Nonlinear Saturation of Centrifugal Instability in Inviscid Taylor-Couette Flow

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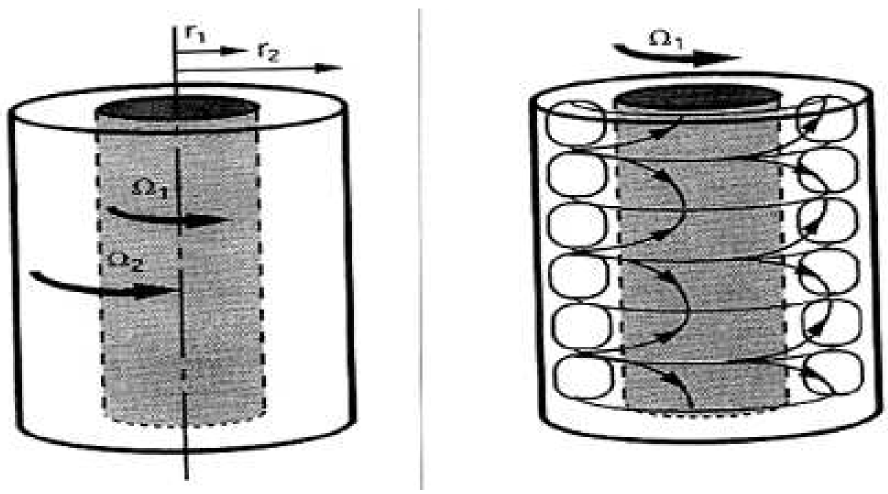
(from Tagg, 1994. Nonlin. Sci. Today)



History

- Rayleigh (1916): inviscid stability requires that angular momentum increases away from axis of rotation

- Taylor (1923): linear stability curve of corresponding viscous problem asymptotes to Rayleigh line in the limit of high Re

- Joseph and Hung (1971): nonlinear asymptotic stability of viscous problem shown for *near rigid body* conditions
  (nonlinear extension of Synge, 1938)

from:

Joseph, D. D. *Stability of Fluid Motions.* Springer-Verlag, 1976

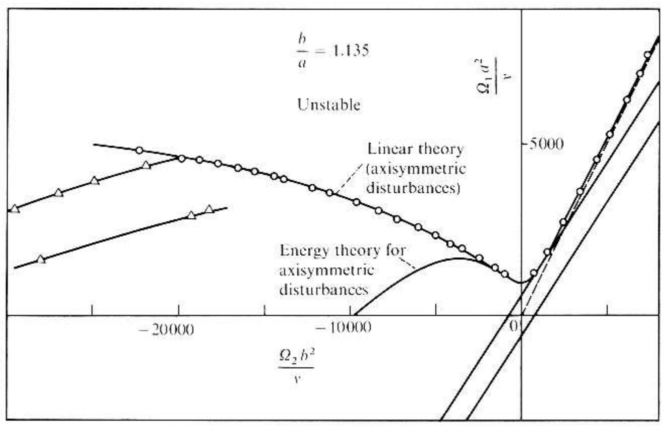


Fig. 37.1.a: Stability regions for Couette flow between rotating cylinders. The circles and triangles are observed points of instability in the experiments of D. Coles (1965) (Joseph and Hung, 1971)

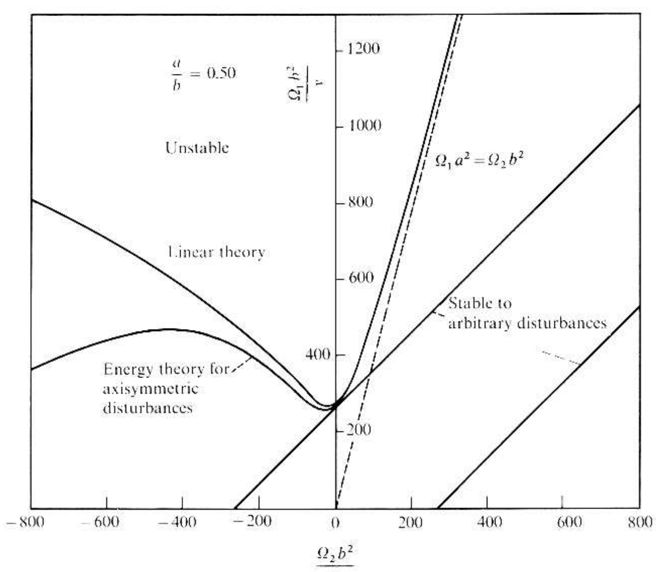


Fig. 37.1.b: Stability regions for Couette flow between rotating cylinders (Joseph and Hung, 1971)

Definitions

- Parameters: $\boxed{\eta \equiv \frac{r_1}{r_2}}$ and $\boxed{\mu \equiv \frac{\Omega_2}{\Omega_1}}$

- Equations nondimensionalized such that

$$\begin{array}{ll} r_1 \equiv 1 & \Omega_1 \equiv 1 \\ r_2 = \eta^{-1} & \Omega_2 = \mu \end{array}$$

- Fluid velocity in cylindrical coordinates:

$$\mathbf{v} = u\hat{\mathbf{e}}_r + v\hat{\mathbf{e}}_\theta + w\hat{\mathbf{e}}_z$$

  Angular momentum: $\boxed{m \equiv rv}$

- State vector:

$$\mathbf{x}(r,z,t) \equiv (u(r,z,t), m(r,z,t), w(r,z,t))^T,$$

- Couette (equilibrium) profile:

$$\mathbf{X}(r) \equiv (0, M(r), 0)^T$$

Nonlinear Stability of Couette Profile

We consider stability of Couette profile

$$\boxed{M(r) = Ar^2 + B}$$

under axisymmetric perturbations. All steady flows in corresponding viscous problem have profile $M(r)$. Here, we consider the inviscid case as a model for high Re flow.

Constants A and B determined by *no-slip* condition on surfaces of cylinders:

$$\boxed{A = \frac{\mu - \eta^2}{1 - \eta^2}, \quad B = \frac{1 - \mu}{1 - \eta^2}}$$

Method based on conservation of kinetic energy:

$$\mathcal{H} = \iint \tfrac{1}{2}|\mathbf{v}|^2 r dr dz$$

and all integrals of the form (Casimirs)

$$\mathcal{C} = \iint C(m) r dr dz$$

where $C(m)$ is any differentiable function.

Choose function $C(m)$ so that first variation of combination $\mathcal{H}+\mathcal{C}$ vanishes when evaluated at equilibrium:

$$\delta(\mathcal{H}+\mathcal{C})|_{\mathbf{X}} = 0$$

(so that $\mathbf{X}$ is a critical point of $\mathcal{H}+\mathcal{C}$).

This is achieved if, for $m \in \text{range}[M(r)]$,

$$\boxed{C'(m) = \frac{-Am}{m-B}}$$

Outside of $\text{range}[M(r)]$, we may extend $C'(m)$ in any way, so long as it is continuous.

Define pseudoenergy as the departure of $\mathcal{H}+\mathcal{C}$ from its equilibrium value. We may write it in the form

$$\mathcal{A}(\mathbf{x},\mathbf{X}) = \iint \tfrac{1}{2}\{u^2+w^2 + \frac{1}{r}[1+r^2 C''(\tilde{m})](m-M)^2\} r\,dr\,dz$$

where $\boxed{\tilde{m}(r,z,t) \in [M(r), m(r,z,t)]}$.

Can claim stability of $\mathbf{X}$ if $0 < \mathcal{A}(\mathbf{x}, \mathbf{X}) < \infty \ \forall \ \mathbf{X}$.

This follows from conservation of $\mathcal{A}$, since

$$||\Delta \mathbf{x}(t)||^2 \leq \mathcal{A}(t) = \mathcal{A}(0) \leq \frac{\lambda_+}{\lambda_-} ||\Delta \mathbf{x}(0)||^2$$

which implies that $||\Delta \mathbf{x}||$ is bounded for all time in terms of its initial value.

The norm is defined by

$$||\Delta \mathbf{x}||^2 \equiv \int\!\!\int \tfrac{1}{2}[u^2 + w^2 + \frac{\lambda_-}{r^2}(m - M)^2] r dr dz,$$

and λ_- and λ_+ are the minimum and maximum values of

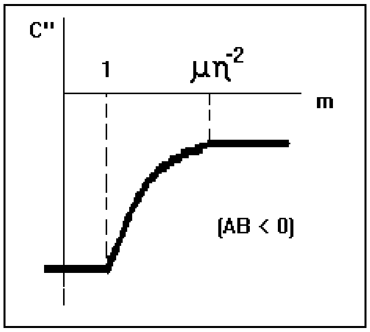
$$\boxed{F(m, r) \equiv 1 + r^2 C''(m)}$$

over all values of m and for all $r \in [1, \eta^{-1}]$.

To account for all possible perturbations, must define $C''(m)$ for all values of m. A simple choice is

$$C''(m) = \begin{cases} \dfrac{B}{A} & m < 1 \\ \dfrac{AB}{(m-B)^2} & 1 < m < \mu\eta^{-2} \\ \dfrac{AB}{(\mu\eta^{-2} - B)} & m > \mu\eta^{-2} \end{cases}$$

Notice that $F(m,r)$ can only be negative if $AB < 0$, in which case the least value of $F(m,r)$ obtains when r is maximized and $C''(m)$ is most negative.



Thus stability assured if

$$F(m=1, r=\eta^{-1}) = \frac{[(\eta^2+1)-\mu](1-\eta^2)}{\eta^2(\mu-\eta^2)} > 0$$

i.e. if $\eta^2 < \mu < \eta^2 + 1$. In dimensional form:

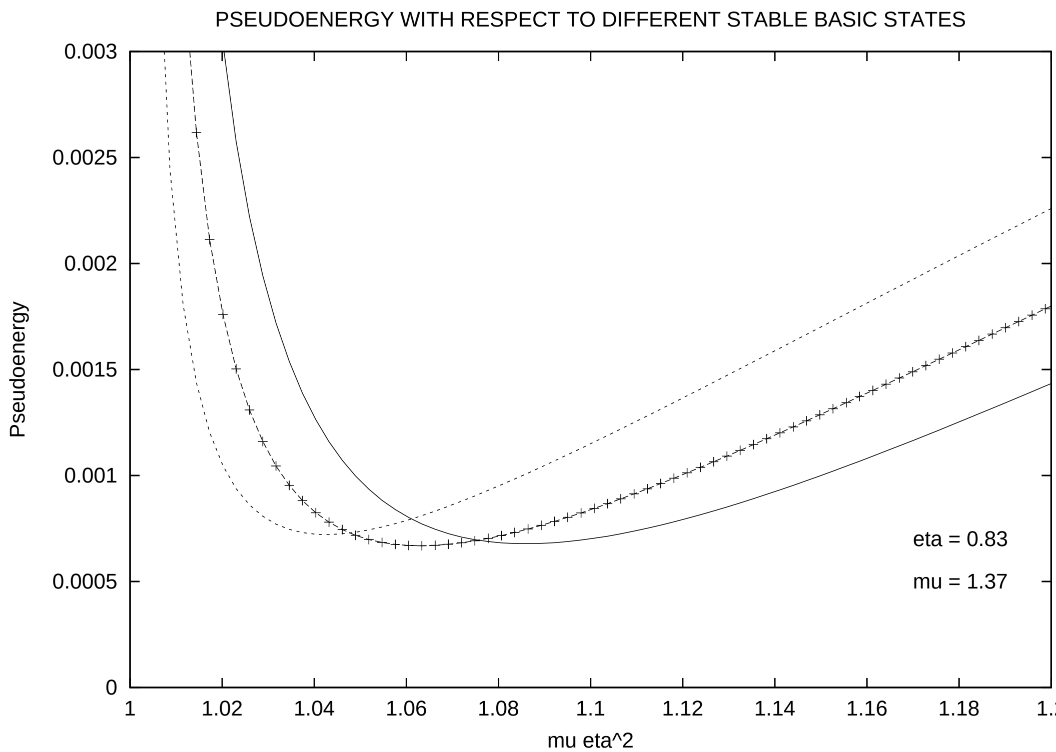
$$\boxed{\left(\frac{r_1}{r_2}\right)^2 < \frac{\Omega_2}{\Omega_1} < \left(\frac{r_1}{r_2}\right)^2 + 1}$$

Saturation of Disturbance Amplitude

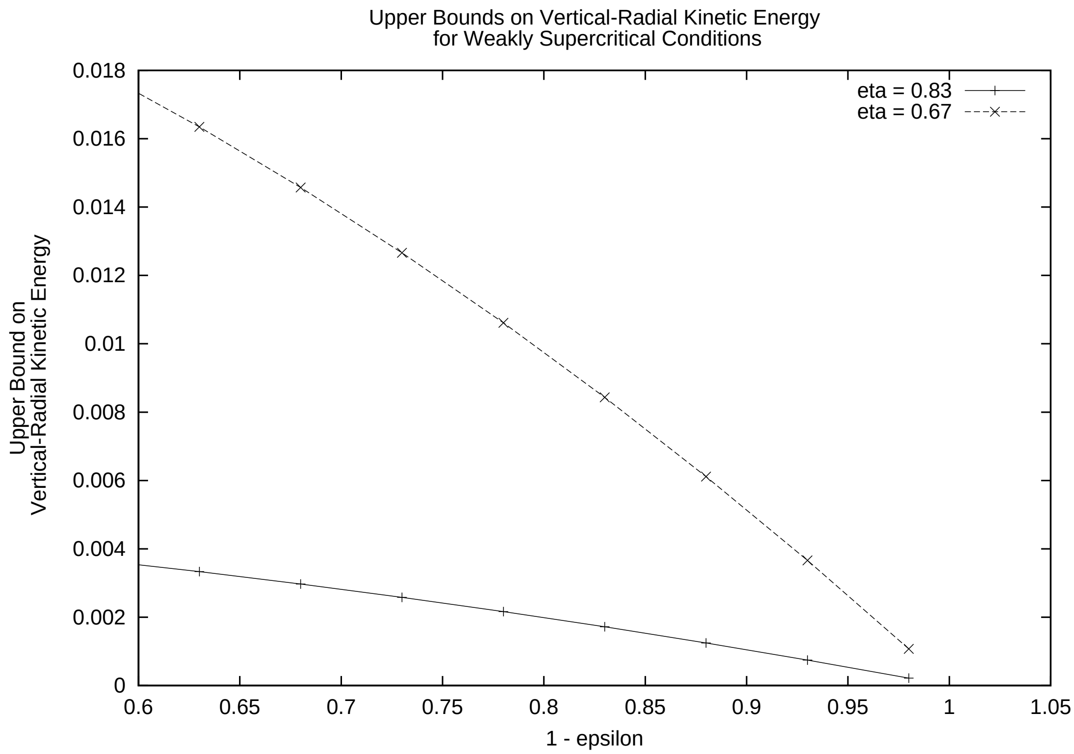
Since, relative to every stable $\mathbf{X}$, the pseudoenergy of every state $\mathbf{x}$ is positive and conserved, the energy released into the overturning flow (e.g. Taylor vortex flow) from an unstable basic state is bounded from above via

$$\mathcal{K}(\mathbf{x}(t)) \equiv \iint \frac{1}{2}[u^2 + w^2] r dr dz \leq \mathcal{A}(\mathbf{x}(0), \mathbf{X}).$$

We consider unstable Couette equilibrium profiles with $\mu_u < \eta^2$ and compute their pseudoenergies relative to a range of stable profiles (characterized by μ and, say, Ω_1):



and we plot the minimum value of all pseudoenergies so obtained:



- Upper bound on $\min \mathcal{A}$ approaches zero as $\epsilon \equiv 1 - \mu\eta^2$ approaches bifurcation point. Hence, the amplitude to which an initially small perturbation can grow is bounded near zero near the critical value of ϵ (characteristic of supercritical bifurcation).

- Landau theory (amplitude equations): equilibrium amplitude $\sim \epsilon^{1/2} \Rightarrow$ energy $\sim \epsilon$

Summary

- Considering only axisymmetric disturbances, we demonstrate a nonlinear generalization to Rayleigh's centrifugal stability condition.

- Growth of small disturbances to weakly supercritical equilibria are bound near zero, consistent with the existence of the stable axisymmetric Taylor vortex state, without the influence of viscosity.

- Hopefully, result can be extended to include viscosity; for example, showing that $\mathcal{A}$ is bounded from above during the evolution from a perturbed state (this might complement asymptotic stability results of Joseph and Hung, etc.)