OUTLINE:

The Hydrostatic Approximation

- Euler Equations in Spherical Coordinates
- The Approximation and the Equations
- Critique of Hydrostatic Approximation

Inertial Instability

- The Phenomenon
- The Taylor-Couette Problem
- Equatorial Inertial Instability

The Euler Equations

- Conservation of momentum by an inviscid fluid in an inertial reference frame:

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla p}{\rho} + \mathbf{g}$$

  In a frame rotating with the Earth:

$$\frac{D\mathbf{v}}{Dt} + 2\mathbf{\Omega} \times \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}$$

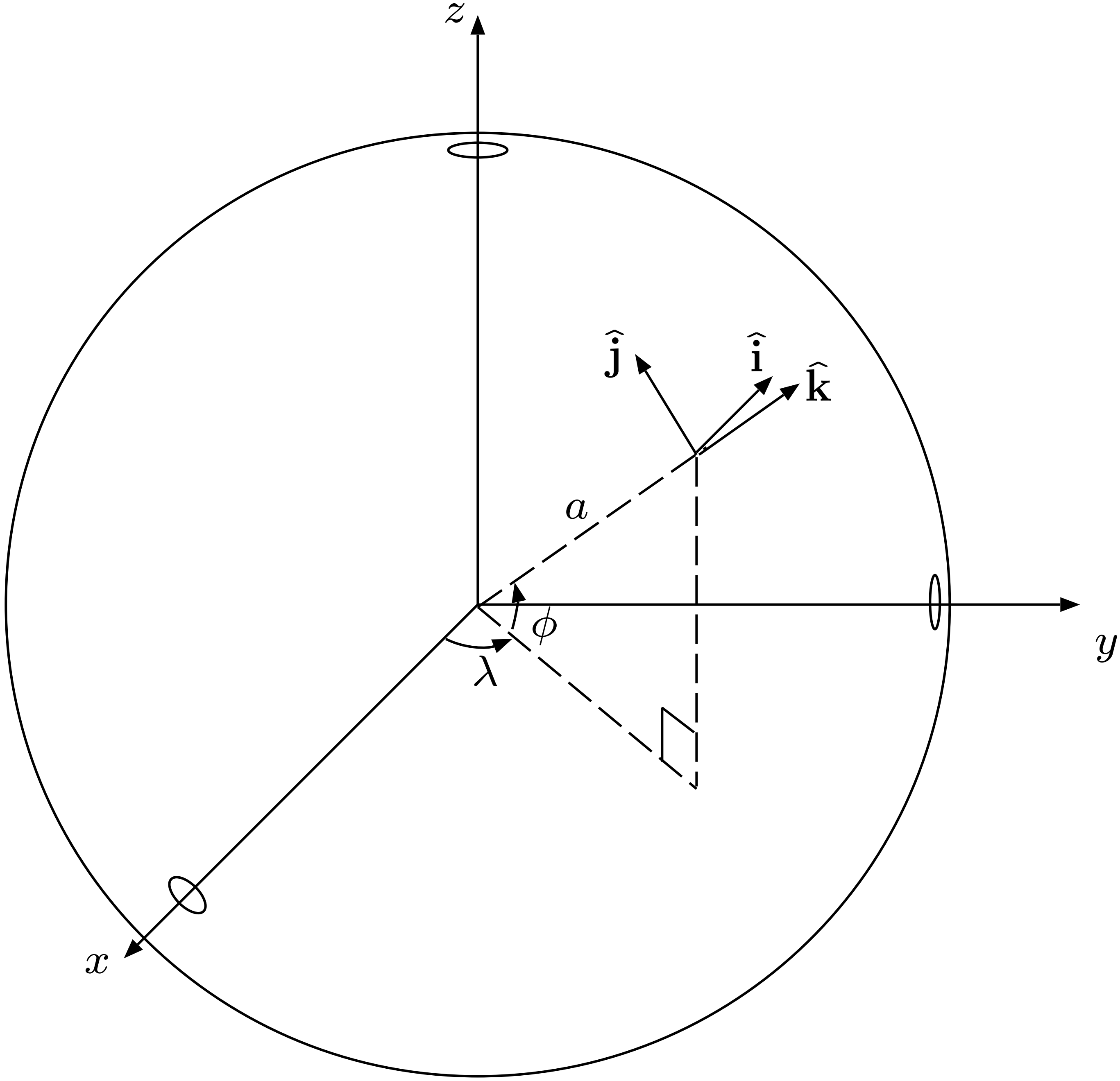
  (the centrifugal acceleration term has been neglected)

- Introduce spherical polar coordinates (λ, ϕ, r), with corresponding unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$, and write the velocity as:

$$\mathbf{v} = u\widehat{\mathbf{i}} + v\widehat{\mathbf{j}} + w\widehat{\mathbf{k}}$$

- Then the material derivative of the velocity can be expanded:

$$\frac{D\mathbf{v}}{Dt} = \widehat{\mathbf{i}}\frac{Du}{Dt} + \widehat{\mathbf{j}}\frac{Dv}{Dt} + \widehat{\mathbf{k}}\frac{Dw}{Dt} + u\frac{D\widehat{\mathbf{i}}}{Dt} + v\frac{D\widehat{\mathbf{j}}}{Dt} + w\frac{D\widehat{\mathbf{k}}}{Dt}$$



z
ĵ
î
k̂
a
φ
y
λ
x

- Derivatives of the unit vectors can be computed by writing them in terms of a fixed coordinate system with unit vectors $\hat{\mathbf{x}}$,$\hat{\mathbf{y}}$,$\hat{\mathbf{z}}$, so that:

$$\begin{aligned}
\hat{\mathbf{i}} &= -\hat{\mathbf{x}}\sin\lambda + \hat{\mathbf{y}}\cos\lambda \\
\hat{\mathbf{j}} &= -\hat{\mathbf{x}}\cos\lambda\sin\phi - \hat{\mathbf{y}}\sin\lambda\sin\phi + \hat{\mathbf{z}}\cos\phi \\
\hat{\mathbf{k}} &= \hat{\mathbf{x}}\cos\lambda\cos\phi + \hat{\mathbf{y}}\sin\lambda\cos\phi + \hat{\mathbf{z}}\sin\phi
\end{aligned}$$

we can show:

$$\begin{aligned}
\frac{D\hat{\mathbf{i}}}{Dt} &= \hat{\mathbf{j}}\frac{u}{r}\tan\phi - \hat{\mathbf{k}}\frac{u}{r} \\
\frac{D\hat{\mathbf{j}}}{Dt} &= -\hat{\mathbf{i}}\frac{u}{r}\tan\phi - \hat{\mathbf{k}}\frac{v}{r} \\
\frac{D\hat{\mathbf{k}}}{Dt} &= \hat{\mathbf{i}}\frac{u}{r} + \hat{\mathbf{j}}\frac{v}{r}.
\end{aligned}$$

- Coriolis term is expanded using $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$:

$$\begin{aligned}
2\mathbf{\Omega}\times\mathbf{v} = 2\Omega\,\Big(&\hat{\mathbf{i}}\,(-v\sin\phi + w\cos\phi) \\
&+\hat{\mathbf{j}}\,u\sin\phi + \hat{\mathbf{k}}\,u\cos\phi\Big)
\end{aligned}$$

- The Euler equations obey the *energy principle*:

$$\frac{D}{Dt}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}+gr+I\right)=-\frac{1}{\rho}\nabla\cdot(p\mathbf{v})+Q$$

- where I am the internal energy, and I obey

$$\frac{DI}{Dt}=-\frac{p}{\rho^2}\frac{D\rho}{Dt}+Q$$

- and the *zonal angular momentum principle*

$$\frac{D}{Dt}\left(\Omega r^2\cos^2\phi+ru\cos\phi\right)=-\frac{1}{\rho}\frac{\partial p}{\partial\lambda}$$

- It is desirable that any approximation to the Euler equations retain similar principles.

The Hydrostatic Approximation

- The observation that the atmosphere is approximately hydrostatic in the vertical allows us to simplify the set of equations considerably.

- The vertical momentum equation is replaced by the *hydrostatic approximation*:

$$\frac{\partial p}{\partial r} = -\rho g$$

- The kinetic energy and material derivatives are redefined to only include horizontal velocities.

- If we were to compute

$$\frac{D}{Dt}\left(\frac{u^2+v^2}{2}+gr+I\right)$$

using the "exact" horizontal momentum equations, we would no longer get the work done by the pressure gradient force.

- To restore the energy principle, we are forced to remove the terms involving w from all of the momentum equations.

- but that upsets the angular momentum principle

- To recover, we are forced to replace the distance from the centre of the earth r with a constant mean radius a wherever it occurs (the *shallow atmosphere approximation*)

- The hydrostatic equations are thus:

$$
\begin{aligned}
\frac{Du}{Dt} &= \frac{uv}{a}\tan\phi + 2\Omega v\sin\phi - \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} \\
\frac{Dv}{Dt} &= -\frac{u^2}{a}\tan\phi - 2\Omega u\sin\phi - \frac{1}{\rho a}\frac{\partial p}{\partial\phi} \\
\frac{\partial p}{\partial z} &= -\rho g \\
\frac{D\rho}{Dt} &= -\rho\nabla\cdot\mathbf{v}
\end{aligned}
$$

- They are the basis for most numerical atmospheric models

- A slightly more detailed set of equations obtains if the shallow atmosphere approximation is made directly in the Euler equations. The angular momentum principle then demands that the w terms be dropped from the horizontal momentum equations and that the vertical acceleration be dropped. (Phillips, 1966)

Critique of Hydrostatic Approximation

- To determine whether or not the hydrostatic approximation is justified, we need to consider the importance of the terms left out compared to the smallest terms left in.

Consider the following typical midlatitude scales (Holton, 1992):

- Horizontal Velocity $\sim U \sim 10$ ms^{-1}

- Vertical Velocity $\sim W \sim 10^{-2}$ ms^{-1}

- Length $\sim L \sim 10^6$ m

- Height $\sim H \sim 10^4$ m

- "Time" $\sim L/U \sim 10^5$ s

- A conservative upper bound on the vertical velocity is provided by the continuity equation (White and Bromley, 1995). In the absence of sound waves, $\nabla \cdot \mathbf{v} \approx 0$ ("quasi-nondivergence"), so that

$$W \leq \frac{UH}{L}$$

- It seems that the most significant of the terms missing from the hydrostatic equations are those proportional to $\cos\phi$.

- For example, in the zonal momentum equation, we have excluded $2w\Omega\cos\phi$. Comparing this term to the zonal acceleration term Du/Dt, using the quasi-nondivergence condition:

$$\frac{2W\Omega\cos\phi}{|Du/Dt|} \sim \frac{2\Omega H\cos\phi}{U}$$

- so we can ignore the $\cos\phi$ term to the extent that

$$\frac{2\Omega H \cos\phi}{U} \ll 1$$

- Similarly, if we compare the omitted $2\Omega u \cos\phi$ term in the vertical momentum equation with the non-hydrostatic part of the pressure gradient force (which is what gives rise to vertical acceleration):

$$\frac{2\Omega u \cos\phi}{\left|\frac{1}{\rho}\frac{\partial(\delta p)}{\partial z}\right|} \sim \frac{2\Omega H \cos\phi}{U}$$

- i.e. the same criterion exists for omitting the $\cos\phi$ terms as in the zonal equation. Using the earlier values,

- Using the typical midlatitude scales,

$$\frac{2\Omega H \cos\phi}{U} \sim 0.14 \cos\phi$$

- Therefore, if the quasi-nondivergence bound is approached, the excluded Coriolis term can be as large as 10% of the horizontal acceleration.

- This is surely too large a term to leave out.

- Most sources (the tabulated typical midlatitude values, for example), estimate that

$$W \leq 0.1 \frac{UH}{L}$$

but they almost always assume, among other things, a statically stable atmosphere.

- For example, the criterion for the neglect of the $\cos\phi$ terms is sometimes expressed in terms of the static stability as $N^2 \gg \Omega^2$, where

$$N^2 \equiv \frac{g}{c_p \Theta} \frac{\partial \Theta}{\partial r}$$

- For a stable atmosphere with a lapse rate of $\partial T / \partial r \approx 6$ K/km,

$$N^2 \approx 10^{-4} \text{ s}^{-2} \text{ and } \Omega^2 \approx 5 \times 10^{-9} \text{ s}^{-2}$$

so the condition is satisfied (Phillips, 1973).

White and Bromley (1995) refer to an "experimental" test of the hydrostatic approximation suggested by Dr. G.J. Shutts.

- Consider an air parcel at rest on the surface of the Earth near the equator.

- If the parcel is lifted to the tropopause (about 15 km), conservation of absolute zonal angular momentum dictates that it gain a zonal velocity of $\Delta U = -2\Omega H \cos\phi$. Then

  $$\frac{|\Delta U|}{U} = \frac{2\Omega H \cos\phi}{U}$$

  (that same expression!)

- At the equator, $\Delta U \approx 2$ m/s, so "the expression" is about 10%-20%.

- Another approach (de Verdière and Schopp, 1994), is to non-dimensionalize the vorticity equation, using different zonal and meridional length scales.

- The neglect of the $\cos\phi$ terms then appears to depend on the horizontal anisotropy of the flow (which is significant at low latitudes).

- They decided that the $\cos\phi$ terms can be neglected if

$$\frac{L_\phi^2}{Ha} \gg 1$$

where L_ϕ is a characteristic meridional length scale of the flow.

- i.e. the flow must evolve over meridional scales much larger than $(Ha)^{1/2} \approx 240$ km

- Having decided that the hydrostatic system is not good enough, White and Bromley (1995) present an alternative system, the *quasihydrostatic system*

- Removing the shallow atmosphere approximation and restoring the $\cos\phi$ Coriolis terms and all of the *metric* terms, the momentum equations are

$$
\begin{aligned}
\frac{Du}{Dt} &= \frac{uv}{r}\tan\phi - \frac{uw}{r} + 2\Omega v\sin\phi - 2\Omega w\cos\phi - \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} \\
\frac{Dv}{Dt} &= -\frac{u^2}{r}\tan\phi - \frac{vw}{r} - 2\Omega u\sin\phi - \frac{1}{\rho r}\frac{\partial p}{\partial\phi} \\
0 &= \frac{u^2}{r} + \frac{v^2}{r} + 2\Omega u\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial r} - g
\end{aligned}
$$

- The only difference from the Euler equations is the omission of the vertical acceleration (and the vertical velocity does not contribute to the kinetic energy).

Inertial Instability

- Consider a circular flow. The horizontal momentum equations (non-rotating coordinate system) are

$$\frac{Du}{Dt} = \frac{v^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$\frac{Dv}{Dt} = -\frac{uv}{r} - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

- If the flow is axisymmetric (all θ derivatives vanish), then

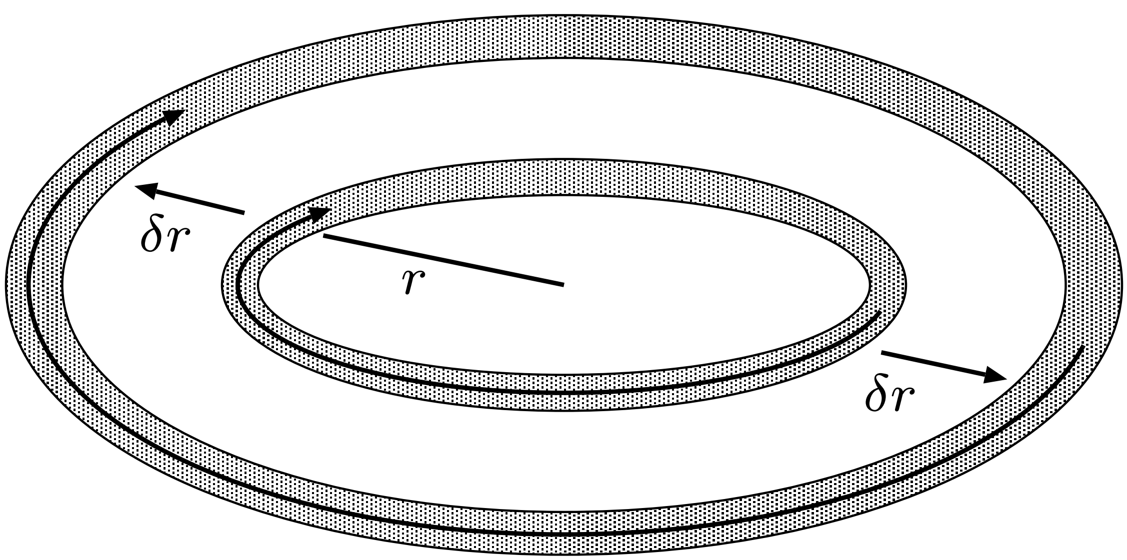
$$\frac{Dm}{Dt} \equiv \frac{D(rv)}{Dt} = 0$$

- If flow is balanced (steady state) then

$$\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

- Define a *fluid parcel* as a thin circular ring of fluid (with a single radial velocity v and a single angular momentum m)

- To determine conditions for stability of the circular flow, consider the response of a parcel to an axisymmetric displacement from r to $r + \delta r$



- The acceleration of the displaced parcel will be

$$\begin{aligned}\frac{Du}{Dt} &= \frac{v^2}{r} - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial r} \\ &= \frac{m^2}{r^3} - \frac{\bar{m}^2}{r^3} \\ &= \frac{1}{r^3}(m^2 - \bar{m}^2)\end{aligned}$$

- for an infinitessimal displacement

$$\left.\frac{Du}{Dt}\right|_{r+\delta r} = -\frac{1}{r^3}\frac{\partial(\bar{m}^2)}{\partial r}\,\delta r$$

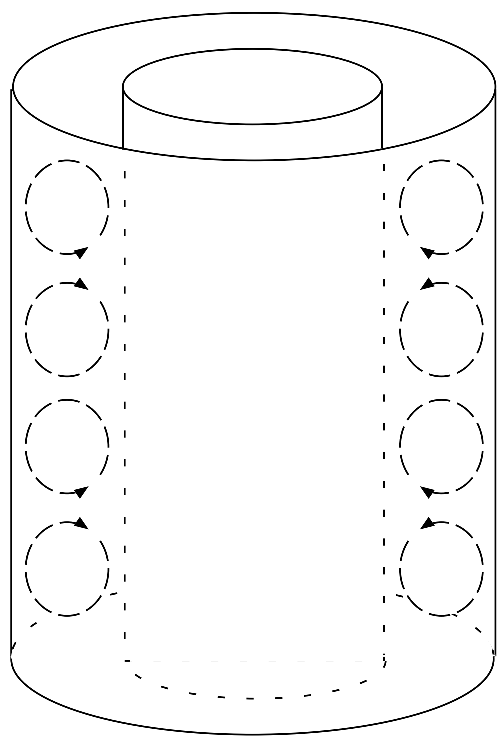
- The flow is stable if the force on a displaced parcel acts in the opposite direction to the displacement.

- Therefore, an axisymmetric circular flow is only stable to arbitrary perturbations if the angular momentum increases with distance from the axis.

- a famous example of inertial instability occurs in Taylor-Couette flow between rotating cylinders

- for high enough rotation rates of the inner cylinder, a negative angular momentum gradient forms

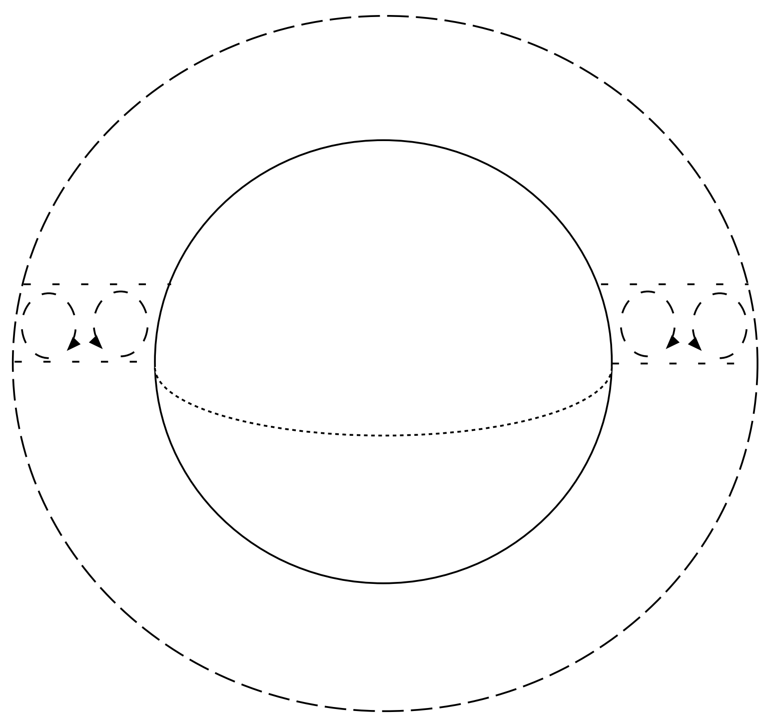
- the flow adjusts by forming vertical rolls



- Inertial instability can also develop in zonal flows in the ocean and middle atmosphere

- The radius of the flow rings decreases with latitude. For stability, the angular momentum must decrease with latitude $\rightarrow$ maximum angular momentum must fall at the equator

- An unstable flow can give rise to meridional-vertical rolls



Current and Future Work

- determine stability criteria for symmetric zonal flows in the quasi-hydrostatic system (c.f. Bowman and Shepherd, 1996)

- observe and investigate inertial instability in middle atmosphere models (QH system?)

- equatorial Kelvin waves in QH system

The End