Symmetric stability in the equatorial middle atmosphere

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I.

INERTIAL INSTABILITY

Definition of inertial instability

- A fluid in equilibrium is said to be inertially unstable when disturbances are amplified by an imbalance between the pressure gradient force and the centrifugal force due to the fluid's rotating.
- Axisymmetric systems conserve angular momentum $m \equiv rv$, where r is the perpendicular distance from the axis of rotation, and v is the component of velocity tangent to the corresponding circle of radius r.
- Rayleigh criterion for inertial stability of an axisymmetric fluid is that the magnitude of the angular momentum increases with distance from the axis of rotation.

Inertial instability vs. convection

- Angular momentum plays the rôle in axisymmetric inertial instability that potential temperature plays in adiabatic convection.
- Recall that potential temperature, defined by

$$\theta = \left(\frac{p_{00}}{p}\right)^{\frac{R}{c_p}} T$$

  is the temperature a fluid parcel would have if its pressure were changed adiabatically to p_{00}.
- Since θ is conserved by fluid parcels, a parcel lifted to a new height will be warmer (more buoyant) than the surrounding fluid if its potential temperature is higher than the ambient potential temperature at its new height. It would then keep rising and we conclude that the initial configuration was therefore unstable.

- The condition for static stability is thus $\frac{\partial \theta}{\partial z} > 0$
- Similarly, the angular momentum is a measure of the speed a ring of fluid would have if displaced symmetrically to a radius of unity.
- The centrifugal force on the fluid in the ring is $F_c = \frac{v^2}{r}$.

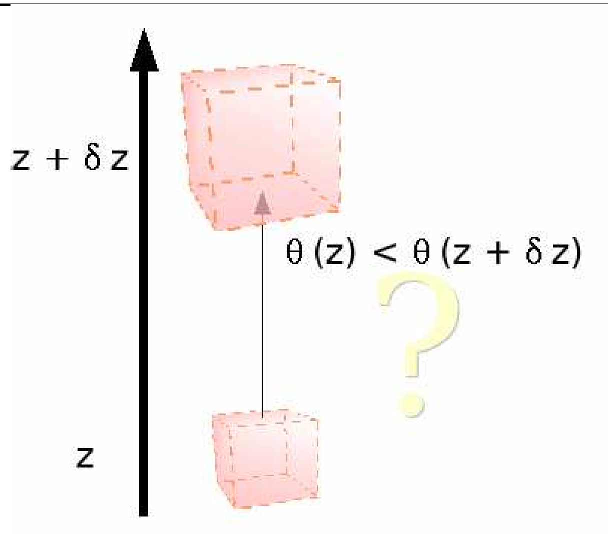
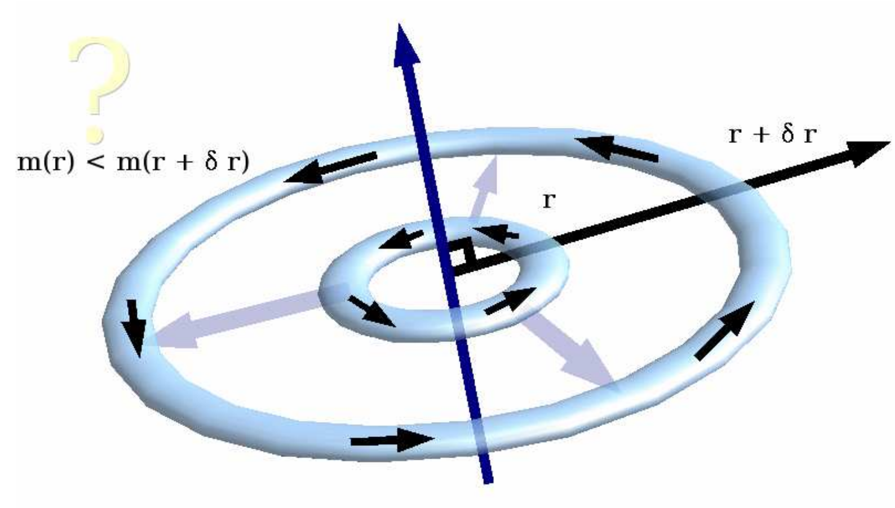
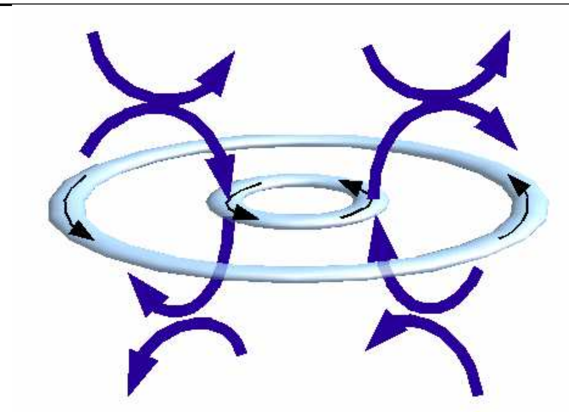
$\Rightarrow$ A ring of fluid displaced outward will have greater absolute velocity than the ambient fluid at its new position, and hence be accelerated outwards, if the magnitude of its angular momentum is greater than that of the ambient fluid.

$\Rightarrow$ Hence the Rayleigh condition for inertial stability $\frac{\partial (m^2)}{\partial r} > 0$

- Similar phenomena observed in laboratory experiments of Rayleigh-Bénard (convection) and Taylor-Couette (flow between rotating cylinders).

Inertial adjustment

- An inertially unstable steady state subject to a small symmetric perturbation will develop Taylor vortex rolls superposed on the tangential flow, thus mixing angular momentum.
- Analogously, a statically unstable steady state subject to a vertical perturbation will lead to cells of rising and descending fluid, mixing potential temperature.
- Evolution of system from unstable basic state towards stable equilibrium called adjustment.
- If forcing which created the basic state is removed (for example by nonlinear interaction between secondary circulation and basic state), adjustment leads to smoothing of offending angular momentum (potential temperature) gradient and removal of instability.

CONVECTIVE STABILITY	INERTIAL STABILITY
	
	

II.

GEOPHYSICAL CONTEXT OF INERTIAL INSTABILITY

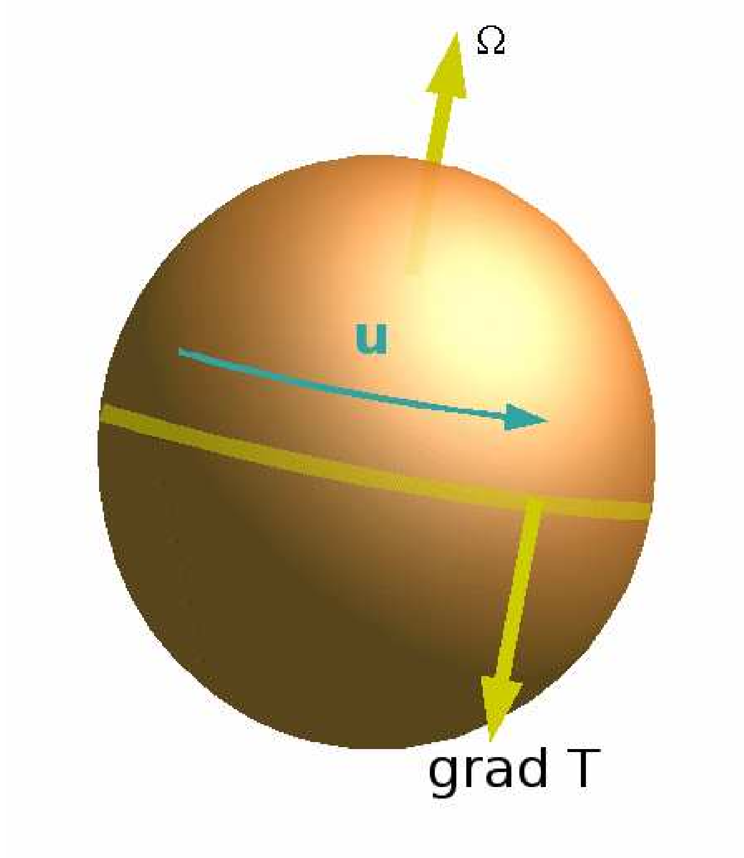
- There is approximately hydrostatic balance in the vertical in the Earth's atmosphere. Motion is predominantly horizontal (at constant distance from the Earth's centre).
- But the Rayleigh criterion refers to distances from the axis of rotation.
- Horizontal motion towards the equator implies motion away from the axis of rotation, and motion away from the equator implies motion towards the axis of rotation.
- Rayleigh criterion in atmosphere becomes: $\phi \frac{\partial m}{\partial \phi}$, where ϕ is latitude and

$$m = r\cos\phi(\Omega r\cos\phi + u)$$

- Notice that the planetary angular momentum is symmetric about the equator, so any latitudinal wind shear $\frac{\partial u}{\partial \phi}$ at equator is inertially unstable.

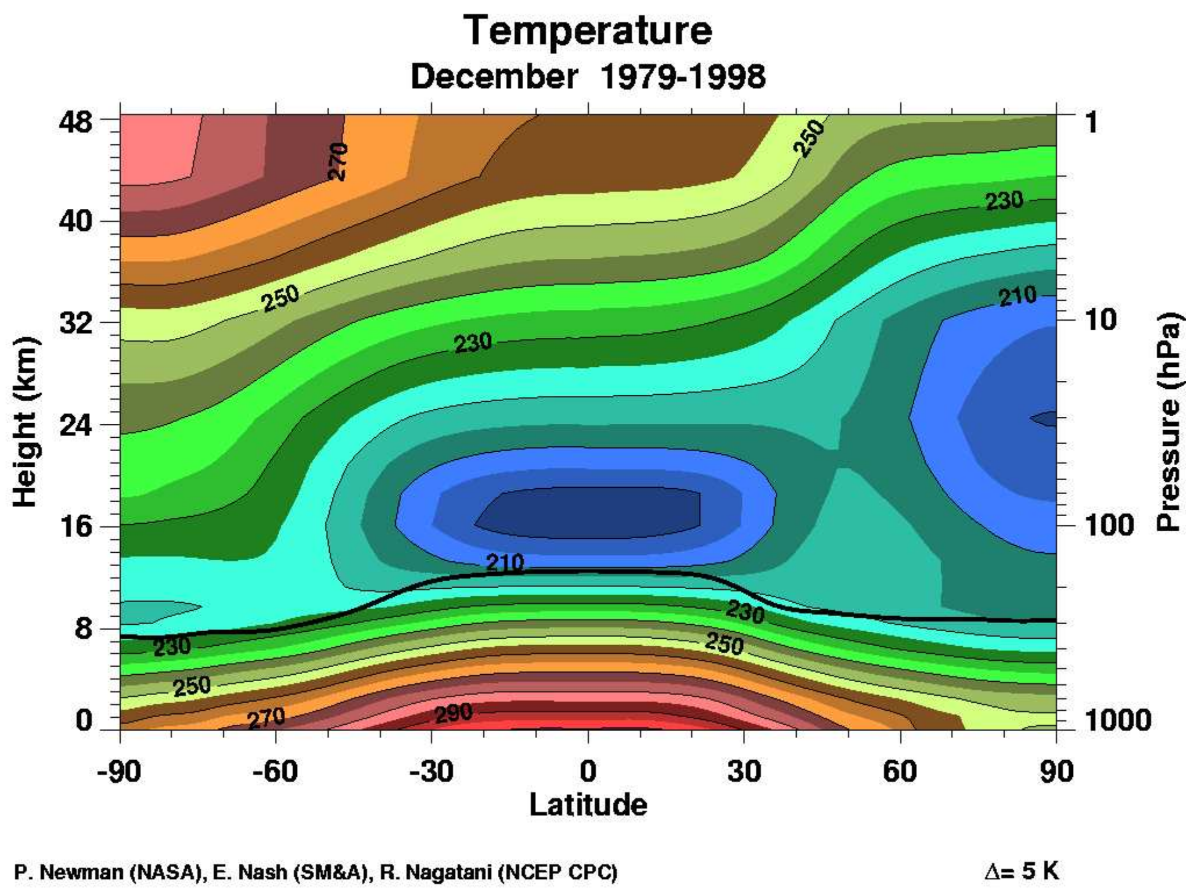
Rôle of inertial instability in solstice dynamics

- The estimated radiative equilibrium temperature distribution in the middle atmosphere (that is the temperature distribution that would obtain due to solar heating, radiatively active chemistry and outgoing radiation - in the absence of dynamics) is symmetric about the equator during the equinox seasons.
- But during the solstices, T_{rad} is decidedly warmer in the summer hemisphere. In particular, the maximum value occurs away from the equator and there is a latitudinal gradient of temperature across the equator.
- The corresponding pressure gradient cannot be balanced by Coriolis forces because it is parallel to the rotation axis.
- Therefore, we don't expect to observe the radiative equilibrium temperature profile at the equator during the solstices.



- Recall $\mathbf{F}_{\text{Coriolis}} = -2\boldsymbol{\Omega} \times \mathbf{u}$, which is necessarily orthogonal to $\boldsymbol{\Omega}$ and so cannot possibly balance the temperature (pressure) gradient.

- Away from the equator, the T_{rad} gradient is approximately in geostrophic balance with the zonal wind.
- On the summer side of the equator, the geostrophically balanced winds would be strong enough so that the maximum angular momentum would be off the equator, violating the Rayleigh criterion.
- This condition is therefore not observed. It is believed that continuous inertial adjustment smooths the temperature around the equator, and a Hadley cell develops preventing the temperature from relaxing towards radiative equilibrium. (Cause and effect are a bit confusing, but this is what is observed in models)
- The Hadley circulation pushes air from summer to winter, smoothing the angular momentum gradient in the winter equatorial region.



- Zonal mean temperature for December, averaged over 16 year period (from NCEP)
- Notice temperature gradients flatten over equatorial region.

(from Semeniuk and Shepherd, 2001)

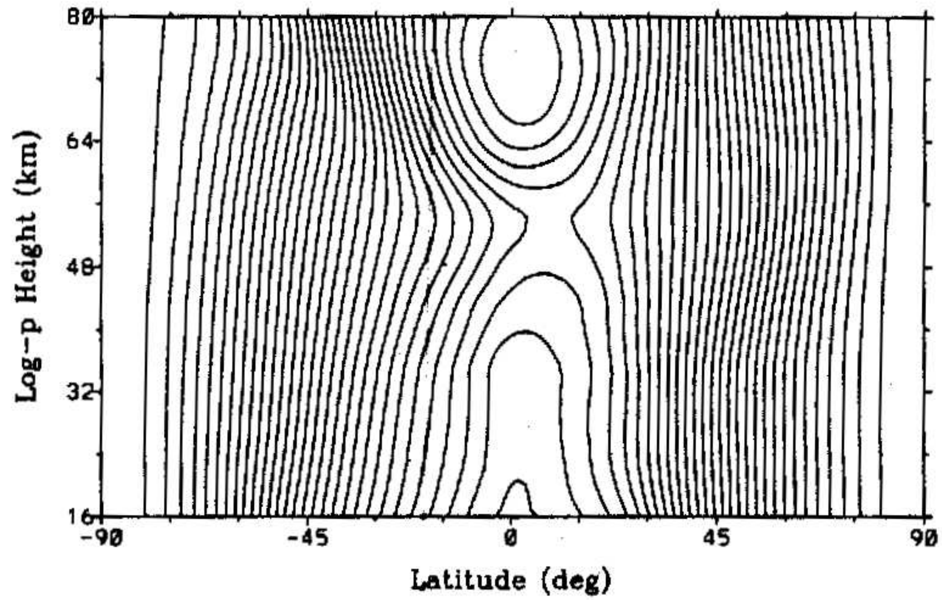


FIG. 19. January mean CMAM absolute angular momentum distribution (10^8 m^2s^{-1}).

- Angular momentum gradient in winter hemisphere weakens due to cross equatorial flow
- Effect most pronounced at stratopause because of maximum ozone heating (and hence maximum gradient in $T_{\rm rad}$) and low density.

Inertial adjustment and the SAO

- The zonal wind at the equatorial stratopause ($\approx 50\text{km}$) undergoes a direction change twice each year (Semi-Annual Oscillation).
- During equinox, the winds are westerly (positive), believed to be driven by upward propagating Kelvin waves.
- During solstices, the winds are easterly (negative). The cross equatorial flow from summer to winter is the cause:
  - The steady state angular momentum is flat across the equator, and there is a persistent meridional flow, and
  - air moving towards the equator while conserving angular momentum must lose relative velocity. The result is an easterly flow at the equator.

III.

DIRECT OBSERVATION

The Dunkerton solution

- A qualitative picture of inertial adjustment comes from solution of a very simplified linear system.
- Dunkerton (1981) solved hydrostatic equations on an equatorial β-plane linearized about a basic state with linear velocity shear across the equator: $M(y) = u_0 + \lambda y - \frac{1}{2}\beta y^2$
- Basic state is unstable in latitude range $0 < y < \lambda/\beta$
- Growing modes of the solution exist for vertical wavenumber above a minimum value depending on the shear in the basic state; growth rate higher for larger shear and smaller vertical scale
- Most unstable mode has largest meridional scale, exhibits vertically stacked Taylor vortices, with zonal jets of alternating sign stacked over equator and temperature anomalies over equator and on the poleward side of the unstable region. (see overhead)

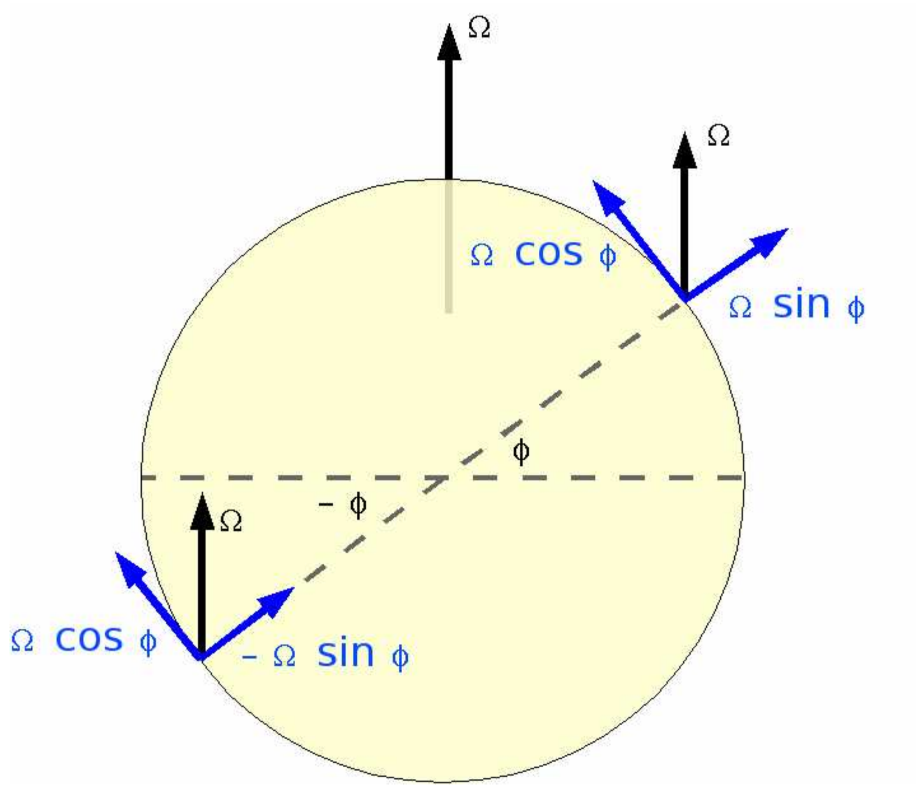
Hayashi et al. 1998

- Hayashi et al. (1998) analysed temperature data from CLAES instrument on UARS satellite and isolated events fitting the Dunkerton theoretical picture of inertial instability.
- Found striking examples of long lived stationary “pancake structures” in the temperature field around the equatorial stratopause during both solstices.
- Events correlated with anomalous potential vorticity (due to angular momentum gradient) on winter side of equator, and with oppositely signed pancake structures at higher winter latitude, consistent with Dunkerton picture.
- Localized in longitude (not axisymmetric)

IV.

CONDITIONS FOR SYMMETRIC STABILITY: EQUATORIAL β-PLANE ANELASTIC SYSTEM

Traditional hydrostatic approximation



- Traditional hydrostatic approximation assumes hydrostatic balance in the vertical direction and neglects the $\cos\phi$ Coriolis force terms due to northward component of rotation vector.

Enhanced equatorial β-plane

- Near equator, can approximate rotation vector by its second order Taylor expansion about $\phi = 0$

$$\boldsymbol{\Omega} = \tfrac{1}{2}(\gamma\,\hat{\mathbf{e}}_y + \beta y\,\hat{\mathbf{e}}_z),$$

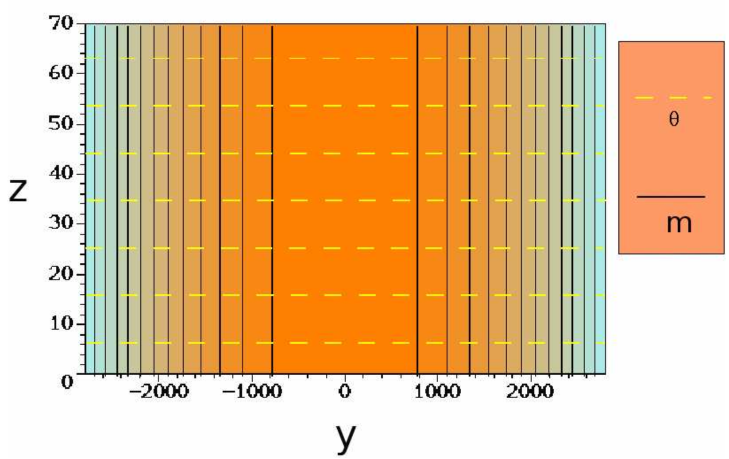
  where $\gamma = 2\Omega$ and $\beta = 2\Omega/a$
  (a being the mean radius of the earth).

- Latitude is replaced by arclength away from equator $y = a\phi$, and y and z are treated as cartesian coordinates.

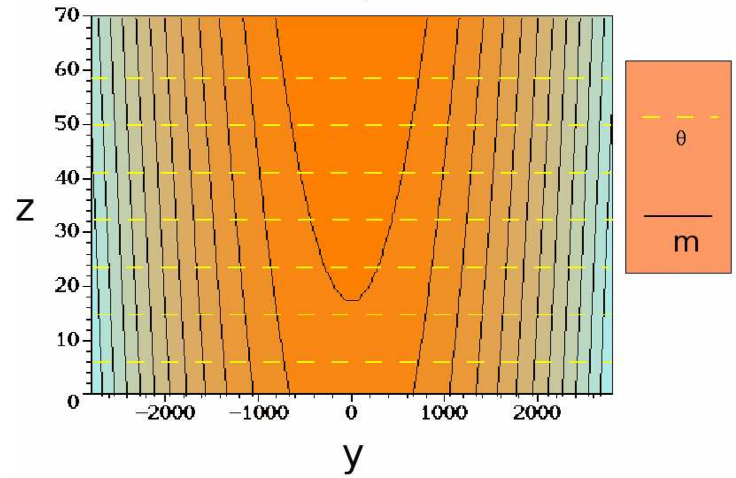
- The inclusion of the γz term has an effect on contours of planetary angular momentum ...

Contours of planetary angular momentum

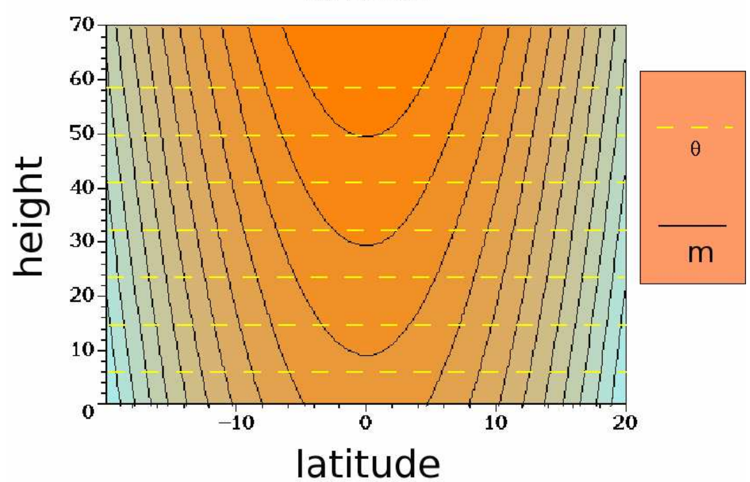
TRADITIONAL β-PLANE



ENHANCED β-PLANE



SPHERE



Anelastic equations

- Anelastic system derives name because the energy that is conserved by the equations omits the elastic energy term which involves pressure perturbations.
- Anelastic equations do not admit sound wave solutions but allow for nonhydrostatic motion; used to model deep convection.
- Based on assumptions that potential temperature varies by a small fraction of its mean value over the domain and that time scale of motions is at least N^{-1} (time scale of gravity waves).
- The middle atmosphere does not strictly satisfy the first assumption because of the strong stratification. We use the anelastic model anyway for a technical reason.
- Continuity equation is $\nabla \cdot (\rho_0 \mathbf{u})$ (quasi-incompressible) (c.f. Boussinesq equations)

Symmetric equations

- If $\dfrac{\partial}{\partial x} \equiv 0$, the resulting equations materially conserve m and θ.
- Symmetric equations have a noncanonical Hamiltonian structure.
- Conserve an energy functional (the Hamiltonian, $\mathcal{H}$) and Casimir invariants $\mathcal{C}$ which depend on m, θ and potential vorticity

$$q = \frac{1}{\rho_0}\left(\frac{\partial\theta}{\partial y}\frac{\partial m}{\partial z} - \frac{\partial\theta}{\partial z}\frac{\partial m}{\partial y}\right),$$

  related to particle relabelling symmetry.
- Can use conserved functionals to calculate stability criteria for an equilibrium.
- Recall that functionals depend on entire functions; they are functions of an infinite number of "independent variables".

Stability of a steady zonal flow

- The state of the system can be specified by the vector $\mathbf{x}(t) = (m, \zeta, \theta)$ (the three entries being angular momentum, the x component of vorticity ζ, and potential temperature)
- Define an equilibrium zonal flow by $\mathbf{X} = (M, 0, \Theta)$ where $M(y,z)$ and $\Theta(y,z)$ are in thermal wind balance.
- We derive conditions for the stability of $\mathbf{X}$ by finding conditions on the Casimir $\mathcal{C}(m,\theta)$ such that $(\mathcal{H}+\mathcal{C})(\mathbf{x})$ is stationary when evaluated at $\mathbf{X}$ (meaning that its functional derivative vanishes),
- and such that the pseudoenergy $\mathcal{A}(\mathbf{x};\mathbf{X}) \equiv (\mathcal{H}+\mathcal{C})(\mathbf{x}) - (\mathcal{H}+\mathcal{C})(\mathbf{X})$ is positive definite and bounded for all $\mathbf{x}$ (so that $\mathbf{X}$ is a minimum of $\mathcal{A}$).

Stability conditions

- (To cut a long story short,) the conditions for stability are

$$\frac{1}{Q}\left[\beta y\frac{\partial\Theta}{\partial z}+\frac{1}{\alpha}\gamma\frac{\partial\Theta}{\partial y}\right]>0 \qquad \text{static stability}$$

$$-\frac{1}{Q}\frac{\partial M}{\partial y}>0 \qquad \text{inertial stability}$$

$$\frac{\beta y}{Q}>0,$$

which correspond to, respectively, static stability, inertial stability and the symmetric stability condition that the potential vorticity Q have the same sign as y.

Remarks

- Conditions agree with our intuition about a stable atmosphere:
  - Q positive (negative) in the northern (southern) hemisphere
  - $|M|$ decreases away from the equator
  - Θ increases with height

- Only explicit contribution by γ to stability conditions is that stability is aided if temperature increases away from the equator.

  (?!) ... centrifugally stable to have more mass (colder air) at equator (widest circle)?

- Of course, γz contributes to $M(y, z)$ and hence $Q(y, z)$ so perhaps there is something hidden.

V.

SUMMARY

Summary

- An axisymmetric rotating fluid can be subject to inertial instability which is analogous to convection, with angular momentum m playing the rôle of potential temperature θ.
- An inertially unstable equilibrium is unlikely to be observed. Rather, evidence of inertial adjustment is observed, and interpretations of middle atmospheric phenomena like the Semi-Annual Oscillation based on its importance agree with what is observed.
- A fuller representation of the Coriolis force vector changes the distribution of planetary angular momentum and hence should have an effect on inertial instability and adjustment.
- A calculation of symmetric stability conditions using an anelastic equations model does not reveal much of an effect; only suggesting that it is more stable to have denser fluid closer to the equator.