A Hamiltonian Representation of the Symmetric Equatorial β-plane Equations

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Outline

- Hamilton's equations in canonical form
- Noncanonical coordinates
  - Example:
    Free rotations of a rigid body
- Symmetric β-plane equations
- Stability

Hamilton's Equations

(for conservative systems)

- State of system specified by:
  - generalized coordinates q_i
  - generalized momenta p_i
- Dynamics governed by Hamilton's Equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q}$$

where $H(q_i, p_i)$ is a conserved function called the *Hamiltonian*

- Example: Plane pendulum

$$q = \theta, \quad p = ml\frac{d\theta}{dt}$$

Hamiltonian is just the total energy:

$$H(q,p) = \frac{p^2}{2m} - mgl\cos(q)$$

- Equations can be written in *symplectic form*:

$$\frac{d\mathbf{x}}{dt} = J\,\nabla H$$

where

$$\mathbf{x} = [q_1, q_2, ..., q_N, p_1, p_2, ..., p_N]^T$$

is a point in the $2N$-dimensional *phase space*, and

$$J = \begin{bmatrix} 0_{N\times N} & I_{N\times N} \\ -I_{N\times N} & 0_{N\times N} \end{bmatrix}$$

$\Rightarrow$ Fixed points (points at which $\frac{d\mathbf{x}}{dt} = 0$) are *critical points* of the Hamiltonian.

- The stability of a fixed point is related to the geometry of H in the neighbourhood of the fixed point.
  - bowl shaped $\iff$ stable
  - saddle shaped $\iff$ unstable

Noether's Theorem

- A *symmetry* of the system is (roughly) a transformation of the coordinates that takes solutions to solutions.
- Noether's theorem says that every distinct symmetry corresponds to a distinct conserved quantity.
- Example: if solutions are invariant under translations in the x direction, the system conserves the component of momentum in the x direction.
- Example: planetary motion conserves the component of angular momentum normal to the plane of motion. This corresponds to invariance under translations in the angle swept out by the orbit of the planet.

Noncanonical Representations

- Sometimes, a reduced set of Hamilton's equations can be found that govern the behaviour of a subset of the generalized coordinates independently of the others.

- The equations can still be written in the symplectic form

$$\frac{d\mathbf{x}}{dt} = J\,\nabla H$$

but now $\mathbf{x}$ has fewer components than before, and J is different from the canonical version.

- In particular, J is not invertible.
$\Rightarrow$ fixed points no longer correspond to critical points of the Hamiltonian.

- There exists a new type of conserved function called a $Casimir$, which does not correspond to an explicit symmetry of the system (as in Noether's theorem).

- What are Casimirs, then?
  - correspond to symmetries associated with coordinates that have been reduced from the system
  - gradients of Casimirs are vectors (sorry Simal) in the nullspace of J

    (i.e. they satisfy $J\,\nabla C = \mathbf{0}$)
  - If $C(\mathbf{x})$ is a Casimir, evolution of system is confined to surfaces of constant C ("symplectic leaves")

- Also interesting:
  - gradient of H evaluated at a fixed point is equal to the gradient of a Casimir $-C$.
  - A fixed point is therefore a critical point of the combined invariant $(H + C)(\mathbf{x})$.
  - etc.

EXAMPLE:
Free Rotations of a Rigid Body

- Canonical version:
  - 6 generalized coordinates representing the orientation (3 *Euler angles*) and the angular momentum about each of the "principal axes" (3 component vector)
  - invariance under rotations about principal axes leads to conservation of total angular momentum
- Noncanonical version:
  - time evolution of angular momenta calculated independently of Euler angles via a noncanonical set of equations
  - time evolution of Euler angles calculated separately using solutions for angular momenta
  - total angular momentum becomes a Casimir

- For the record:

$$\frac{d}{dt}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{bmatrix} \begin{bmatrix} \partial H/\partial L_1 \\ \partial H/\partial L_2 \\ \partial H/\partial L_3 \end{bmatrix}$$

where the Hamiltonian is

$$H(L_1, L_2, L_3) = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$$

(an ellipsoid in (L_1, L_2, L_3) space)

and the Casimir is

$$C(L_1, L_2, L_3) = L_1^2 + L_2^2 + L_3^2$$

(a sphere in (L_1, L_2, L_3) space)

... so system is "completely integrable" - phase curves are the intersections of the sphere with the ellipsoid

Hamiltonian Fluid Systems

- governed by *partial* differential equations, so the phase space is "infinite-dimensional"

- properties of Hamiltonian systems generalize with a few modifications:

  - independent variables become functions of space

  - Hamiltonian and Casimirs (if any) become conserved "functionals" of the independent variables

  - gradients become "functional derivatives"

  - J becomes a linear operator (involving partial derivatives, for example)

- Fluid systems written in terms of Eulerian variables (the usual way) are necessarily noncanonical because of the *particle relabelling* symmetry.

Symmetric, Hydrostatic, Adiabatic, β-plane ... Equations

- approximation for low latitude axisymmetric dynamics in a spherical shell

- neglects complications of curvature of earth, but retains variation of the Coriolis parameter with latitude

For the record:

$$
\begin{aligned}
u_t &= -vu_y - \omega u_p + \beta y v \\
v_t &= -vv_y - \omega v_p - \beta y u + \Phi_y \\
\theta_t &= -v\theta_y - \omega\theta_p \\
0 &= v_y + \omega_p,
\end{aligned}
$$

where pressure coordinates (x, y, p) have been used, (u, v, ω) is the velocity, and θ is potential temperature.

Hamiltonian Form of Equations

By changing variables from $(u, v, \omega, \theta, \Phi)$ to (ζ, m, θ), where

$$\zeta = \frac{\partial v}{\partial p}, \quad m = u - \tfrac{1}{2}\beta y^2$$

the equations can be written in the Hamiltonian form

$$\frac{\partial}{\partial t}\begin{bmatrix} \zeta \\ m \\ \theta \end{bmatrix} = \begin{bmatrix} \partial(\zeta, \cdot) & \partial(m, \cdot) & \partial(\theta, \cdot) \\ \partial(m, \cdot) & 0 & 0 \\ \partial(\theta, \cdot) & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\mathcal{H}/\delta\zeta \\ \delta\mathcal{H}/\delta m \\ \delta\mathcal{H}/\delta\theta \end{bmatrix}$$

with Hamiltonian

$$\mathcal{H}(\zeta, m, \theta) = \int\int \Big\{ \tfrac{1}{2}v^2 + \tfrac{1}{2}\beta y^2 m + \mathcal{E}(\rho, \theta) + p/\rho^2 \Big\}\, dy dp$$

and Casimirs

$$\mathcal{C}(m, \theta) = \iint C(m, \theta) dy dp$$

Stability of Equilibria

- Fixed points of the equations satisfy the thermal wind balance

$$\beta y \left(\frac{\partial m}{\partial p}\right)_y = \frac{1}{\rho\theta}\left(\frac{\partial \theta}{\partial y}\right)_p$$

- to determine stability criteria
  - determine Casimirs such that

$$\frac{\delta \mathcal{H}}{\delta \mathbf{x}} = -\frac{\delta \mathcal{C}}{\delta \mathbf{x}}$$

    at equilibrium

  - further restrict $\mathcal{C}$ such that $\mathcal{H} + \mathcal{C}$ is "bowl-shaped" at equilibrium
  - check for finite amplitude stability

We find the stability conditions

- static stability

$$\left(\frac{\partial\theta}{\partial p}\right)_y < 0$$

(i.e. parcels displaced upwards must be colder than surroundings so that they fall back, etc.)

- inertial stability

$$y\left(\frac{\partial m}{\partial y}\right)_p < 0$$

(i.e. angular momentum must be maximum at the equator and decrease towards the poles)

Summary

- Hamilton's equations provide an elegant geometric way of looking at the stability of fixed points in conservative systems

- some systems can be written in a reduced, but "non-canonical" Hamiltonian form (in particular, conservative fluid systems)

- stability analysis can be adapted to noncanonical systems with the use of Casimir invariants

- Example: static and inertial stability of equilibria in symmetric, equatorial β-plane system.