On springs and (non-) conservation laws

The problem we discussed in the tutorial looks like a textbook problem to me, but I couldn't find a reference for it. Anyway, this is what we want to solve:

> Let us consider a dish of mass M hanged on a spring of constant k. There is an object of mass m, suspended at height h over the level of equilibrium of the dish (see Fig 1 for the setup). One releases the body m, which falls on the dish and sticks to it ("perfectly inelastic collision").
>
> Determine the depth d of the maximum elongation of the spring, as measured relative to the level of collision (C-level) of M and m.

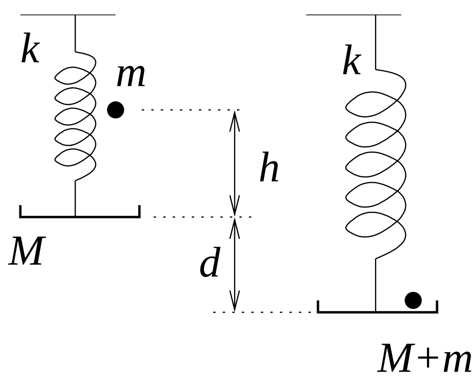

Figure 1: The setup.

To solve the problem we have to figure out what is conserved and what is not. For one, it should be clear that the *total mechanical energy* of the system (spring+ dish + dropped object) is *not* conserved through the whole process because we have an inelastic collision at some point.

Nevertheless, we do not discard the conservation laws – we may, and we are going to use them piecewise.

We start by noticing that the setup in Fig 1 is a bit too simplified, and may trick someone into missing important pieces of information. Namely, since we are told the collision takes place at the level of *equilibrium* for the dish M, it is clear that the spring was *already stretched* at that time and position – there was some sagging of the spring because of the dish. On the far left on Fig. 2 the sag is denoted by b.

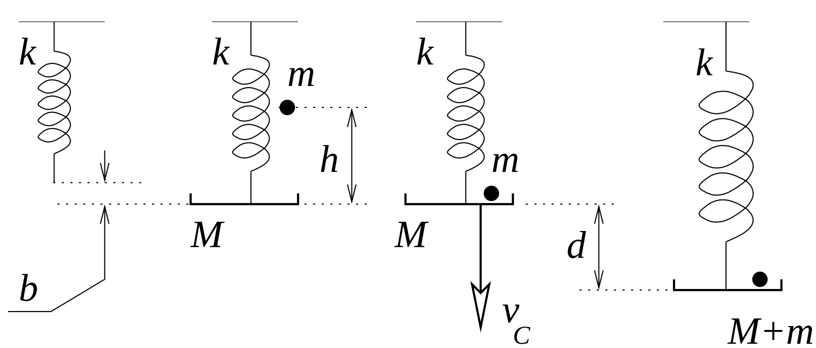

Figure 2: Some more details on the setup

During the collision between M and m, the total kinetic energy is not conserved (part of m's kinetic energy goes into heat; we are not given details about this venue, so we stop thinking of it), but the total momentum (of the system $m + M$) *is conserved*. Of course, to claim conservation of the momentum during the collision we must assume that the impulse $\Delta P = F_{ext}\Delta\tau$ of any net external force F_{ext} acting on M and m during the time interval of collision ($\Delta\tau$) is negligible compared to the total momentum. F_{ext} stands in this case for the weights of the colliding bodies and the spring force acting on M.

Anyway, to move on with the solution let us note that we can calculate the speed v_m of m just before it hits M. The speed's value is $v_m = \sqrt{2gh}$.

Since we know that the momentum of the system $m + M$ is conserved during the collision, we may write $\vec{P}_i = \vec{P}_f$, or in detail

$$\vec{P}_i = m\vec{v}_m + M\underbrace{\vec{v}_M}_{=0} = (m+M)\vec{v}_C = \vec{P}_f, \qquad (1)$$

where by $\vec{v}_C$ I denoted the velocity of the compound system $m + M$. We project (1) on a coordinate axis oriented vertically downward, and get for the speed v_C

$$m\sqrt{2gh} = (m+M)v_C \;\rightarrow\; v_C = \frac{m}{m+M}\sqrt{2gh}. \qquad (2)$$

After the collision, between the C-level and the lowest level reached by the $m + M$ system there is a distance d. The total mechanical energy *is conserved* on this part of the "road", so we may write $E_C = E_f$, or in extended form:

$$\begin{aligned} E_C &= E_{pC} + E_{kC} + E_{eC} \\ &= (m+M)gd + \frac{1}{2}(m+M)v_C^2 + \frac{1}{2}kb^2, \qquad (3) \\ E_f &= \underbrace{E_{pf}}_{=0} + \underbrace{E_{kf}}_{=0} + E_{e\,f} = \frac{1}{2}k(b+d)^2. \qquad (4) \end{aligned}$$

Above I denoted by subscripts p, k, e the potential, kinetic and elastic potential energy respectively, while the subscript f stands for the final state values of the corresponding quantities. The lowest ("final") level is taken to have 0 potential (gravitational) energy. For the elastic potential energy the 0-level corresponds to the unstressed spring.

What is left is mostly a bit of math: from (3) and (4) we obtain a quadratic equation for d.

Before doing that, we need to calculate b. Indeed, since M was at equilibrium just before the collision, the elastic force of the spring was balancing the weight of M. Hence,

$$kb = Mg \;\rightarrow\; b = \frac{Mg}{k}. \qquad (5)$$

With the above expression for b the quadratic equation for d becomes, after a bit of algebraic manipulation,

$$\frac{1}{2}kd^2 - mgd - \frac{m^2gh}{m+M} = 0. \qquad (6)$$

The discriminant of this quadratic equation is

$$\mathcal{D} = m^2g^2 + \frac{2km^2gh}{m+M} = m^2g^2\left(1 + \frac{2kh}{(m+M)g}\right), \qquad (7)$$

where I forced out the common factor m^2g^2 for cosmetic reasons.

The solutions of (6) are

$$d_\pm = \frac{-(-mg) \pm \sqrt{\mathcal{D}}}{2 \times \frac{1}{2}k} = \frac{mg}{k}\left(1 \pm \sqrt{1 + \frac{2kh}{(m+M)g}}\right). \qquad (8)$$

From the setup of the problem (see the diagrams) it is clear that d is expected to be positive, therefore the final solution for d is

$$d = \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2kh}{(m+M)g}}\right). \qquad (9)$$

First, we need to check the dimensions of our result. The bracket is just a number, with no units. The mg/k factor is indeed a length (see, e.g. (5)). So we are Ok here.

What about some limit cases? Say, if $h \to 0$, then (9) gives $d \to 2\frac{mg}{k}$. It is not quite obvious why d should be twice the equilibrium elongation of the spring with only m attached to it. Please, solve this particular case from scratch, to convince yourself that this is what you get if body m is just put on the dish, with no initial momentum and no kinetic energy.

Last revised: October 30, 2004
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