{(y, z) | -1 \leq y \leq 1, 0 \leq z \leq 1\}$$

Seek conditions for stability of zonal basic state X with

$$\psi = \zeta = 0,\ m = M(y,z),\ \theta = \Theta(y,z)$$

in thermal wind balance

$$yM_z + M_y = -\Theta_y$$

As a first step, we seek conditions for stability with respect to small perturbations (linear stability). To that end, linearize equations about $\mathbf{X}$:

$$\begin{aligned} m'_t &= \frac{1}{\rho_0}\,\partial(\psi', M) \\ \zeta'_t &= \partial(\tfrac{1}{2}y^2 - z, m') - \partial(z, \theta') \\ \theta'_t &= \frac{1}{\rho_0}\,\partial(\psi', \Theta), \end{aligned}$$

Suppose that $Q(y,z) = \frac{1}{\rho_0}\,\partial(\Theta, M)$ is nonzero everywhere except on a finite set of curves. Partition $\mathcal{D}$ accordingly:

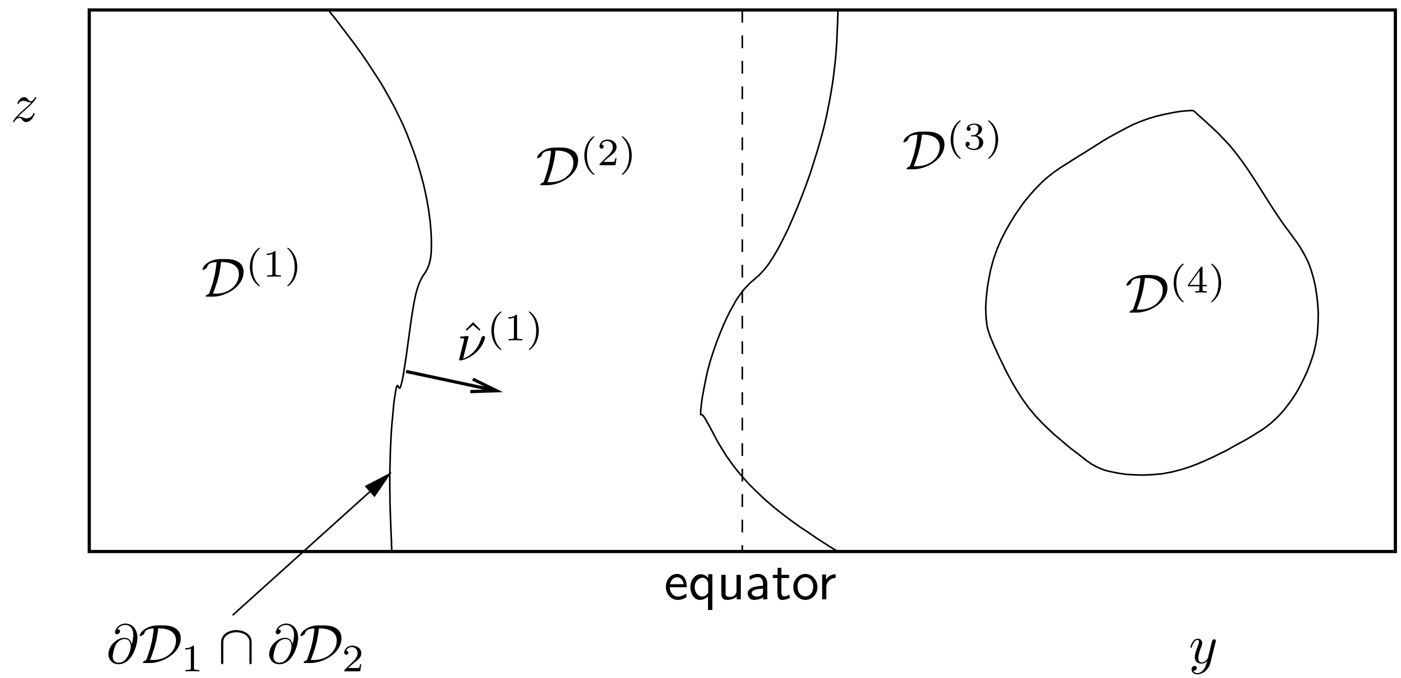


Figure 1: Sample partition of $\mathcal{D}$ into regions with nonzero Q.

Consider

$$\mathcal{C}_L = \sum_{i=1}^{n} \iint_{\mathcal{D}^{(i)}} \rho_0 C^{(i)}(m, \theta) dydz$$

which is not conserved by the nonlinear equations, since

$$\frac{d}{dt}\mathcal{C}_L = \frac{1}{2} \sum_{i,j=1}^{n} \int_{\partial\mathcal{D}^{(i)} \cap \partial\mathcal{D}^{(j)}} \psi \left\{ \hat{\mathbf{i}} \times \left[(C_m^{(j)} - C_m^{(i)}) \nabla m \right.\right.$$

$$\left.\left. + (C_\theta^{(j)} - C_\theta^{(i)}) \nabla \theta \right] \right\} \cdot \hat{\nu}^{(i)} dl^{(i)}(y, z).$$

In general, the $C^{(i)}$ are not the same, so the values will not match on the boundaries for all values of m and θ.

However, the following functional, based on the second variation of $\mathcal{H} + \mathcal{C}_L$,

$$\mathcal{H}_L = \iint_{\mathcal{D}} \frac{1}{\rho_0} \left[(\psi_y')^2 + (\psi_z')^2\right] dydz$$

$$+ \sum_{i=1}^{n} \iint_{\mathcal{D}^{(i)}} \rho_0 \left[C_{mm}^{(i)}(M,\Theta)m'^2 + 2C_{\theta m}^{(i)}(M,\Theta)\theta' m' + C_{\theta\theta}^{(i)}(M,\Theta)\theta'^2\right] dydz$$

is conserved by the linear equations, provided

$$C_m^{(i)}(M,\Theta) = -(\tfrac{1}{2}y^2 - z), \qquad C_\theta^{(i)}(M,\Theta) = z$$

for all i, and the $C^{(i)}$ and their first and second partial derivatives all match on the inside boundaries.

Then, if the matrices

$$\Lambda^{(i)} = \begin{bmatrix} C^{(i)}_{mm} & C^{(i)}_{m\theta} \\ C^{(i)}_{\theta m} & C^{(i)}_{\theta\theta} \end{bmatrix}$$

are all positive definite at all points $[y(M,\Theta), z(M,\Theta)]$, then $\mathbf{X}$ is linearly stable. More precisely, $\mathbf{x}' = 0$ is stable with respect to the norm

$$\begin{aligned} ||\mathbf{x}'||^2_{(L)\lambda} &= \iint_{\mathcal{D}} \tfrac{1}{\rho_0} \left[(\psi'_y)^2 + (\psi'_z)^2 \right] dydz \\ &\quad + \sum_{i=1}^{n} \iint_{\mathcal{D}^{(i)}} \rho_0 \lambda^{(i)} \left[m'^2 + \theta'^2 \right] dydz, \end{aligned}$$

where $\lambda^{(i)}$ is the minimum of the eigenvalues of $\Lambda^{(i)}$ for all of the values of M and Θ inside $\mathcal{D}^{(i)}$. (i.e. the norm will stay small if it starts small enough.)

The conditions for linear stability can be written

$$
\begin{aligned}
C^{(i)}_{mm}(M,\Theta) &= \frac{1}{\rho_0 Q}(\Theta_y + y\Theta_z) > 0 \\
C^{(i)}_{\theta\theta}(M,\Theta) &= -\frac{1}{\rho_0 Q} M_y > 0 \\
C^{(i)}_{mm}(M,\Theta) C^{(i)}_{\theta\theta}(M,\Theta) - {C^{(i)}_{m\theta}}^2(M,\Theta) &= \frac{y}{\rho_0 Q} > 0.
\end{aligned}
$$

which are conditions for STATIC, INERTIAL, and “SYMMETRIC” stability.

Notice that for linear stability, we needed $Q = 0$ on the equator and $Q \neq 0$ everywhere else, so we only need two partitions. To derive nonlinear stability, we partition the domain not along a curve fixed in space, but along the line $q = 0$ which can move as the flow changes.

Define

$$\mathcal{C} = \iint_{\mathcal{D}} \rho_0 \left\{ C^-(m,\theta) + H(q) \left[C^+(m,\theta) - C^-(m,\theta) \right] \right\} dydz,$$

where

$$H(q) = \begin{cases} 0, & q < 0 \\ 1, & q \geq 0 \end{cases},$$

C^- and C^+ and their first and second partial derivatives all match along the curve defined by $Q = 0$, and

... C^- and C^+ satisfy

$$\begin{aligned} C_m^-(M,\Theta) &= -\left[\tfrac{1}{2}\left(Y^-(M,\Theta)\right) - Z^-(M,\Theta)\right] \\ C_\theta^-(M,\Theta) &= Z^-(M,\Theta) \\ C_m^+(M,\Theta) &= -\left[\tfrac{1}{2}\left(Y^+(M,\Theta)\right) - Z^+(M,\Theta)\right] \\ C_\theta^+(M,\Theta) &= Z^+(M,\Theta), \end{aligned}$$

where (Y^-, Z^-) and (Y^+, Z^+) are the inverse functions defined by $(M(y,z), \Theta(y,z))$ in the regions with $Q < 0$ and $Q \geq 0$ respectively.

Define the pseudoenergy

$$\mathcal{A} = (\mathcal{H} + \mathcal{C})(\zeta, m, \theta) - (\mathcal{H} + \mathcal{C})(0, M, \Theta)$$

By construction, $\mathcal{A}$ evaluated at $\mathbf{X}$ vanishes, and it is conserved. If we can show that $\mathcal{A}$ is positive for all $\mathbf{x}$, then we can conclude that $\mathbf{X}$ is nonlinearly stable.

$\mathcal{A}$ can be rewritten using Taylor's Remainder Theorem,

$$\begin{aligned}\mathcal{A} \;=\; \iint_{\mathcal{D}} \Big\{ & \tfrac{1}{2\rho_0}\left[(\psi_y)^2+(\psi_z)^2\right] \\ & + \tfrac{1}{2}\rho_0 \Big[C^-_{mm}(\tilde{m},\tilde{\theta})(m-M)^2 + 2C^-_{m\theta}(\tilde{m},\tilde{\theta})(m-M)(\theta-\Theta) \\ & \qquad + C^-_{\theta\theta}(\tilde{m},\tilde{\theta})(\theta-\Theta)^2 \Big] + H(q)\left(C^+(m,\theta)-C^-(m,\theta)\right) \\ & \qquad - H(Q)\left(C^+(M,\Theta)-C^-(M,\Theta)\right)\Big] \Big\}\, dydz.\end{aligned}$$

where $\tilde{m}(y,z,t) \in [M,m]$ and $\tilde{\theta}(y,z,t) \in [\Theta,\theta]$. If linear conditions are satisfied by $C^-(m,\theta)$ for all values of m and θ, all terms in $\mathcal{A}$ are positive except the last part which depends on the extent to which C^- and C^+ have been mixed, and hence on the asymmetry of $\mathbf{X}$:

$$\begin{aligned}\mathcal{A}_a \;=\; \iint_{\mathcal{D}} & \tfrac{1}{2}\rho_0 \big[H(q)\left(C^+(m,\theta)-C^-(m,\theta)\right) \\ & - H(Q)\left(C^+(M,\Theta)-C^-(M,\Theta)\right)\big]\, dydz.\end{aligned}$$

While $\mathcal{A}_a$ is not sign definite, it can be bounded:

$$\begin{aligned}
\mathcal{A}_a &> \iint_{\mathcal{D}} \Big[-\max_{(M,\Theta)} \tfrac{1}{2}\rho_0 \big| C^+(m,\theta) - C^-(m,\theta) \big| \\
&\qquad - \tfrac{1}{2}\rho_0 H(Q) \left(C^+(M,\Theta) - C^-(M,\Theta) \right) \Big] \, dydz
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_a &< \iint_{\mathcal{D}} \Big[\max_{(M,\Theta)} \tfrac{1}{2}\rho_0 \big| C^+(m,\theta) - C^-(m,\theta) \big| \\
&\qquad - \tfrac{1}{2}\rho_0 H(Q) \left(C^+(M,\Theta) - C^-(M,\Theta) \right) \Big] \, dydz,
\end{aligned}$$

and hopefully C^- and C^+ can be extended to all values of m and θ in such a way that the bounds can be written in terms of $M(y,z)$ and $\Theta(y,z)$. Note also that the roles of C^- and C^+ can be interchanged to improve on the bounds.

In the case of $\mathbf{X}$ being symmetric about $y = 0$, $C^- = C^+$, and $\mathcal{A}_a = 0$. We then define a norm on the space $\Delta\mathbf{x} = \mathbf{x} - \mathbf{X} = (m - M, \zeta, \theta - \Theta)$

$$\begin{aligned} ||\Delta\mathbf{x}||_\lambda^2 &= \iint_{\mathcal{D}} \left\{ \tfrac{1}{2\rho_0} \left[(\psi_y)^2 + (\psi_z)^2 \right] \right. \\ &\qquad \left. + \tfrac{1}{2}\rho_0 \lambda \left[(m - M)^2 + (\theta - \Theta)^2 \right] \right\} dydz. \end{aligned}$$

If λ is the minimum of the eigenvalues of the matrix $\Lambda(m, \theta)$ over all possible values of m and θ (since $\Lambda(m, \theta)$ is assumed to be positive definite, λ is positive), then we have that

$$||\Delta\mathbf{x}(t)||_\lambda^2 \leq \mathcal{A}(\mathbf{x}(t)) = \mathcal{A}(\mathbf{x}(0)) \leq \frac{\lambda_+}{\lambda} ||\Delta\mathbf{x}(0)||_\lambda^2,$$

where λ_+ is the maximum of the eigenvalues of Λ, which implies that $||\Delta\mathbf{x}(t)||_\lambda$ is bounded for all t in terms of its initial value $||\Delta\mathbf{x}(0)||_\lambda$. In particular, $\mathbf{x}(t)$ can be bounded as close to $\mathbf{X}$ as desired by setting $\mathbf{x}(0)$ close enough to $\mathbf{X}$.

V.

SOLUTION TO ANELASTIC EQUATIONS LINEARIZED ABOUT EQUILIBRIUM WITH CONSTANT VELOCITY SHEAR

Consider basic state with

$$\begin{aligned} M &= -\tfrac{1}{2}by^2 + \lambda y \\ \Theta &= (\epsilon\gamma)(\tfrac{1}{2}by^2 - \lambda y) + \Gamma z \end{aligned}$$

with b, λ and Γ positive. $M_y = -by + \lambda$, so state is unstable in interval $0 < y < \lambda/b$ (violates condition that M decrease away from the equator). Linearized equations are then

$$\begin{aligned} m'_t &= \frac{1}{\rho_0}(by - \lambda)\psi'_z \\ \zeta'_t &= \frac{1}{\epsilon}\theta'_y + \beta y m'_z + \gamma m'_y \\ \theta'_t &= -\frac{1}{\rho_0}(\epsilon\gamma)(by - \lambda)\psi'_z + \frac{1}{\rho_0}\Gamma\psi'_y \end{aligned}$$

($\beta \to S\delta\beta$, $\gamma \to \alpha\gamma$), which are combined to get

$$\rho_0(\frac{1}{\rho_0}\psi'_z)_{ztt} + \frac{1}{\alpha^2}(\psi'_y)_{ytt} = -\beta b y(y - \frac{\lambda}{b})\rho_0(\frac{1}{\rho_0}\psi'_z)_z + \gamma\Gamma\psi'_{yy}$$

Seek separable solution: $\psi'(y,z,t) = Y(y)Z(z)T(t)$, and find

$$\begin{aligned} Y'' + k^2\left[\omega^2 - \beta b y(y - \frac{\lambda}{b})\right] Y &= 0 \\ \rho_0 \frac{d}{dz}(\tfrac{1}{\rho_0}\, Z') + k^2(\tfrac{1}{\alpha^2} - \gamma\Gamma)Z &= 0 \\ T'' + \omega^2 T &= 0, \end{aligned}$$

where k and ω are constants, and $\rho_0 = R_0(1-Bz)^{c_v/c_p}$. Solution is

$$\begin{aligned} \psi'(y,z,t) &= \exp(i\omega t) \\ &\times \exp\left[-\tfrac{1}{2}\sqrt{\beta b}k(y - \tfrac{\lambda}{2b})^2\right] H_n\left[(\beta b k^2)^{-\frac{1}{4}}(y - \frac{\lambda}{2b})\right] \\ &\times (1-Bz)^{\frac{1}{2}(1+\frac{c_v}{c_p})}\left\{c_{1kn} J_{\frac{1}{2}(1+\frac{c_v}{c_p})}\left[\frac{k}{B}\sqrt{\frac{1}{\alpha^2} - \gamma\Gamma(1-Bz)}\right]\right. \\ &+ \left. c_{2kn} Y_{\frac{1}{2}(1+\frac{c_v}{c_p})}\left[\frac{k}{B}\sqrt{\frac{1}{\alpha^2} - \gamma\Gamma(1-Bz)}\right]\right\} \end{aligned}$$

where n is an integer satifying:

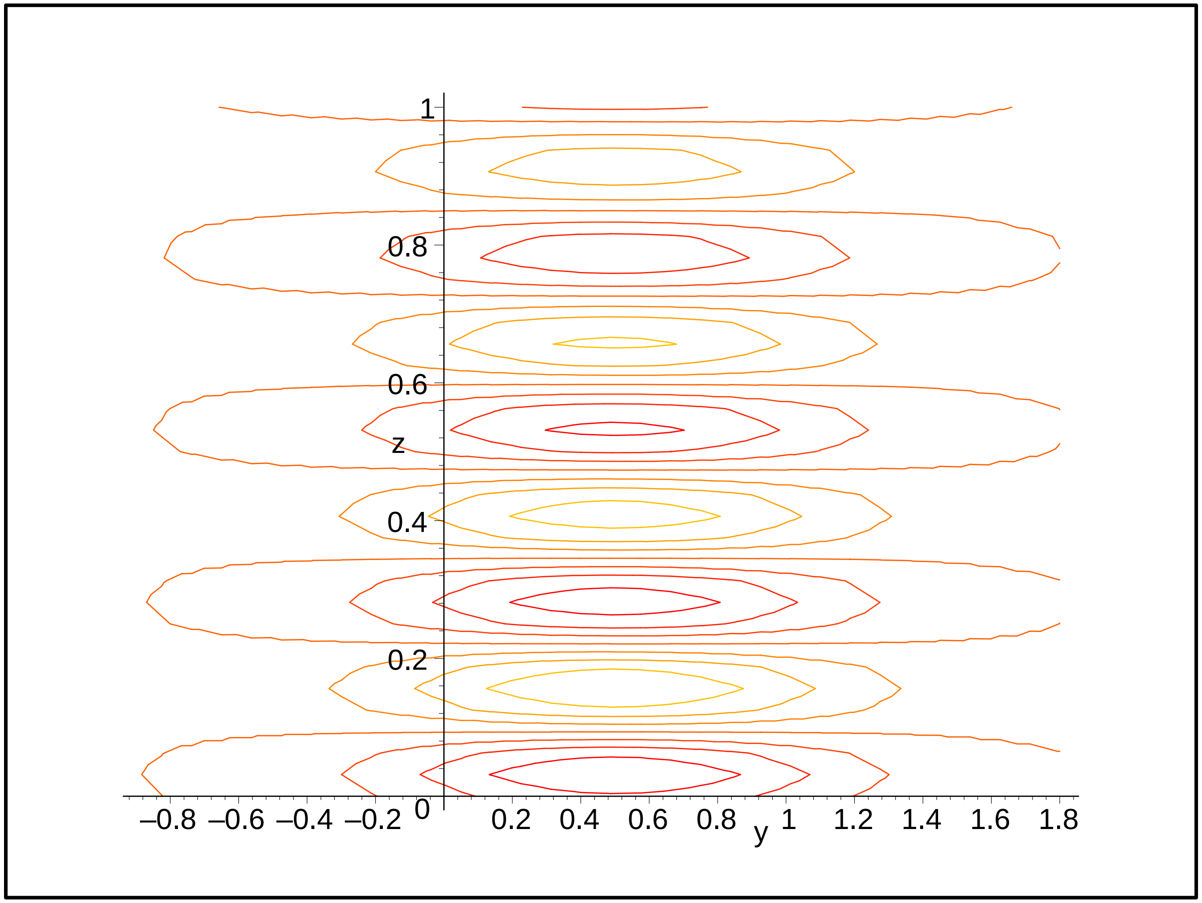
$$2n + 1 = \frac{k}{\sqrt{\beta b}} \left[\omega^2 + \beta b \left(\frac{\lambda}{2b} \right)^2 \right]$$

Solution is unstable (growing) for $\omega^2 < 0$, i.e. for

$$\omega^2 = \frac{\sqrt{\beta b}}{k}(2n + 1) - \beta b \left(\frac{\lambda}{2b} \right)^2 < 0$$

So most unstable modes have smallest meridional index n and largest vertical "wavenumber" k, so they are wide and short "pancakes". A value of $k = 2500$ gives unstable mode with four cells in vertical centred on y=0.3.

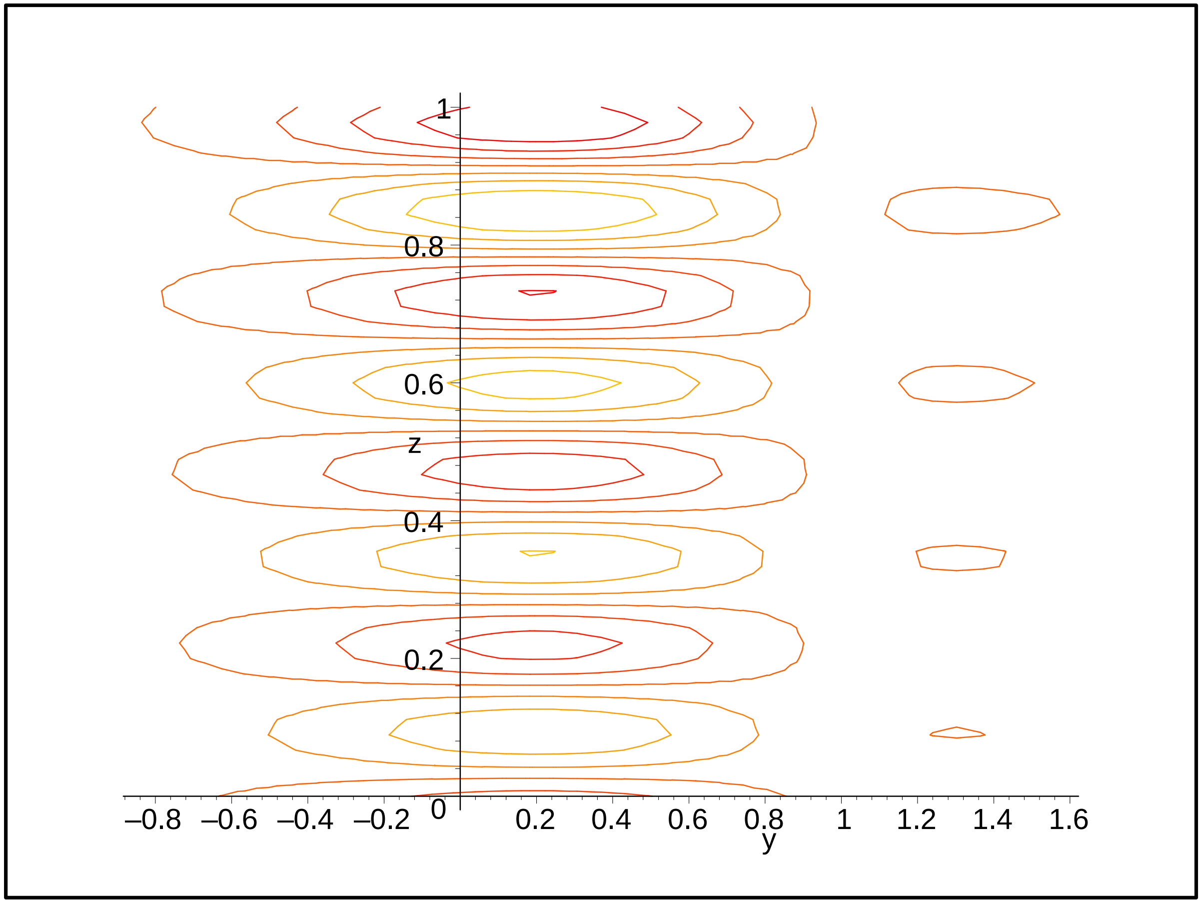
STREAM FUNCTION CONTOURS



$l = 10^6$ m, $h = 10^4$ m, $y_c = 0.3$, $S = 10^5$, $\delta = 6$, $k = 2500$

RED = NEGATIVE, ORANGE = POSITIVE

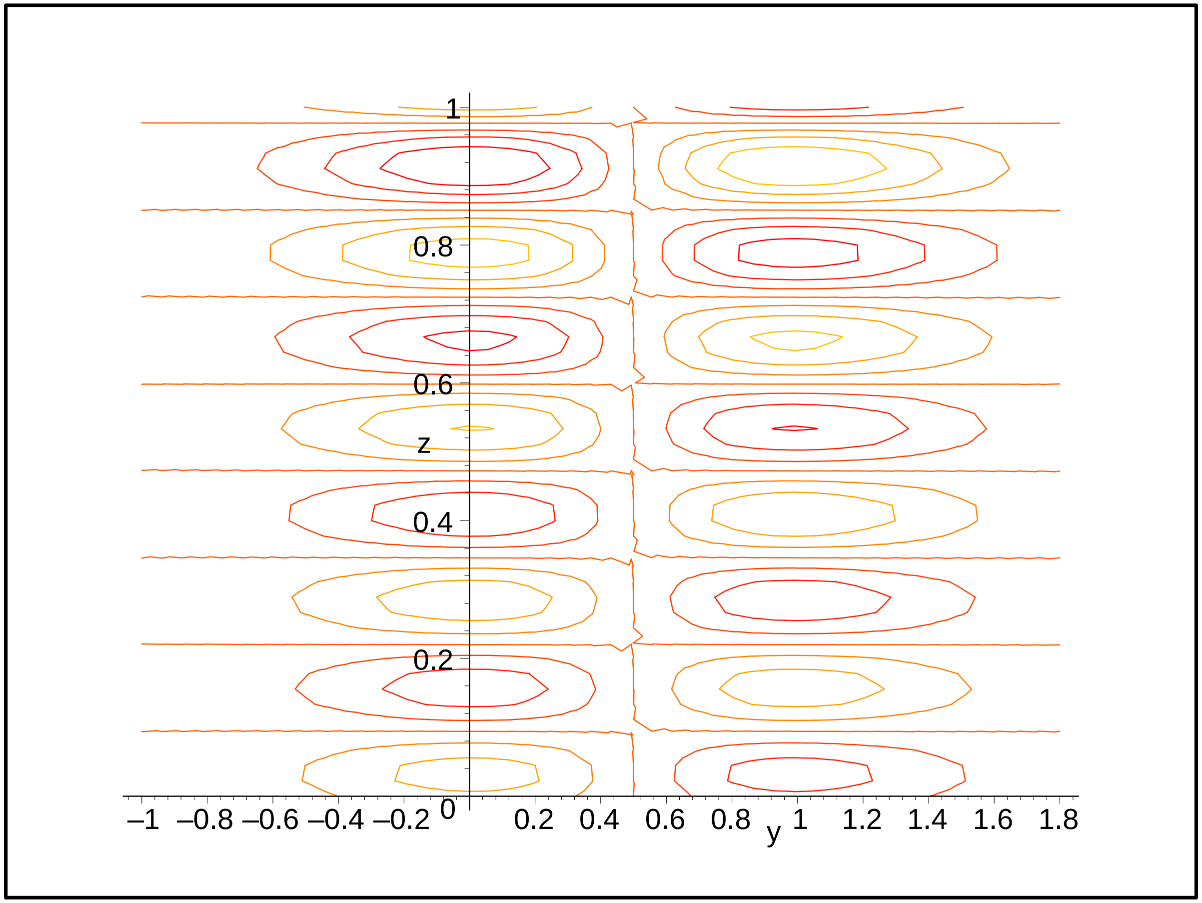
ANGULAR MOMENTUM PERTURBATION CONTOURS



$l = 10^6$ m, $h = 10^4$ m, $y_c = 0.3$, $S = 10^5$, $\delta = 6$, $k = 2500$

RED = NEGATIVE, ORANGE = POSITIVE

POTENTIAL TEMPERATURE PERTURBATION CONTOURS



$l = 10^6$ m, $h = 10^4$ m, $y_c = 0.3$, $S = 10^5$, $\delta = 6$, $k = 2500$

RED = NEGATIVE, ORANGE = POSITIVE