PROGRAMME SCHEDULE

Tuesday, December 7$^{\rm th}$, 2004

12:50-13:40	 **Lunch** 
13:40-13:58	Mark Fruman, U of T The effects of nonhydrostatic terms on conditions for equatorial symmetric stability
13:58-14:16	Kirill Semeniuk, York University Modelling the October-November 2003 solar proton event in CMAM

The effects of nonhydrostatic terms on conditions for equatorial symmetric stability

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12^{th} Annual GCC Workshop, Toronto

Tuesday, December 7^{th}, 2004

OUTLINE

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1. Introduction to SYMMETRIC STABILITY

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2. Importance of “NONHYDROSTATIC” CORIOLIS TERMS

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2. Importance of “NONHYDROSTATIC” CORIOLIS TERMS
3. Reconsidered CONDITIONS FOR STABILITY

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4. EXAMPLES

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2. Importance of “NONHYDROSTATIC” CORIOLIS TERMS
3. Reconsidered CONDITIONS FOR STABILITY
4. EXAMPLES
5. Further with ANELASTIC MODEL

1. SYMMETRIC STABILITY

- Refers to the stability of an equilibrium which is symmetric in one direction under disturbances which have the same symmetry.
- In this case, we consider stability of zonally symmetric solutions to adiabatic Euler Equations in atmosphere to zonally symmetric disturbances
- System is 2 dimensional: (ϕ, r), with 2 material invariants:
  - Absolute angular momentum $m \equiv \Omega r^2 \cos^2 \phi + ur \cos \phi$ (because of zonal symmetry)
  - Potential temperature θ (or entropy) (because flow is adiabatic)
- and 2 forces acting on air parcels: gravity and the Coriolis force

- Most geophysical applications use the Primitive Equations, in which the Coriolis force is strictly horizontal ($\perp$ gravity).
  $\Rightarrow$ In that case, m is to displacement in latitude as
  θ is to displacement in height.

- The conditions for symmetric stability are:

  * θ increases with height (static stability)

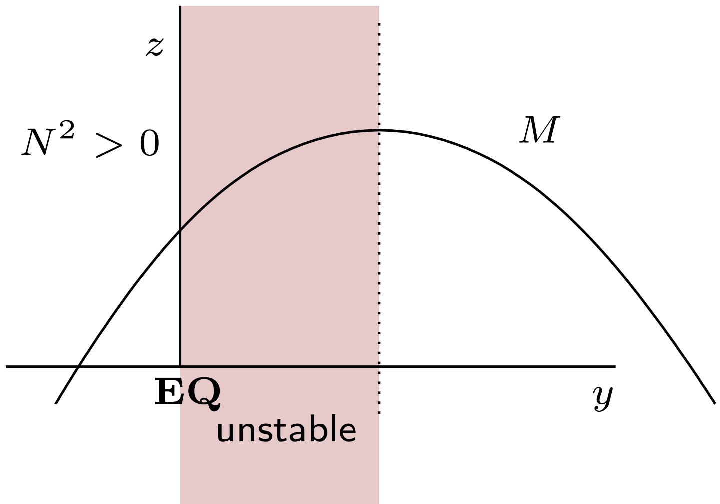
  * m increases towards the equator at constant altitude (Rayleigh centrifugal stability theorem)

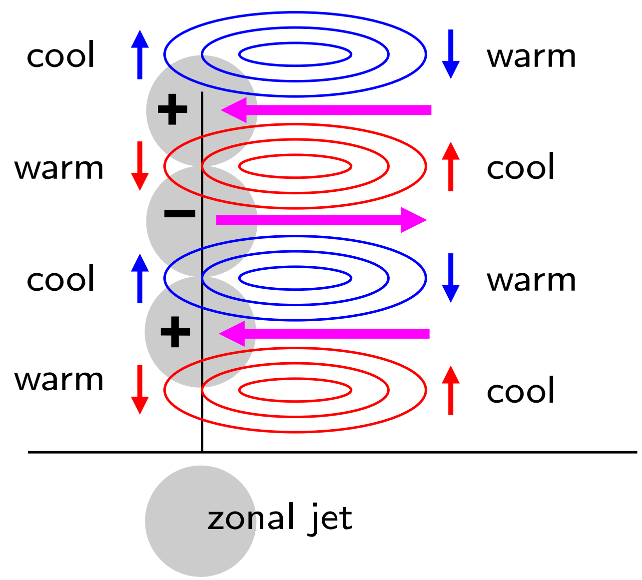
  * Potential vorticity has the same sign as latitude

- If Coriolis force is not orthogonal to gravity, conditions do not decouple so neatly, but potential vorticity condition generalizes.

Dunkerton problem

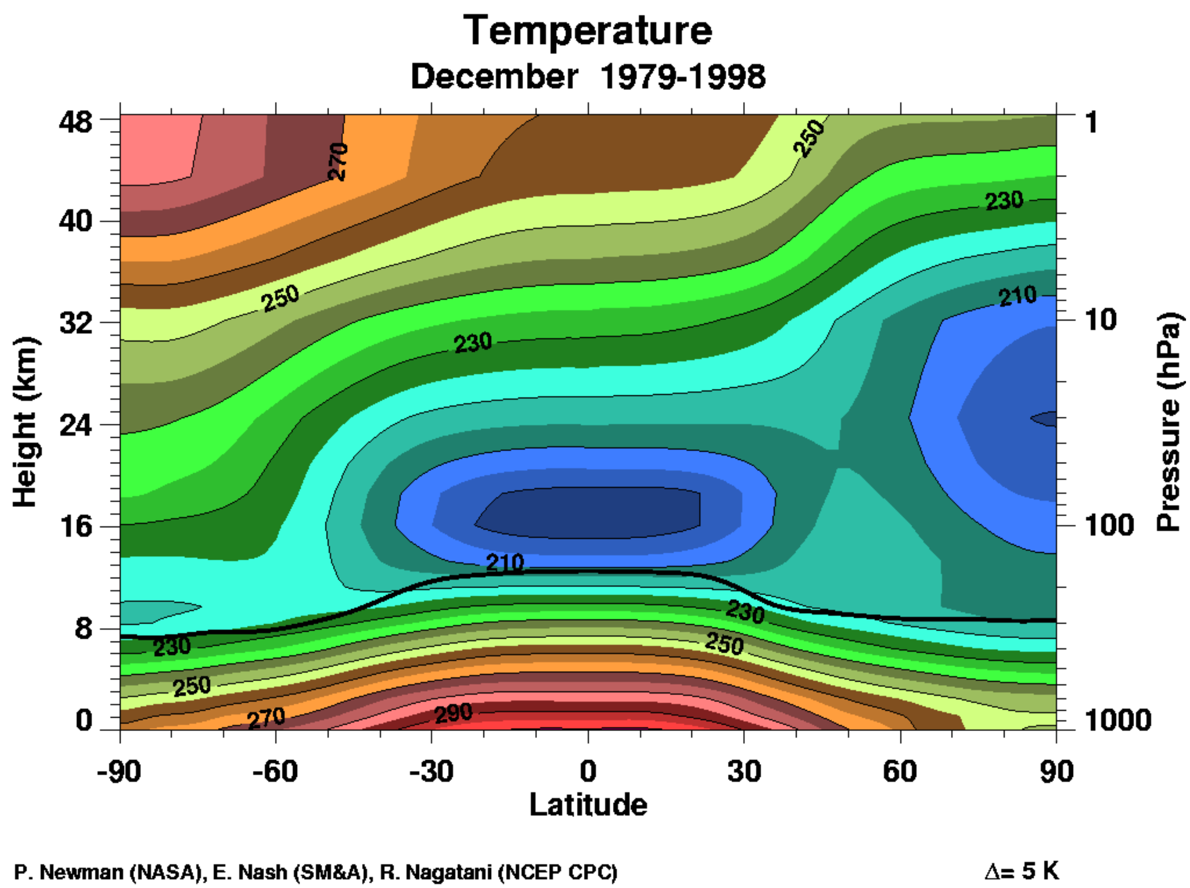
- Meridional velocity shear $U = \lambda y$ at the equator violates Rayleigh stability condition in interval $0 < y < \lambda/\beta$
- Dunkerton (1981) solved linearized, hydrostatic equations on β-plane
- Solution exhibits
  - "Taylor Vortices" in unstable region
  - zonal jets over equator
  - pancake structures in temperature perturbation field





Geophysical Context

- Under solstice conditions radiative equilibrium temperature has meridional gradient at equator
  - ⇒ can only balance with cross equatorial flow
  - ⇒ advects angular momentum maximum (and zero potential vorticity line) across equator
  - ⇒ drives system towards inertially unstable state
- presumably, undetectable adjustment continuously taking place
  - ⇒ flattens temperature and angular momentum across equatorial region
- In models and satellite data, see evidence of inertial adjustment having taken place (pancake structures in temperature field, stacked rolls and jets in velocity field)



- Zonal mean temperature for December, averaged over 16 year period (from NCEP)
- Notice temperature gradients flatten over equatorial region.

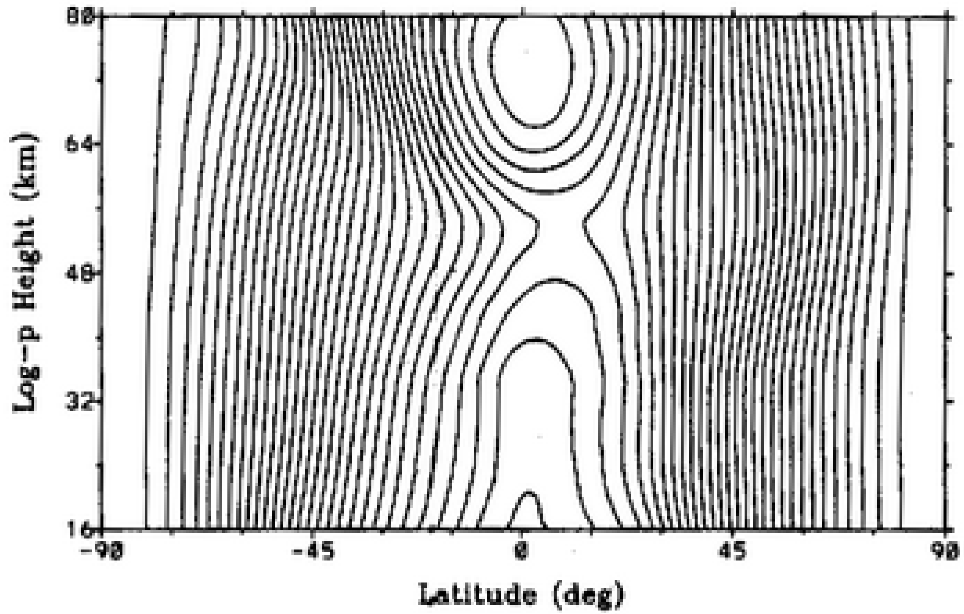


FIG. 19. January mean CMAM absolute angular momentum distribution (10^8 m^2s^{-1}).

- Angular momentum gradient in winter hemisphere weakens due to cross equatorial flow
- Effect most pronounced at stratopause because of maximum ozone heating (and hence maximum gradient in $T_{\rm rad}$) and low density.

2. NON-HYDROSTATIC CORIOLIS TERMS

- Vertical pressure gradient very nearly balances gravity so vertical momentum equation is often replaced by hydrostatic balance
- Neglecting nonhydrostatic terms upsets conservation of energy and angular momentum principles unless more changes are made:
  - Shallow atmosphere: $r \rightarrow a$
  - Vertical velocity w dropped from kinetic energy
  - Neglect metric terms involving w
  - Neglect Coriolis force terms proportional to $\cos\phi$

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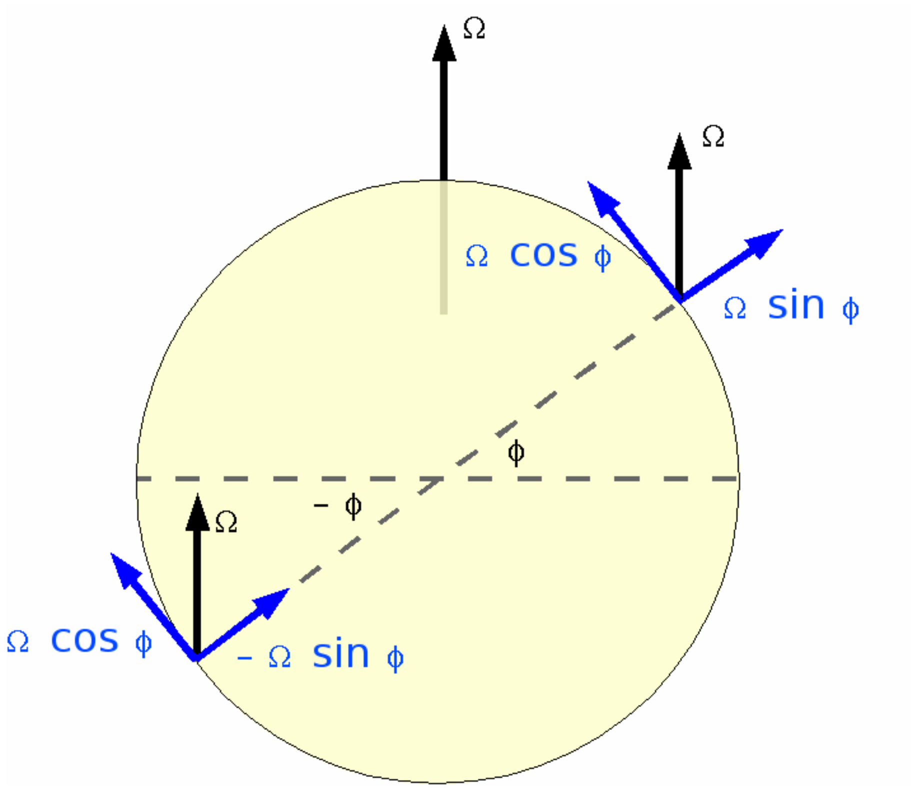
i.e.: $$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + \boxed{\frac{u^2}{r} + \frac{v^2}{r} + 2\Omega u \cos\phi}$$

$$\frac{Du}{Dt} = \frac{uv}{\boxed{a}}\tan\phi - \boxed{\frac{uw}{r} - 2\Omega w \cos\phi} + 2\Omega v \sin\phi - \frac{1}{\rho \boxed{a} \cos\phi}\frac{\partial p}{\partial \lambda}$$



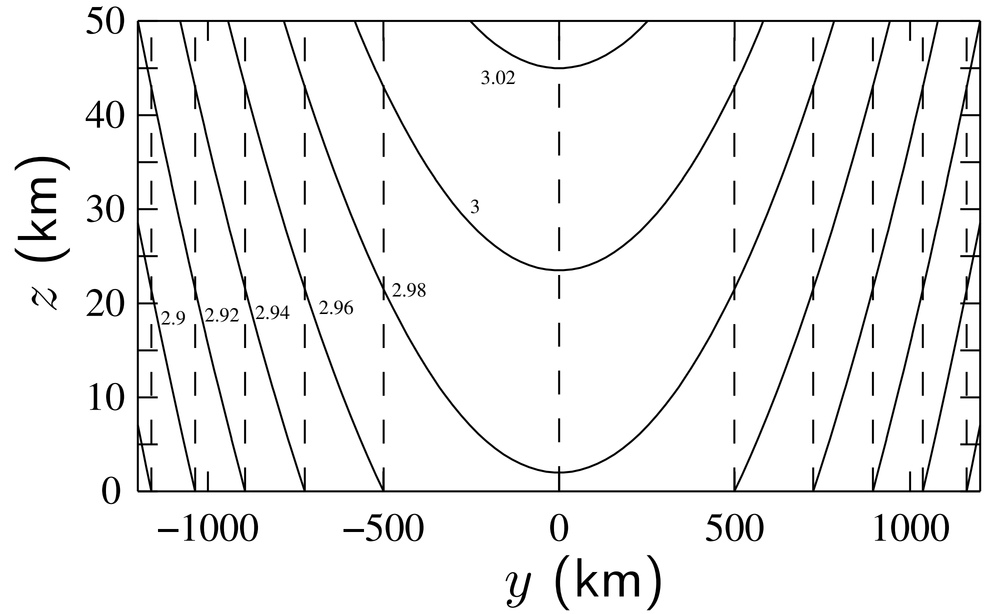
- Neglecting $\cos\phi$ terms is equivalent to neglecting the component of $\mathbf{\Omega}$ parallel to the surface.
- Most significant near equator

- Near equator, one might use (extended) equatorial β-plane

  Let $\beta \equiv \dfrac{2\Omega}{a}$ and $\gamma \equiv 2\Omega$

  Then $\mathbf{\Omega} = 2\Omega\cos\phi\,\hat{\mathbf{e}}_\phi + 2\Omega\sin\phi\,\hat{\mathbf{e}}_r \approx \gamma\,\hat{\mathbf{e}}_y + \beta y\,\hat{\mathbf{e}}_z$

- On (extended) β-plane, absolute angular momentum is $m \equiv u - \frac{1}{2}\beta y^2 + \gamma z$
- Non-hydrostatic terms make significant difference near equator - behold contours of "planetary angular momentum"



- Dashed lines are contours for hydrostatic case
- And symmetric stability is linked with gradients of angular momentum, so

3. STABILITY CONDITIONS

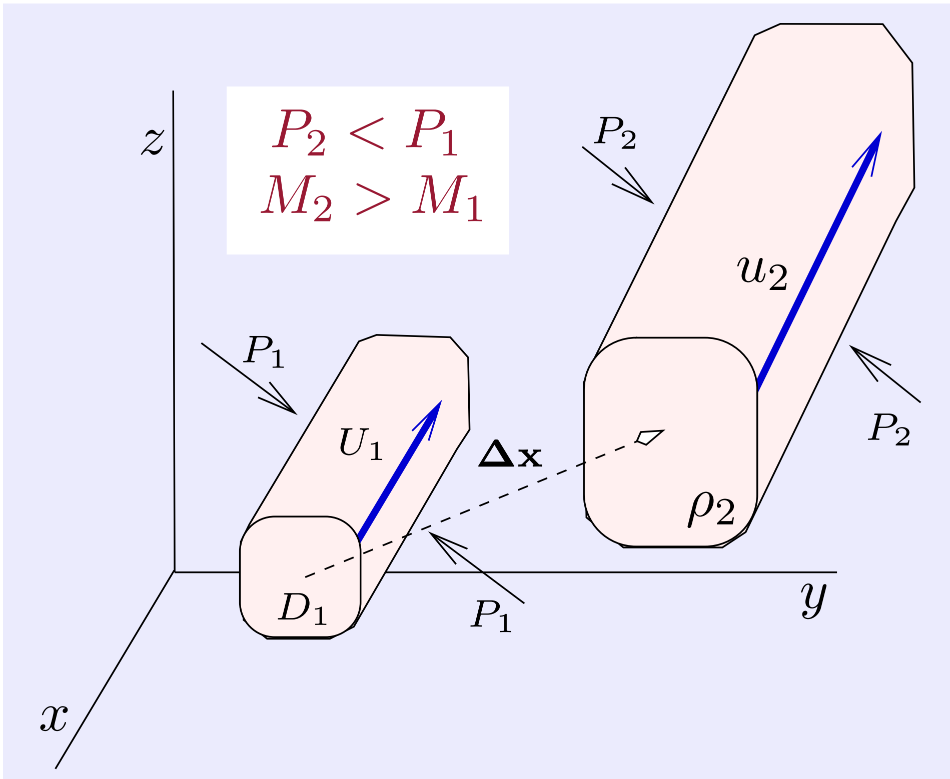
- Seek conditions for stability under small amplitude disturbances.
- Task is to derive conditions on a zonally symmetric equilibrium solution
  - · $u = U(y, z)$ ($m = M(y, z)$)
  - · $v = w = 0$
  - · $\theta = \Theta(y, z)$ and $p = P(y, z)$
    (temperature $T(y, z)$, density $D(y, z)$)

  to the zonally symmetric equations of motion such that small amplitude disturbances do not grow.
- Use two complementary approaches:
  - Energy method
  - Parcel displacement method

Energy method

- Construct energy-like functional $\mathcal{A}_L(u, v, w, \rho, \theta; U, D, \Theta)$ ("pseudoenergy") of the dependent variables and their equilibrium values such that $\mathcal{A}_L$:
  - is conserved
  - vanishes when evaluated at the equilibrium
  - has a critical point at the equilibrium
- If $\mathcal{A}_L$ is positive for all states, then equilibrium is stable (sufficient condition).
- Usually a method for finite amplitude stability, but in this case, only applies to linear equations (large amplitude perturbations to density field can make pseudoenergy negative)

Parcel displacement method



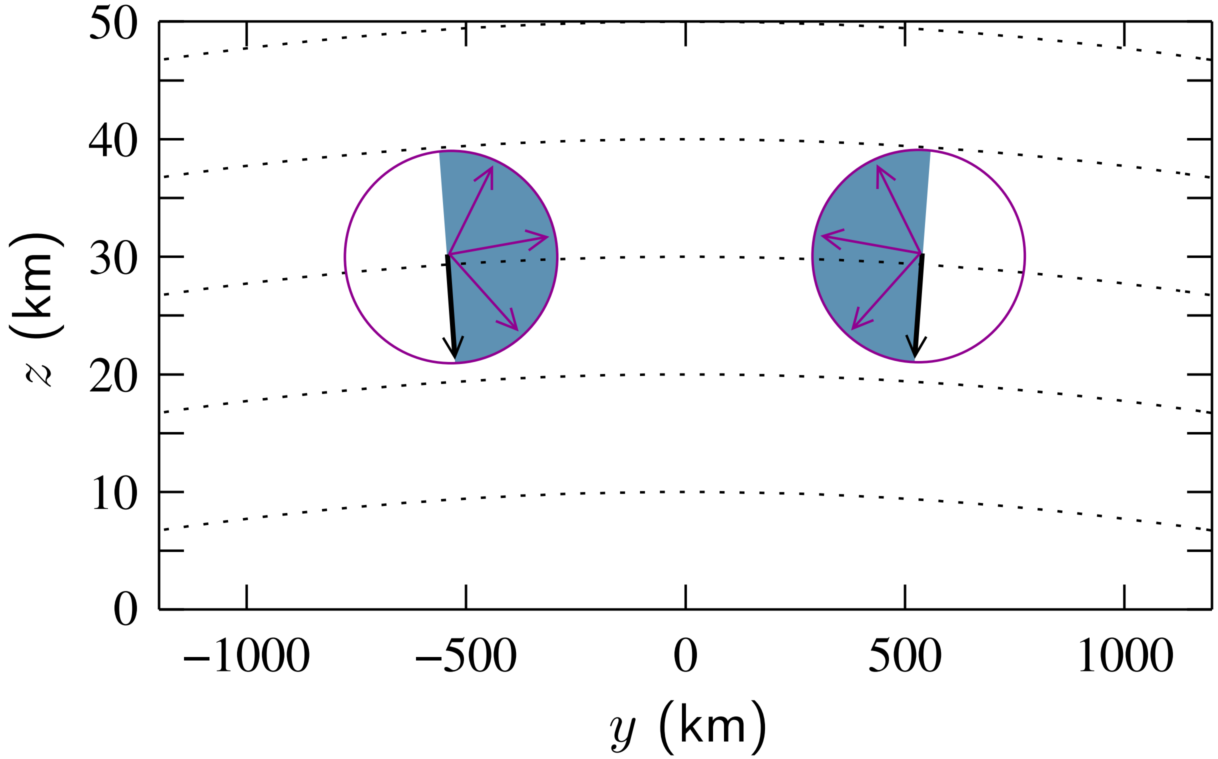
- Tube/ring displaced by $\Delta\mathbf{x} \equiv (\Delta y, \Delta z)$.
- Expands and accelerates, conserves θ and m.
- Does not disturb pressure field.

- Project acceleration onto $\Delta\mathbf{x}$: $\left[\frac{Dv_2}{Dt}, \frac{Dw_2}{Dt}\right] \cdot \Delta\mathbf{x} = \Delta\mathbf{x}^T \mathcal{S}\, \Delta\mathbf{x}$

$\Rightarrow$ If negative $\forall \Delta\mathbf{x}$ ($\iff$ $\mathcal{S}$ is negative definite), force on displaced tube is restoring and steady state is stable.

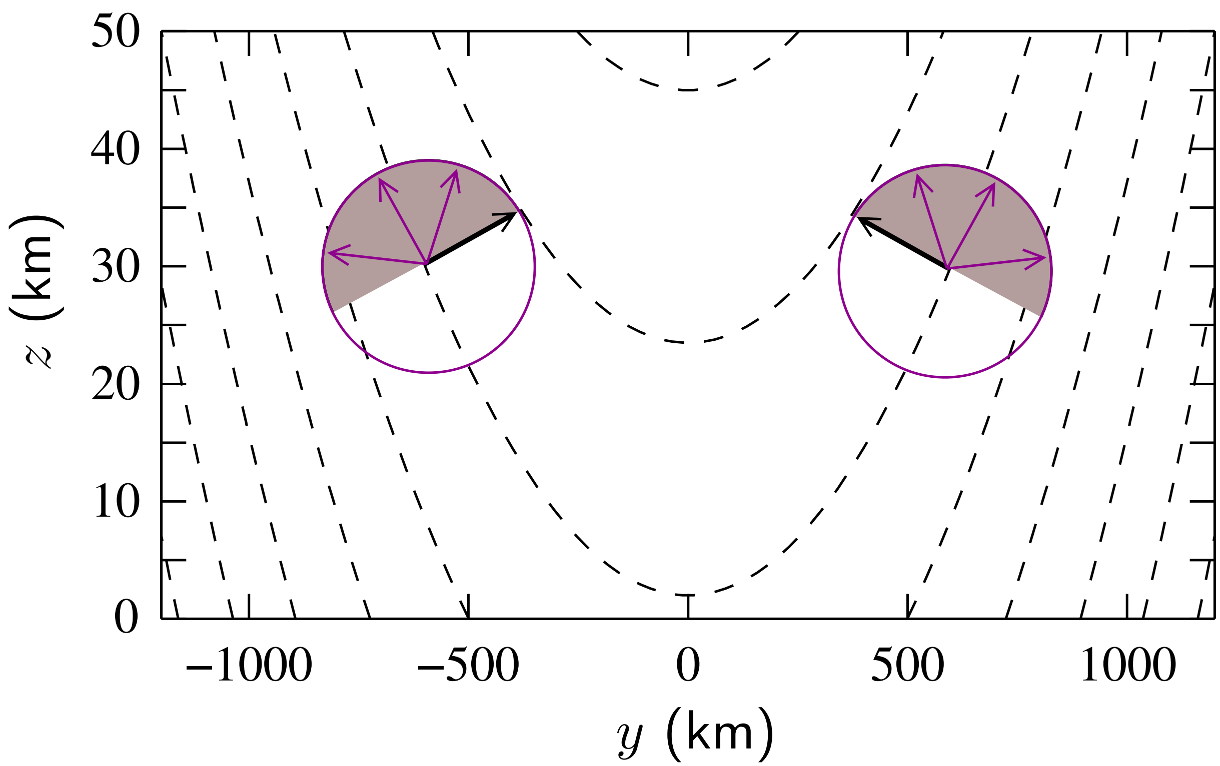
- The two methods give equivalent results.
- Notation: $\partial(F,G) \equiv \frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}$
- Sign of $\partial(F,G)$ is given by right hand rule applied to ∇F and ∇G: $\partial(F,G) > 0$ if ∇F is "clockwise" of ∇G
- Conditions for stability are
  - $\frac{1}{Q}\partial(M,P) > 0$ (inertial stability)
  - $\frac{1}{Q}\partial(\Theta,M^{(p)}) > 0$ (static stability)
  - $yDQ \equiv y\partial(\Theta,M) > 0$ (symmetric stability)

"Inertial Stability"



- Contours are curves of constant pressure.
- ∇M must be in coloured semicircle for static stability.
- Condition very close to hydrostatic condition $yM_y < 0$.

"Static Stability"

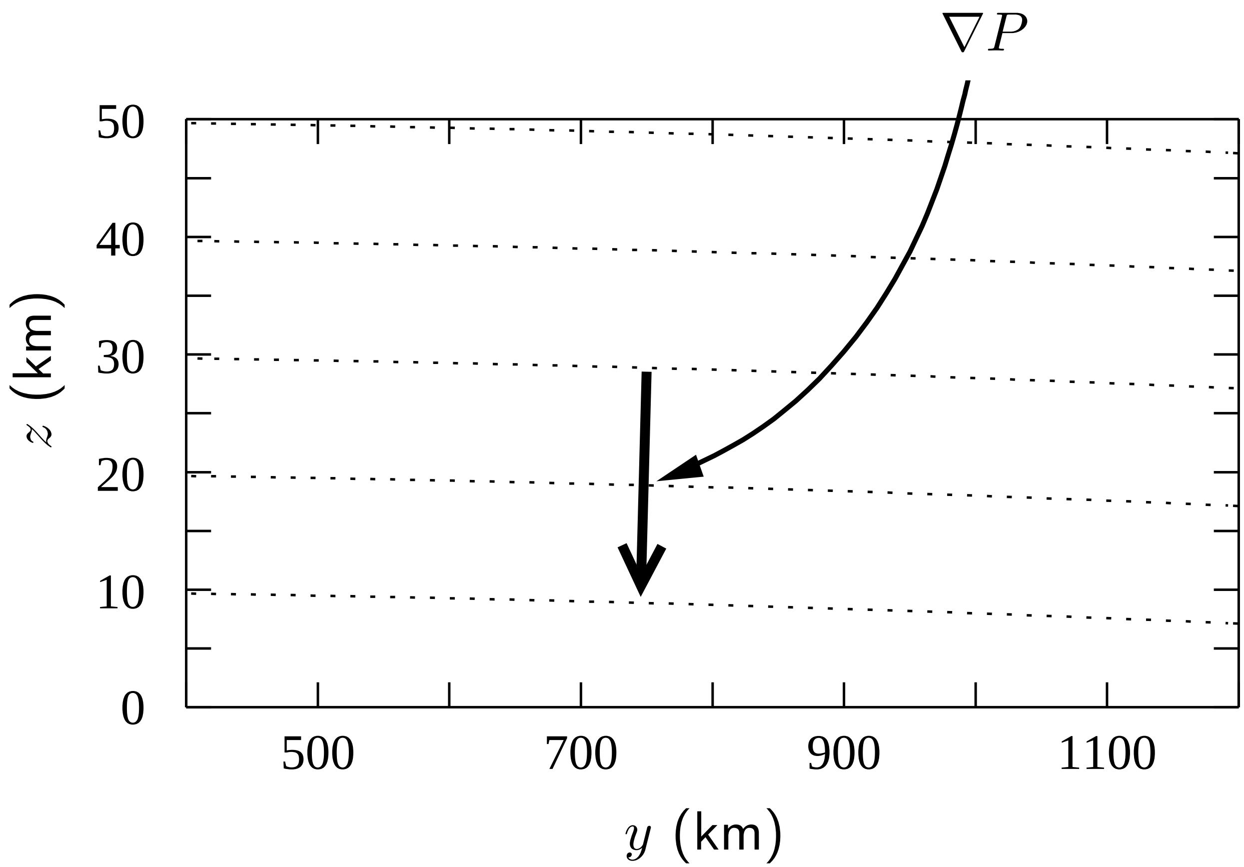


- Contours are curves of constant $M^{(p)} \equiv -\frac{1}{2}\beta y^2 + \gamma z$, tangent to local rotation vector $\mathbf{\Omega} \equiv \gamma \hat{\mathbf{e}}_y + \beta y \hat{\mathbf{e}}_z$.
- $\nabla\Theta$ must be in coloured semicircle for static stability.

A symmetrically unstable case:

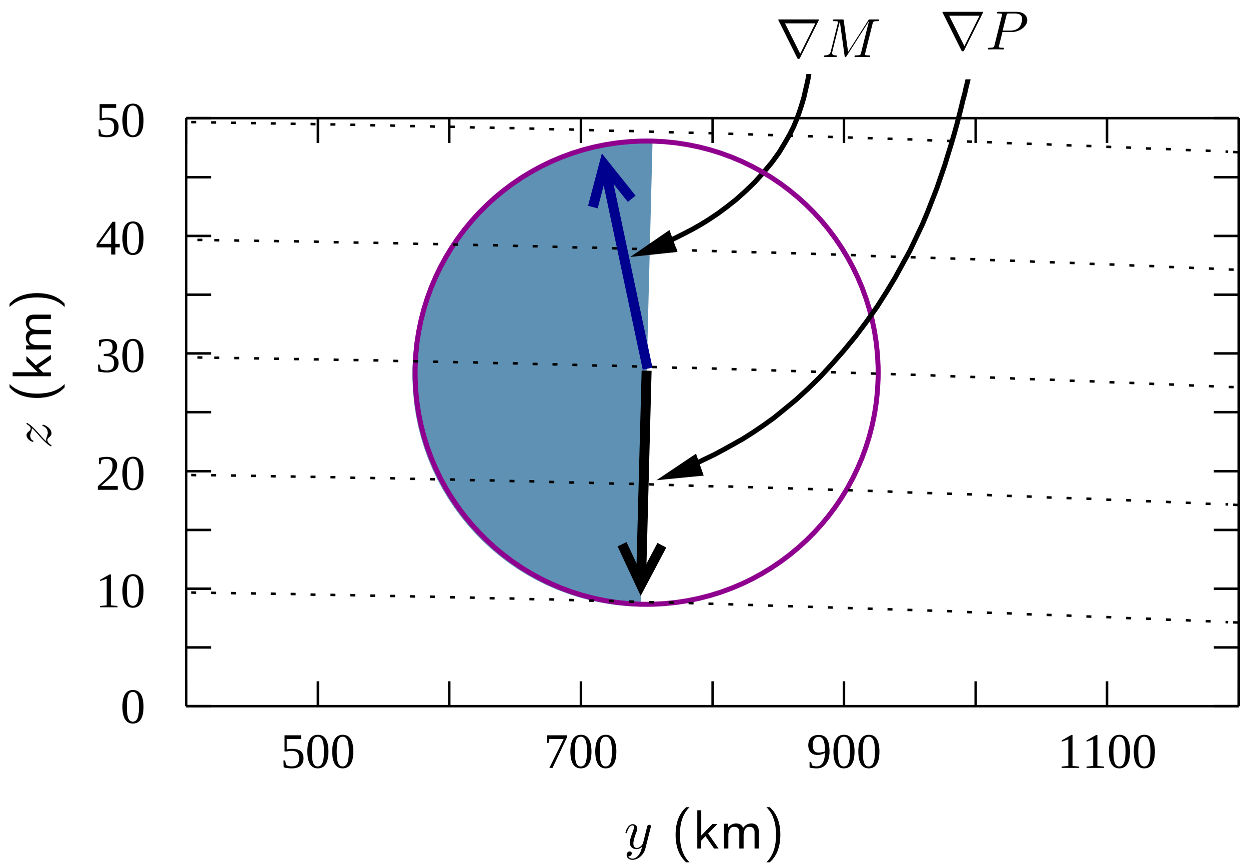
- It is possible to satisfy the inertial and static stability conditions but to fail the "symmetric" stability condition.
- This occurs if the Θ surfaces tilt up (enough, but not overturning) and the M surfaces tilt down (enough, but not overturning) so that Q is of the wrong sign.

A symmetrically unstable case:



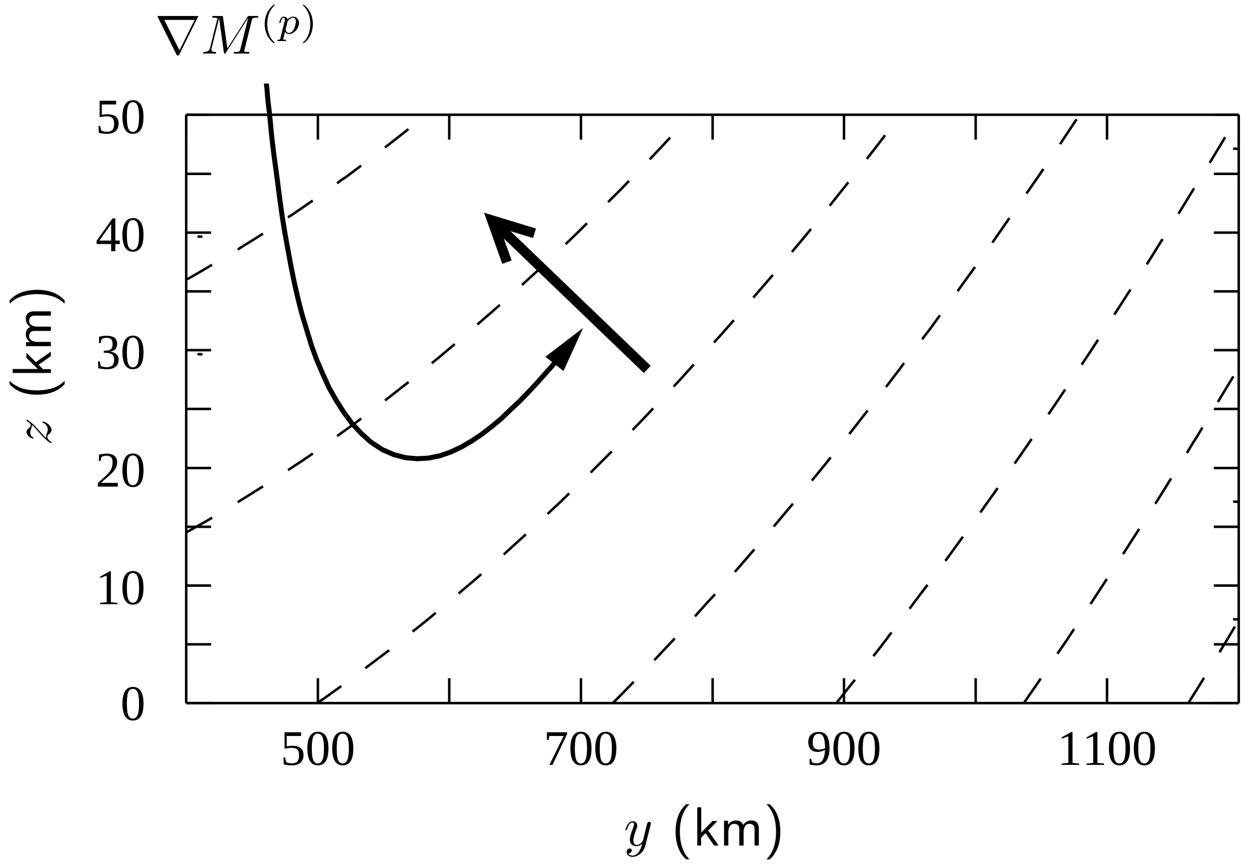
- Dotted lines are lines of constant pressure

A symmetrically unstable case:



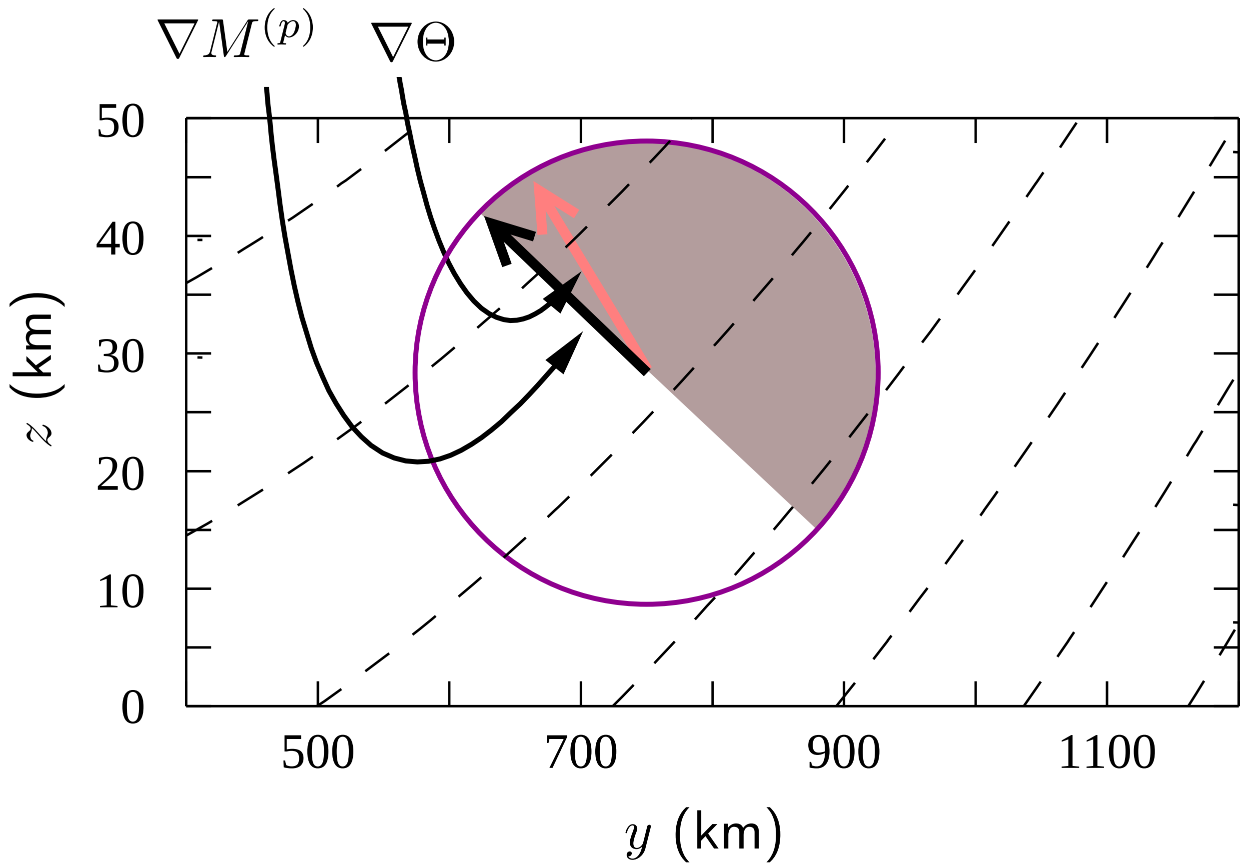
- Dotted lines are lines of constant pressure
- ∇M must be "clockwise" of ∇P in N. hemisphere for stability.

A symmetrically unstable case:



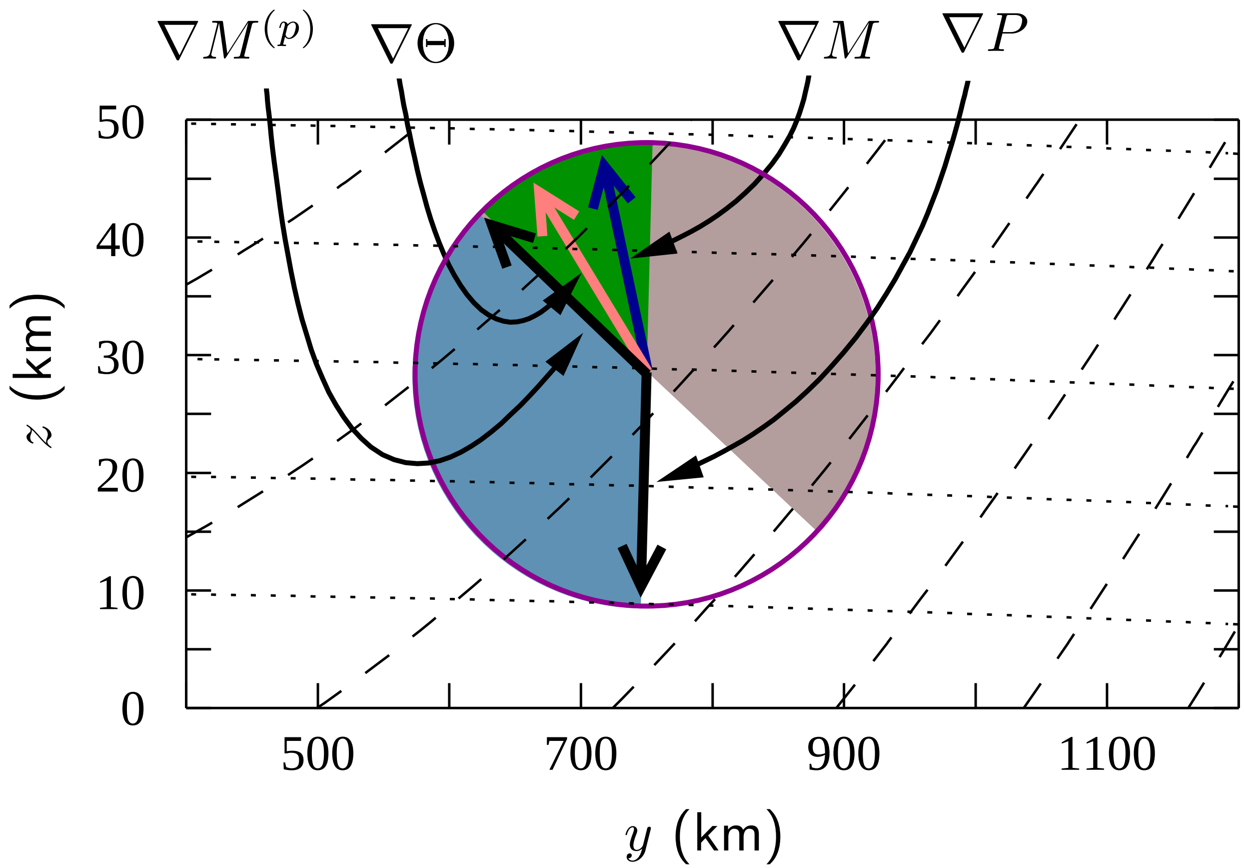
- Dashed lines are lines of constant planetary angular momentum.

A symmetrically unstable case:



- Dashed lines are lines of constant planetary angular momentum.
- $\nabla\Theta$ must be clockwise of $\nabla M^{(p)}$ in N. hemisphere for stability.

A symmetrically unstable case:



- But $\nabla\Theta$ must be clockwise of ∇M for stability

$\Rightarrow$ This state is unstable!.

4. EXAMPLES

- Not every combination of U, Θ and P is possible for an equilibrium - must satisfy (D = Density)

$$
\begin{aligned}
-\beta y U - \frac{1}{D} P_y &= 0 \\
\gamma U - g - \frac{1}{D} P_z &= 0
\end{aligned}
$$

- Exact solutions to balance equations can be calculated for simple cases - e.g. specifying velocity and temperature and solving for pressure.

- In each example, it is assumed that the atmosphere is an ideal gas; i.e. $p = \rho R T$.

E1. Isothermal, solid-body rotation

- Uniform temperature: $T = T_{00}$
- Uniform zonal velocity: $U = U_{00}$
- If P_{00} is the pressure at the origin,

$$P(y,z) = P_{00} \exp\left[-\frac{1}{2}\left(\frac{\beta U_{00}}{RT_{00}}\right) y^2 - \left(\frac{g - \gamma U_{00}}{RT_{00}}\right) z\right]$$

- Can write $\Theta = \Theta(P,T)$, and since $\nabla T = 0 \;\; \nabla\Theta \,||\, -\nabla P$
- The potential vorticity Q satsifies

$$DQ = \left(\frac{\beta g}{T_{00}}\right) y$$

$\Rightarrow$ Symmetrically stable!

E2. Linear horizontal velocity shear

- Zonal velocity: $U = \lambda y$
- Assume constant temperature over equator. Then pressure is

$$P(y,z) = P_{00}\left(1 - \frac{\gamma\lambda}{g}y\right)^{\left(\frac{g}{RT_{00}}\right)\left(\frac{\beta}{\gamma}\right)\left(\frac{g}{\gamma\lambda}\right)^2} \times \exp\left\{-\frac{g}{RT_{00}}\left[z + \frac{\beta}{\gamma}\left(\frac{g}{\gamma\lambda}y - \frac{1}{2}y^2\right)\right]\right\}.$$

- Potential vorticity:

$$Q = \frac{\beta g}{DT}\left[\left(1 + \frac{\gamma\lambda^2}{\beta g}\right)y - \frac{\lambda}{\beta}\left(1 + \frac{\gamma^2}{g^2}c_p T_{00}\right)\right]$$

$\Rightarrow$ Unstable in $0 < y \lesssim \dfrac{\lambda}{\beta}$

E3. Vertical temperature gradient

- Let $T(0,z) = T_{00}(1+\sigma z)$ and $U(y) = \frac{1}{2}\beta' y^2$. Pressure is:

$$P(y,z) = P_{00}\left\{1 + \sigma\left[z - \frac{\beta g}{\beta'\gamma^2}\left(\ln\left(1 - \frac{\beta'\gamma}{2g}y^2\right) + \frac{\beta'\gamma}{2g}y^2\right)\right]\right\}^{-\dfrac{g}{RT_{00}\sigma}}$$

- and potential vorticity

$$Q = \frac{1}{DT}\left\{\left[(g+\sigma c_p T_{00})(\beta-\beta') - \beta'\gamma^2\left(\frac{c_p T_{00}}{g}\right)(1+\sigma z)\right]y\right.$$
$$\left. + \frac{1}{2}\left[\gamma\beta'^2 + \sigma\gamma\beta'\left(\frac{c_p T_{00}}{g}\right)(\beta+\beta')\right]y^3 + [\sigma\beta c_p T_{00}]\,y\ln\left(1 - \frac{\beta'\gamma}{2g}y^2\right)\right\}$$

$\Rightarrow$ Stable if $\beta' < \left\{\dfrac{g+\sigma c_p T_{00}}{g + c_p T_{00}\left[\sigma + \frac{\gamma^2}{g}(1+\sigma H)\right]}\right\}\beta \approx \beta$

E4. $|y|^k$ angular momentum profile

- Let $M = -\alpha|y|^k + \gamma z$ and $T(0,z) = T_{00}$

$$Q = \frac{1}{DT}\left\{-\beta\gamma^2\left(\frac{c_pT_{00}}{g}\right)y + \frac{1}{2}\beta^2\gamma y^3 + k\alpha\left[g + \gamma^2\left(\frac{c_pT_{00}}{g}\right)\right]\frac{|y|^k}{y} - \beta\gamma\alpha\left(1+\frac{k}{2}\right)y|y|^k + \gamma k\alpha^2\frac{|y|^{2k}}{y}\right\}$$

- For $1 < k < 2$, state is stable.
- For $k > 2$, state is symmetrically unstable in a very small interval about the equator (but satisfies separate inertial and static stability conditions).
  - At equator $\nabla M,\ \nabla\Theta \,||\, \hat{\mathbf{e}}_z$
  - As y increases, $\nabla\Theta$ tips towards equator faster than ∇M.

5. ANELASTIC MODEL

- Certain results from hydrostatic case can be achieved in nonhydrostatic case using anelastic equations
- Assumes that fastest time scale is that of gravity waves (filters sound wave modes) and that θ departs relatively little from prescribed reference profile $\theta_0(z)$ (c.f. Boussinesq system).
- Only 4 prognostic variables - (u, v, w) and θ instead of the 5 in Euler equations (ρ given by continuity equation)
- Can extend small amplitude result to finite amplitude for certain basic states
- Can solve linear equations exactly for Dunkerton problem in the case of $\theta_0(z) =$ constant

SUMMARY

- Symmetric instability plays a role in solstice season dynamics in equatorial middle atmosphere.
- Hydrostatic primitive equations neglect $\cos\phi$ Coriolis terms, which are significant near equator.
- Stability of a steady solution to Euler equations depends on directions of ∇M and $\nabla\Theta$ relative to each other, ∇P and $\mathbf{\Omega}$.
- Examples: - Isothermal atmosphere is stable
  - Meridional velocity shear at equator is unstable
  - Vertical temperature gradient does not significantly affect symmetric stability conditions
  - Shallower than quadratic M is unstable at equator
- Can find finite amplitude result and exact linear solution to Dunkerton problem using anelastic equations

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13:58-14:16	Kirill Semeniuk, York University Modelling the October-November 2003 solar proton event in CMAM
14:16-14:34	Michel Bourqui, ETH-Zurich A new fast stratospheric ozone chemistry scheme in an intermediate GCM