Linear and Nonlinear Symmetric Stability in the Equatorial Middle Atmosphere

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1. Introduction to SYMMETRIC STABILITY

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2. ENERGY and STABILITY

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3. Conditions for LINEAR stability

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1. SYMMETRIC STABILITY

- Refers to the stability of an equilibrium which is symmetric in one direction under disturbances which have the same symmetry.
- In this case, we consider stability of zonally symmetric solutions to adiabatic Euler/Anelastic Equations in atmosphere to zonally symmetric disturbances
- System is 2 dimensional: (ϕ, r), with 2 material invariants:
  - Absolute angular momentum $m \equiv \Omega r^2 \cos^2 \phi + ur \cos \phi$ (because of zonal symmetry)
  - Potential temperature θ (or entropy) (because flow is adiabatic)
- and 2 forces acting on air parcels: gravity and the Coriolis force

- Most geophysical applications use the Primitive Equations, in which the Coriolis force is strictly horizontal ($\perp$ gravity).

  $\Rightarrow$ In that case, m is to displacement along pressure surfaces as

  θ is to displacement in height.

- The (Primitive Equations) conditions for symmetric stability are:

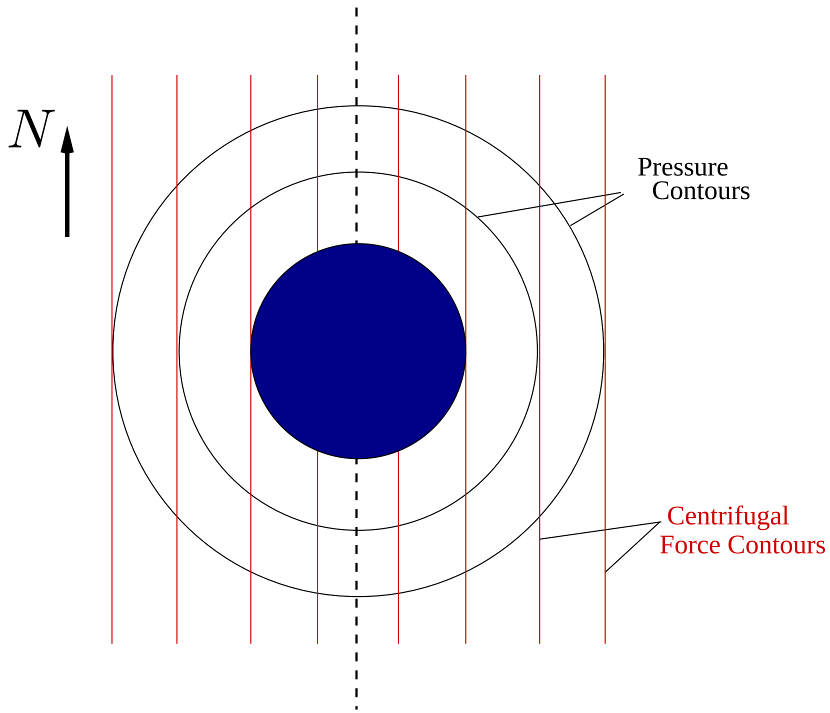
  * θ increases with height (static stability)

  * m increases towards the equator at constant pressure (Rayleigh centrifugal stability theorem)

  * Potential vorticity has the same sign as latitude

Nonhydrostatic Terms

- Pressure contours are approximately spheres.
- Centrifugal force contours are approximately cylinders (solid-body rotation)
- Near equator, they are actually parallel.



- Nonhydrostatic terms are neglected in Primitive Equations in order to retain conservation of energy and angular momentum principles
- Near equator, terms not negligible (error of $\sim 10\%$)

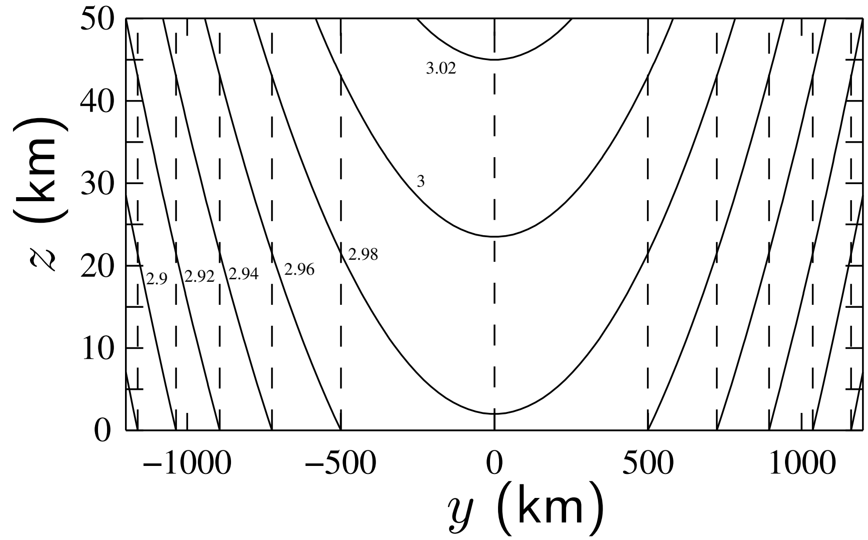
- We use generalized equatorial β-plane
- Expand $\mathbf{\Omega}$ to second order in latitude about equator:

$$2\mathbf{\Omega} = 2\Omega\cos\phi\,\hat{\mathbf{e}}_\phi + 2\Omega\sin\phi\,\hat{\mathbf{e}}_r \approx \gamma\,\hat{\mathbf{e}}_y + \beta y\,\hat{\mathbf{e}}_z$$

- Approximation to absolute angular momentum is proportional to

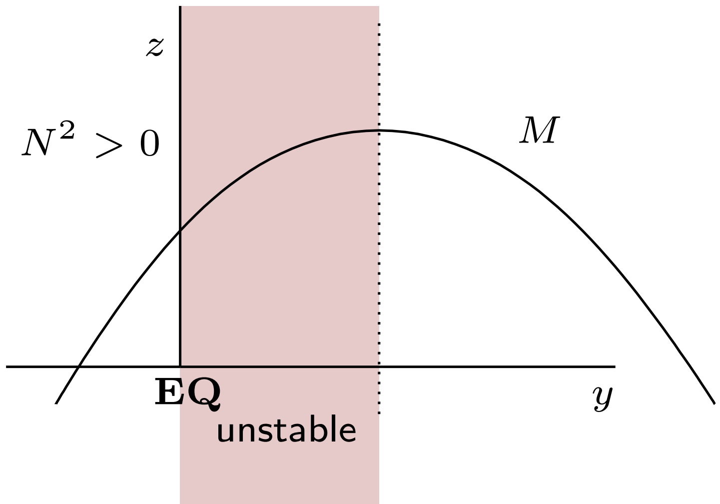
$$m \equiv u - \tfrac{1}{2}\beta y^2 + \gamma z$$

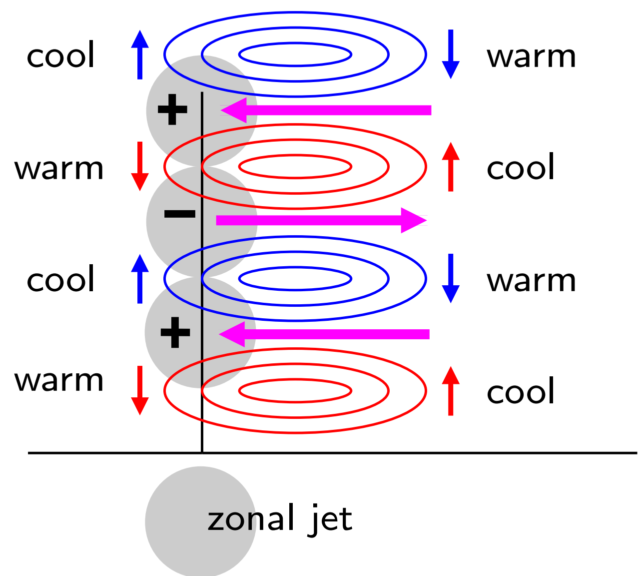
- Planetary angular momentum contours



Dunkerton problem

- Meridional velocity shear $U = \lambda y$ at the equator violates Rayleigh stability condition in interval $0 < y < \lambda/\beta$
- Dunkerton (1981) solved linearized, hydrostatic equations on β-plane
- Solution exhibits
  - "Taylor Vortices" in unstable region
  - zonal jets over equator
  - pancake structures in temperature perturbation field





Geophysical Context

- Under solstice conditions radiative equilibrium temperature has meridional gradient at equator

  $\Rightarrow$ can only balance with cross equatorial flow

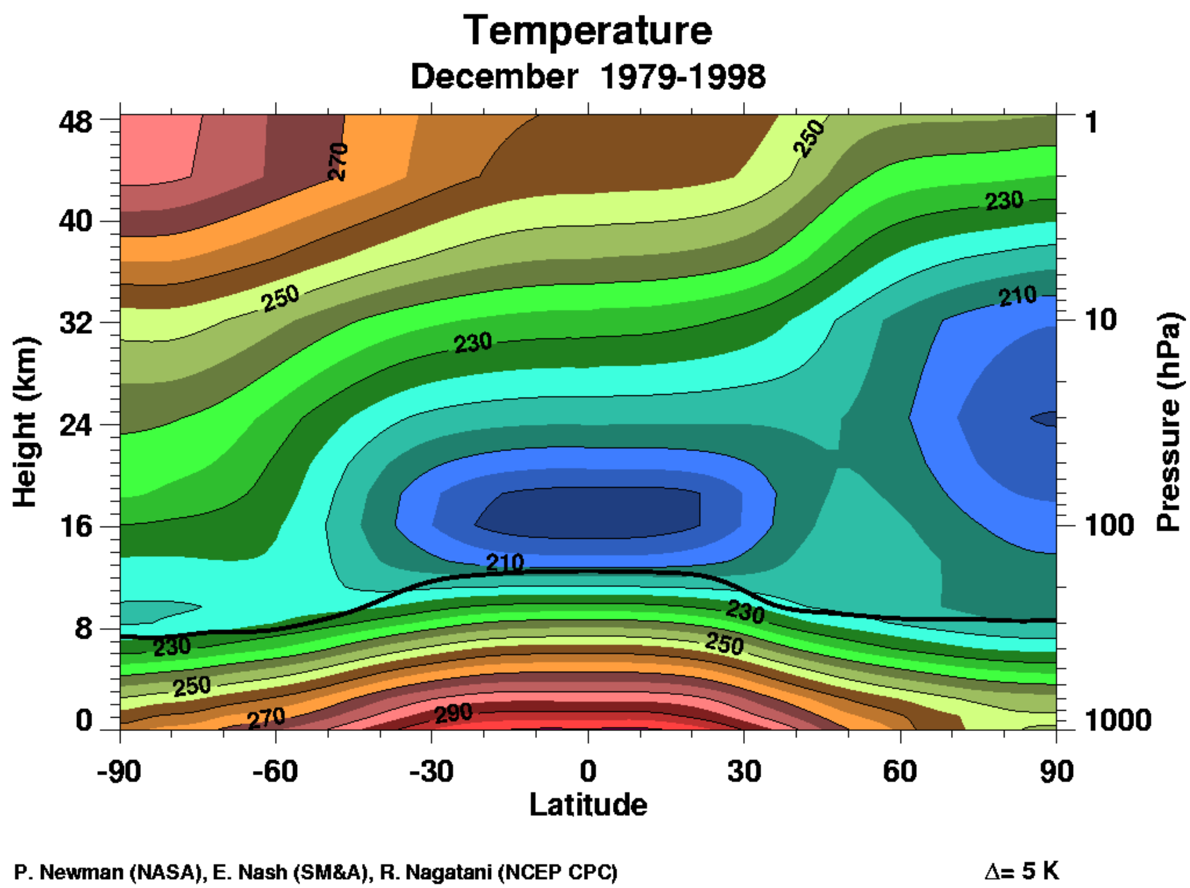
  $\Rightarrow$ advects angular momentum maximum (and zero potential vorticity line) across equator

  $\Rightarrow$ drives system towards inertially unstable state

- presumably, undetectable adjustment continuously taking place

  $\Rightarrow$ flattens temperature and angular momentum across equatorial region

- In models and satellite data, see evidence of inertial adjustment having taken place (pancake structures in temperature field, stacked rolls and jets in velocity field)



- Zonal mean temperature for December, averaged over 16 year period (from NCEP)
- Notice temperature gradients flatten over equatorial region.

from Semeniuk and Shepherd, 200?

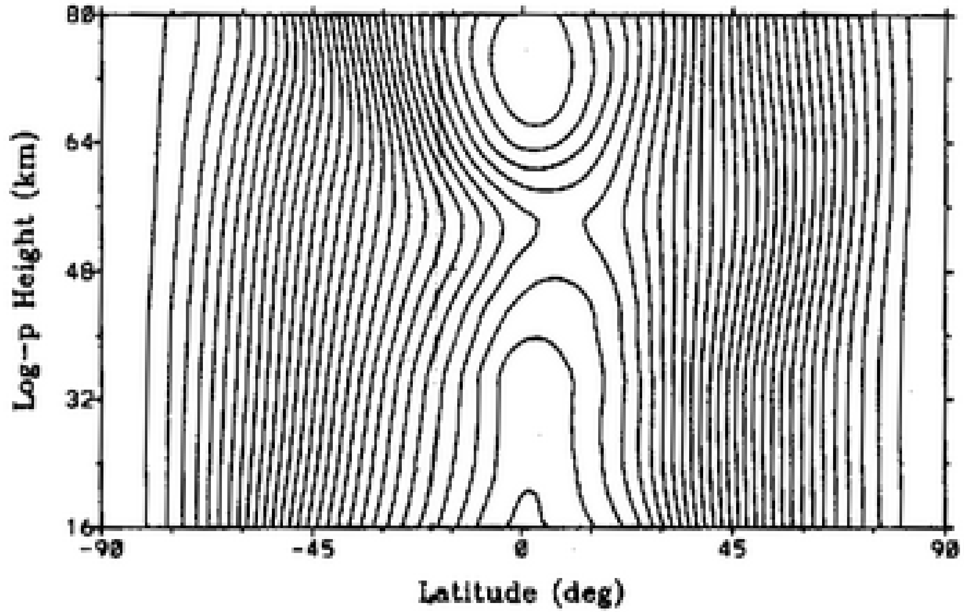


FIG. 19. January mean CMAM absolute angular momentum distribution (10^8 m^{2}s^{-1}).

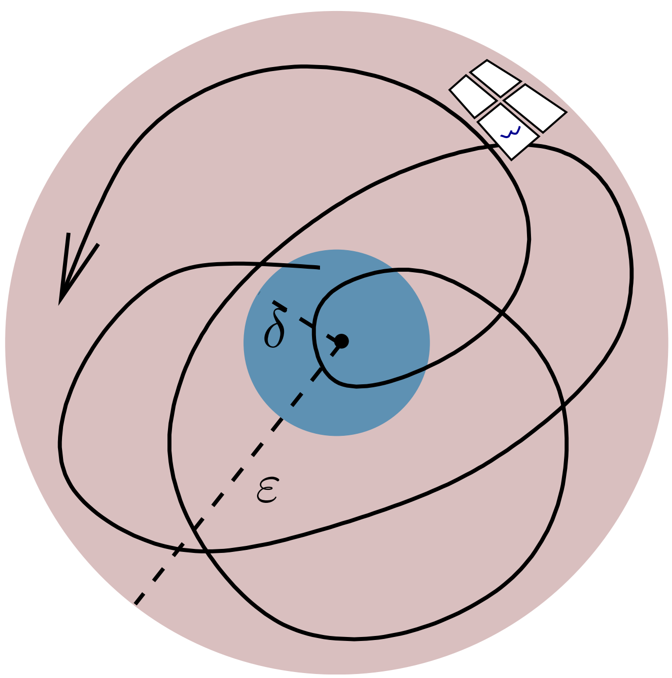
- Angular momentum gradient in winter hemisphere weakens due to cross equatorial flow
- Effect most pronounced at stratopause because of maximum ozone heating (and hence maximum gradient in T_{rad}) and low density.

2. ENERGY AND STABILITY

- State of the system represented by a point in phase space (e.g. position and momentum of a particle)
- Evolution of system in time corresponds to phase curve
- Phase space of fluid system is infinite dimensional
- Hamiltonian systems conserve (at least) Hamiltonian function/functional (energy)
- Steady solutions (equilibria) are called fixed points in phase space, and correspond to critical points of a conserved functional
- Stability of equilibrium related to geometry of the conserved functional near the fixed point

Stability Definition (Lyapunov)

Equilibrium $\mathbf{X}$ is stable with respect to the norm $||\mathbf{x} - \mathbf{X}||$ if for every ε, there is a δ such that if $||\mathbf{x}(t = 0) - \mathbf{X}|| < \delta$, then $||\mathbf{x}(t) - \mathbf{X}|| < \varepsilon$ for all times t.



- Black line is trajectory of system through phase space
- In finite dimensions, norm might be Euclidean (distance) norm

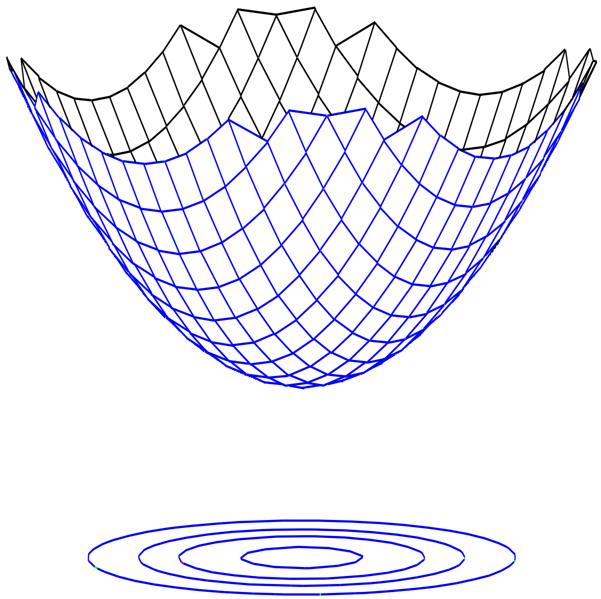
$$||\mathbf{x} - \mathbf{X}|| = \sqrt{\sum_i (x_i - X_i)^2}$$

(← these balls are at least 3-dimensional)

Finite Dimensional Systems

- In finite dimensions, there are only two geometries near fixed point:

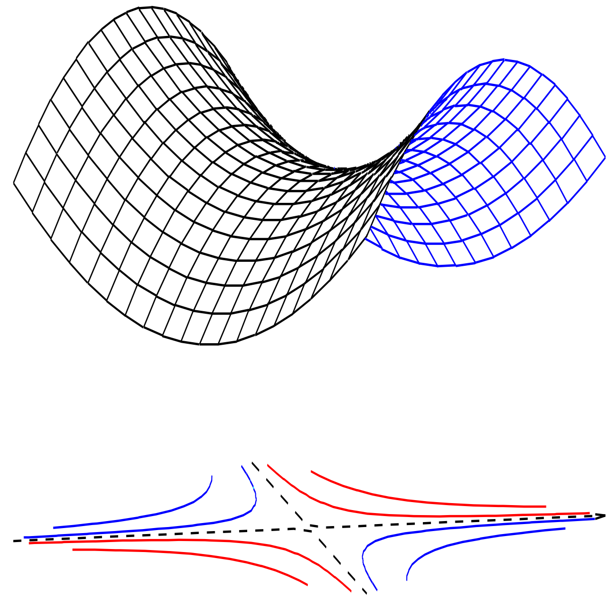
$E = (x - X)^2 + (y - Y)^2$



“bowl”

(X, Y) is stable

$E = -(x - X)^2 + (y - Y)^2$



“saddle”

(X, Y) is unstable

Infinite Dimensional Systems

- Infinite dimensional systems are more subtle
  (and impossible to visualize!)

- Stability depends on the particular norm being used

- Small amplitude stability results
  (i.e. stability of linearized equations - hence "linear stability")
  can sometimes be obtained using variational calculus

- Hamiltonian fluid systems described by Eulerian variables are noncanonical

  ⇒ fixed points are critical points of pseudoenergy: a combination of the Hamiltonian and a Casimir invariant

  - In symmetric stability problem, Casimirs depend on m and θ

3. LINEAR STABILITY CONDITIONS

- Linearize equations about steady solution
  $\mathbf{X} = (U(y,z), v = 0, w = 0, \rho = D(y,z), \theta = \Theta(y,z))$
  satisfying

$$
\begin{aligned}
-\beta y U - \frac{1}{D} P_y &= 0 \\
\gamma U - g - \frac{1}{D} P_z &= 0
\end{aligned}
$$

- Conserved functional for the linearized equations is of the form

$$
\begin{aligned}
\mathcal{A}_L(\mathbf{x};\mathbf{X}) &= \iint\limits_{\mathrm{SH}} (\mathbf{x}-\mathbf{X})^T \Lambda_{\mathrm{SH}}(\mathbf{X})(\mathbf{x}-\mathbf{X})\,\mathrm{d}y\,\mathrm{d}z \\
&\quad + \iint\limits_{\mathrm{NH}} (\mathbf{x}-\mathbf{X})^T \Lambda_{\mathrm{NH}}(\mathbf{X})(\mathbf{x}-\mathbf{X})\,\mathrm{d}y\,\mathrm{d}z,
\end{aligned}
$$

- $\mathbf{X}$ is stable if matrices $\Lambda_{\mathrm{SH}}(\mathbf{X})$ and $\Lambda_{\mathrm{NH}}(\mathbf{X})$ are positive definite

  $\Rightarrow$ $\mathcal{A}_L$ is "shaped" like a bowl near $\mathbf{X}$

- Define family of norms:

$$||\mathbf{x}-\mathbf{X}||_\lambda^2 = \iint D \left\{ \lambda \left[\left(\frac{\rho - D}{D_0} \right)^2 + \left(\frac{\theta - \Theta}{\Theta_0} \right)^2 + \left(\frac{u - U}{U_0} \right)^2 \right] + v^2 + w^2 \right\} \mathrm{d}y \, \mathrm{d}z$$

- Let λ_- and λ_+ be the minimum and maximum eigenvalues of the $\Lambda(\mathbf{X})$ matrices. Then for all time,

$$||\mathbf{x} - \mathbf{X}||_{\lambda_-}^2 \leq \mathcal{A}_L \leq ||\mathbf{x} - \mathbf{X}||_{\lambda_+}^2$$

- Stability follows from conservation of $\mathcal{A}_L$

$$||\mathbf{x}(t) - \mathbf{X}||_{\lambda_-}^2 \leq \mathcal{A}_L(t) = \mathcal{A}_L(0) \leq ||\mathbf{x}(0) - \mathbf{X}||_{\lambda_+}^2 \leq \frac{\lambda_+}{\lambda_-} ||\mathbf{x}(0) - \mathbf{X}||_{\lambda_-}^2$$

- Notation: $\partial(F,G) \equiv \dfrac{\partial F}{\partial y}\dfrac{\partial G}{\partial z} - \dfrac{\partial F}{\partial z}\dfrac{\partial G}{\partial y}$

- Sign of $\partial(F,G)$ is given by right hand rule applied to ∇F and ∇G:

$\partial(F,G) > 0$ if ∇F is "clockwise" of ∇G

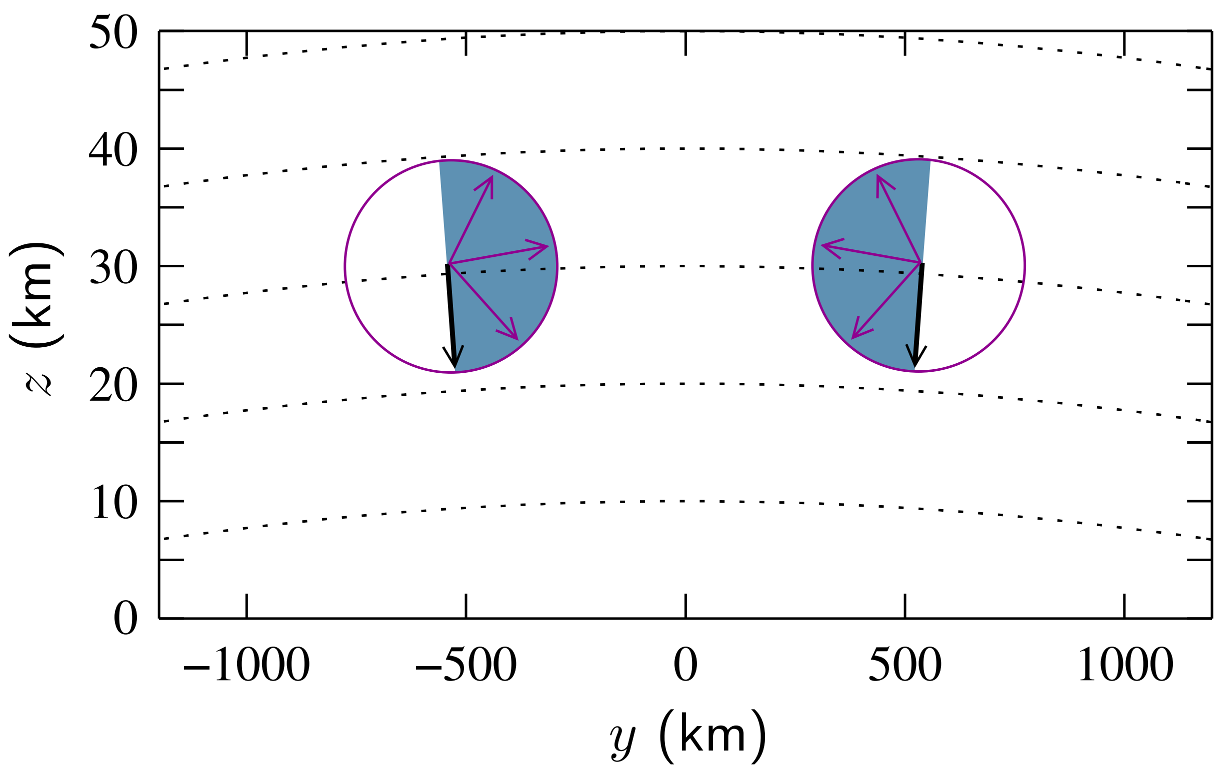
- Conditions for linear stability are

$$\frac{1}{Q}\partial(M,P) > 0 \qquad \text{(inertial stability)}$$

$$\frac{1}{Q}\partial(\Theta, M^{(p)}) > 0 \qquad \text{(static stability)}$$

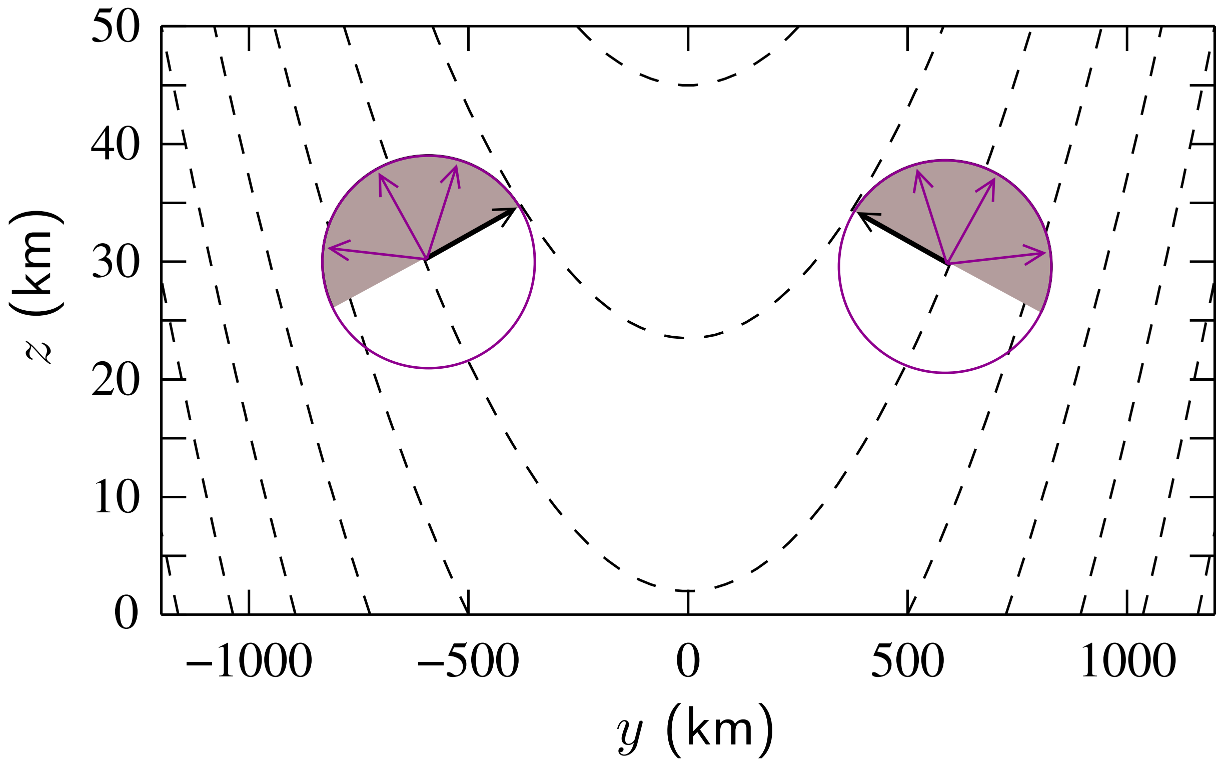
$$yDQ \equiv y\partial(\Theta, M) > 0 \qquad \text{(symmetric stability)}$$

"Inertial Stability"



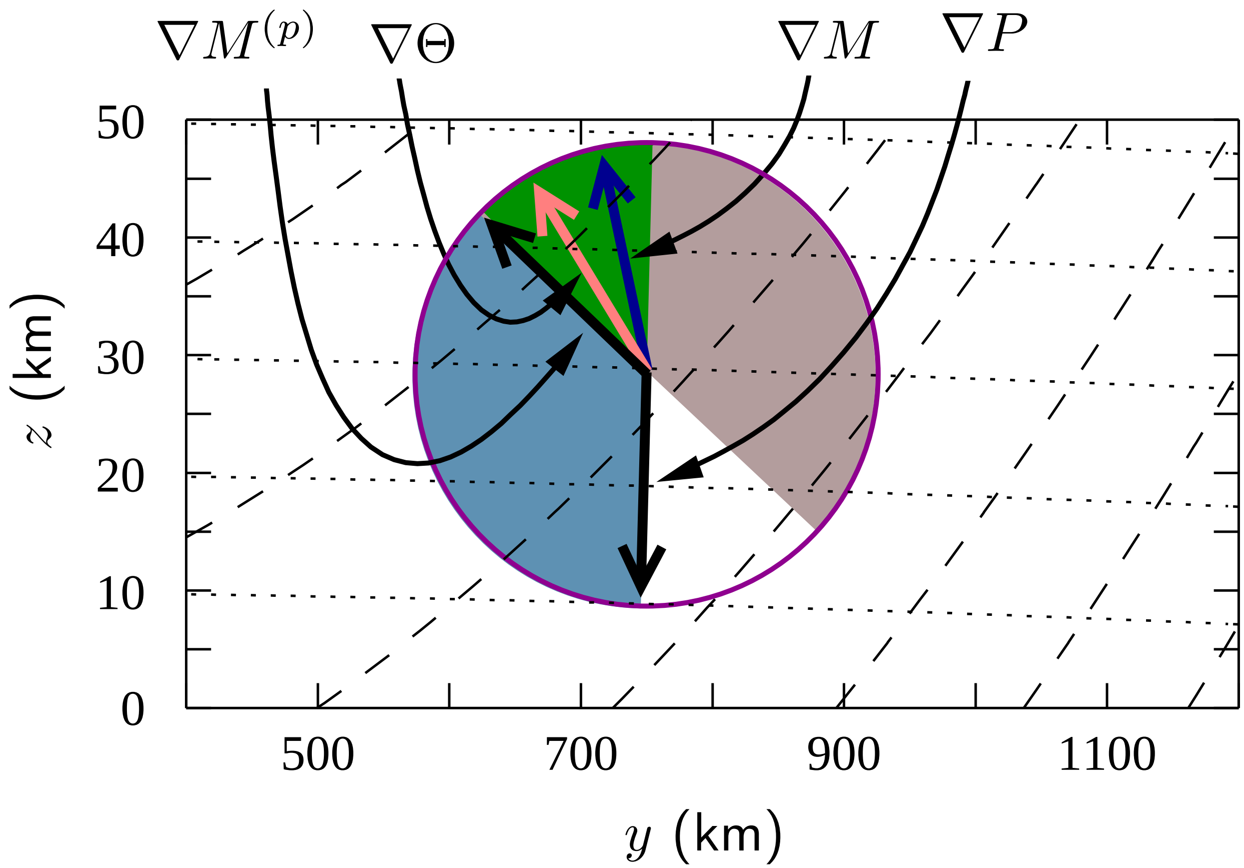
- Contours are curves of constant pressure.
- ∇M must be in coloured semicircle for static stability.
- Condition identical to hydrostatic condition $y\, M_y\, |_p < 0$.

"Static Stability"



- Contours are curves of constant $M^{(p)} \equiv -\frac{1}{2}\beta y^2 + \gamma z$, tangent to local rotation vector $\boldsymbol{\Omega} \equiv \gamma \hat{\mathbf{e}}_y + \beta y \hat{\mathbf{e}}_z$.
- $\nabla\Theta$ must be in coloured semicircle for static stability.

A symmetrically unstable case:



- But $\nabla\Theta$ must be clockwise of ∇M for stability

$\Rightarrow$ This state is unstable!.

4. ANELASTIC SYSTEM and NONLINEAR STABILITY

- Certain results from hydrostatic case can be achieved in nonhydrostatic case using anelastic equations
- Assumes that fastest time scale is that of gravity waves (filters sound wave modes) and that θ departs relatively little from prescribed reference profile $\theta_0(z)$ (c.f. Boussinesq system).
- Only 4 prognostic variables - (u, v, w) and θ (3 independent), instead of the 5 in Euler equations
- Can extend small amplitude result to finite amplitude for certain basic states
- Can solve linear equations exactly for Dunkerton problem in the case of $\theta_0(z) =$ constant

(talk to CONS**T**AN**TI**NE about Anelastic Equations)

- Conserved functional for the nonlinear anelastic equations is ($\mathcal{K}_\perp$ is kinetic energy in (u, v) components):

$$\begin{aligned} \mathcal{A}(\mathbf{x};\mathbf{X}) &= \mathcal{K}_\perp + \iint \rho_0 \left[(\tfrac{1}{2}\beta_\delta y^2 - \gamma_\alpha z)(m - M) + \tfrac{1}{\epsilon B}\pi_0(\theta - \Theta) \right] \mathrm{d}y\,\mathrm{d}z \\ &\quad + \iint_{q<0} \rho_0 \left[C^-(m,\theta) - C^-(M,\Theta) \right] \mathrm{d}y\,\mathrm{d}z \\ &\quad + \iint_{q>0} \rho_0 \left[C^+(m,\theta) - C^+(M,\Theta) \right] \mathrm{d}y\,\mathrm{d}z \end{aligned}$$

- Notice that the domains of the last two integrals change with time as the sign of potential vorticity q changes

- Define the norm

$$||\mathbf{x} - \mathbf{X}||_\lambda^2 = \mathcal{K}_\perp + \iint \left\{ \lambda \frac{\rho_0}{2} \left[(m - M)^2 + (\theta - \Theta)^2 \right] \right\} \mathrm{d}y\,\mathrm{d}z$$

- Steady states which are even functions of y
  (so that $C^- = C^+ \equiv C$) and satisfy the linear conditions
  (similar to the Euler equations case)
  are candidates for nonlinearly stability

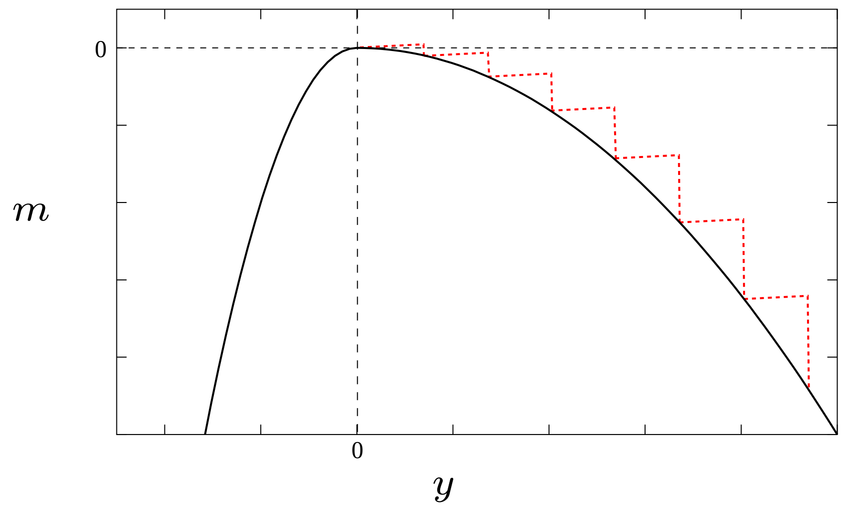
- Must test that $C(m, \theta)$ and $C^+(m, \theta)$ functions can be constructed such that $\mathcal{A}$ has a global minimum at $\mathbf{X}$

- Simplest example of stable state is:

$$
\begin{aligned}
M(y, z) &= M_0 - \tfrac{1}{2} b y^2 \\
\Theta(y, z) &= \Theta_0 + (\epsilon \gamma_\alpha)(\tfrac{1}{2} b y^2) + \epsilon \Gamma z
\end{aligned}
$$

- The required $C(m, \theta)$ is a quadratic function of m and θ

- Steady states which are not even functions of latitude cannot be Lyapunov stable (in our norm)
- Consider the perturbation below:
  - red curve is $m(y)$ and black curve is $M(y)$
  - red curve has $q < 0$ everywhere
  - as steps $\to 0$, dominant term in $\mathcal{A}$ is $C^-(m, \theta) - C^+(M, \Theta) < 0$



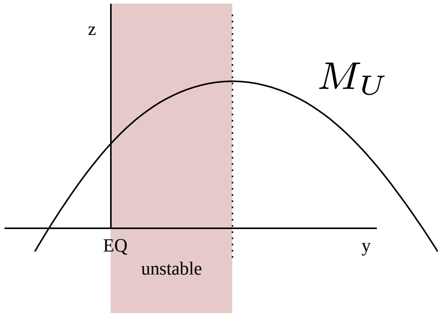
- But isn't that small amplitude? No, look at $\dfrac{\partial m}{\partial y} - \dfrac{\partial M}{\partial y}$ but it does satisfy $||\mathbf{x} - \mathbf{X}|| \to 0$

Anelastic Dunkerton Problem

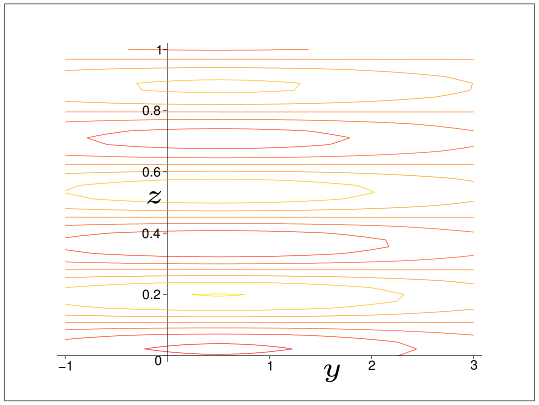
- Consider again the unstable equilibrium $\mathbf{X}_U(y,z)$:

$$M_U(y,z) = -\tfrac{1}{2}b_U y^2 + \lambda_U y$$

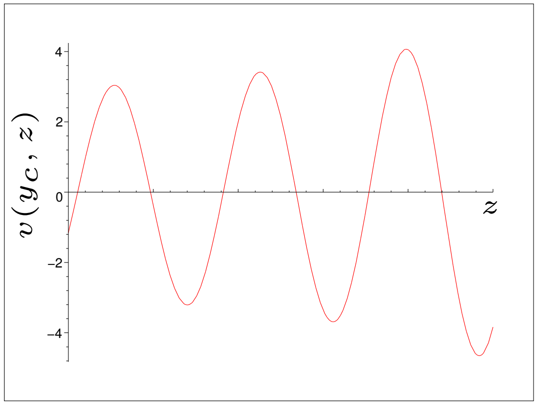
$$\Theta_U(y,z) = (\epsilon\gamma_\alpha)(\tfrac{1}{2}b_U y^2 - \lambda_U y) + \epsilon\Gamma_U z$$



- Can solve anelastic equations linearized about $\mathbf{X}_U(y,z)$:



Streamfunction



$v(y_c,z)$

- Taylor Vortices
- v increases with z depending on slope of $\rho_0(z)$

5. SATURATION BOUNDS

- Recall that when $\mathbf{X}$ is nonlinearly stable, $\mathcal{A}(\mathbf{x};\mathbf{X})$ is the sum of two positive terms: the kinetic energy term $\mathcal{K}_\perp(\mathbf{x})$ and what we might call the available potential energy term $\mathcal{APE}(\mathbf{x};\mathbf{X})$
- Since $\mathcal{A}$ is conserved, its initial value is a rigorous upper bound on $\mathcal{K}_\perp(\mathbf{x}(t))$
- Given $\mathbf{x}(0)$ close to an unstable equilibrium $\mathbf{X}_U$, seek the smallest $\mathcal{A}(\mathbf{X}_U;\mathbf{X})$ among all nonlinearly stable $\mathbf{X}$
- This is a measure of how large the instability can grow before it saturates. $\mathcal{A}$ is called a saturation bound.
- Can be used as part of a parameterization scheme for subgridscale adjustment in numerical models

- Again, consider Dunkerton state $\mathbf{X}_U$:

$$
\begin{aligned}
M_U(y,z) &= -\tfrac{1}{2}b_U y^2 + \lambda_U y \\
\Theta_U(y,z) &= (\epsilon\gamma_\alpha)(\tfrac{1}{2}b_U y^2 - \lambda_U y) + \epsilon\Gamma_U z
\end{aligned}
$$

- Minimize $\mathcal{A}(\mathbf{X}_U;\mathbf{X})$ over the class of nonlinearly stable states of the form

$$
\begin{aligned}
M(y,z) &= M_0 - \tfrac{1}{2}by^2 \\
\Theta(y,z) &= \Theta_0 + (\epsilon\gamma_\alpha)(\tfrac{1}{2}by^2) + \epsilon\Gamma z
\end{aligned}
$$

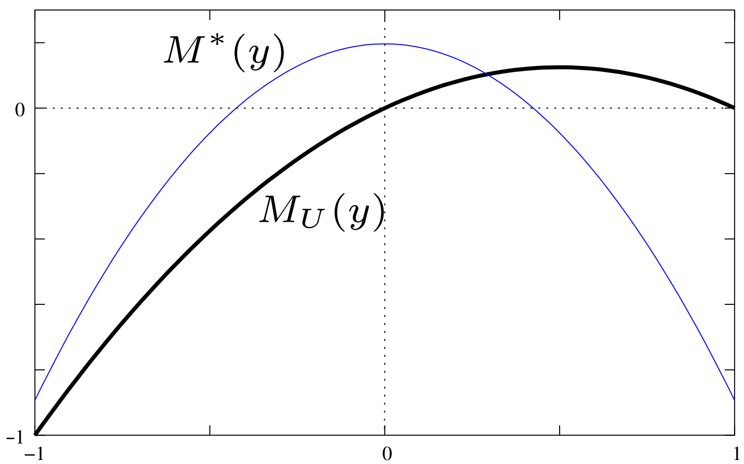
- i.e. find M_0, b, Θ_0, Γ which minimize $\mathcal{A}$

Example 1 - Inertial Instability

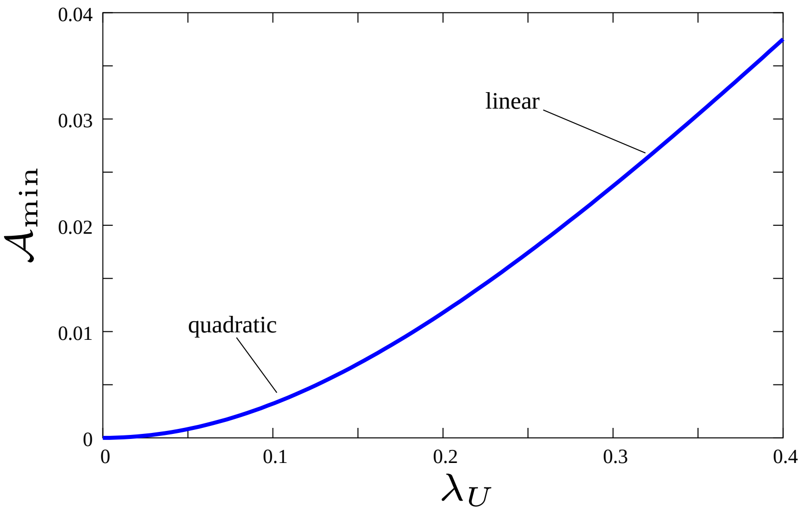
- Consider statically stable ($\Gamma_U > 0$), inertially unstable ($\lambda_U > 0$)
- minimizing $\mathbf{X}$ has:

$$\Gamma_{\min} = \Gamma_U\,, \qquad b_{\min} = |b_U|\sqrt{1 + 15\left(\frac{\lambda_U}{b_U}\right)^2}$$

$$(M_0)_{\min} = \frac{1}{6}(b_{\min} - b_U), \qquad (\Theta_0)_{\min} = \frac{1}{6}(\epsilon\gamma_\alpha)(b_{\min} - b_U)$$



Stable $M(y)$ (blue) which minimizes $\mathcal{A}$



Minimum $\mathcal{A}$ as function of λ_U

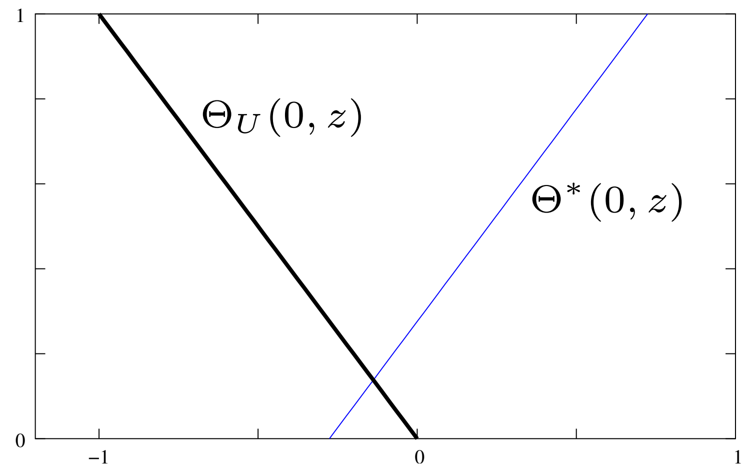
Notice $\mathcal{A}_{\min} \to 0$ as $\lambda_U \to 0$

Example 2 - Static Instability

- Consider statically unstable ($\Gamma_U < 0$), inertially stable ($\lambda_U = 0$)
- minimizing $\mathbf{X}$ has:

$$\Gamma_{\min} = -|\Gamma_U|, \quad b_{\min} = b_U$$

$$(M_0)_{\min} = 0, \quad (\Theta_0)_{\min} = -2\frac{I_1}{I_0}\epsilon|\Gamma_U|$$



Minimizing $\Theta(0, z)$ (blue) (z is vertical axis)

- Saturation bound is $\mathcal{A}_{\min} = \frac{4}{I_0}(I_0 I_2 - I_1^2)|\Gamma_U|$

SUMMARY

- Symmetric instability plays a role in solstice season dynamics in equatorial middle atmosphere
- Nonhydrostatic Coriolis terms are significant near equator and classical symmetric stability results (e.g. Dunkerton, 1981) can be generalized to incorporate them
- Linear stability of a steady solution to Euler equations depends on directions of ∇M and $\nabla \Theta$ relative to each other, ∇P and $\mathbf{\Omega}$
- Can find finite amplitude stability result and exact linear solution to Dunkerton problem using anelastic equations
- Finite amplitude stability result can be used to find saturation bounds on energy conversion during inertial/convective adjustment