Relative velocity

The notes below present the solution to the stripped down version of a problem I encountered in high school.

On a straight road a police car P is traveling with velocity $\mathbf{u}$. On the opposite line two cars travel with velocity $\mathbf{v}$. When each of the cars pass by the police car they slow down to velocity $\mathbf{v}'$. If the initial distance between the cars **1** and **2** is d, find the new distance d' between them after they both passed the police car.

First, we solve the problem in the fixed reference frame (FRF). The FRF is linked to the road, so the situation is as depicted on Fig. 1.

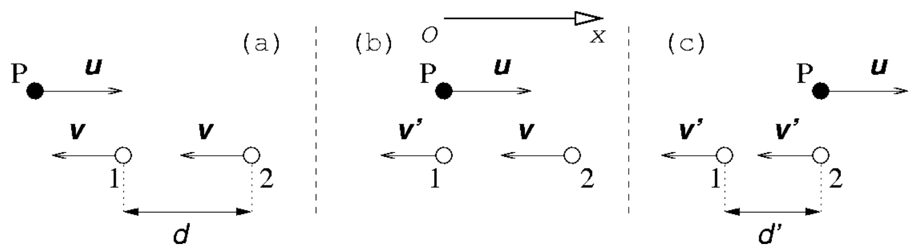

Figure 1: The setup in the fixed reference frame.

(*i*) A *"colloquial"* solution in the FRF

The distance between **1** and **2** changes because in the time interval τ which starts right after the car **1** passes by P and before car **2** does the same, **2** "catches up" car **1**. The speed of this "chase" is $\Delta v = v - v'$. Observe that I use loosely (hence the *"colloquial"*), not rigorously the FRF (no vectors, just scalars and every day language).

The second car "catches up"the first one only until **2** slows down to v' on it's turn, and that happens when **2** meets P. Hence, the time interval τ is given by $\tau = \dfrac{d}{u+v}$.

The new distance between the cars, after both passed by the police car, is given by

$$d' = d - \Delta d = d\left(1 - \frac{v - v'}{u+v}\right) = d\,\frac{u+v'}{u+v} \tag{1}$$

(*ii*) A *"by the book"* solution in the FRF

We write down the (scalar) equations which govern the motion of cars **1, 2** and P. The orientation of Ox axis is such that the projection of the velocity of the police car is positive.

The origin of the FRF is chosen to be at the point where P and **1** met. The equations of motion are therefore

$$x_1 = -v't, \tag{2}$$
$$x_2 = d - vt, \tag{3}$$
$$x_P = ut. \tag{4}$$

The time interval of interest is determined by the moment when P and **2** meet. Hence, we impose the condition

$$x_2 = x_P, \tag{5}$$

which, from (3) and (4) gives us τ

$$\tau = \frac{d}{u+v}. \tag{6}$$

At this moment, the distance between **1** and **2** is already d' and it doesn't change anymore. Hence

$$d' = |x_1 - x_2|_{t=\tau} = |-v'\tau - d + v\tau| = |(v-v')\frac{d}{u+v} - d| = \frac{u+v'}{u+v}d. \tag{7}$$

(*iii*) A solution in the moving reference frame (MRF)

Of course, since the police car velocity is constant it is tempting to use P as our MRF.

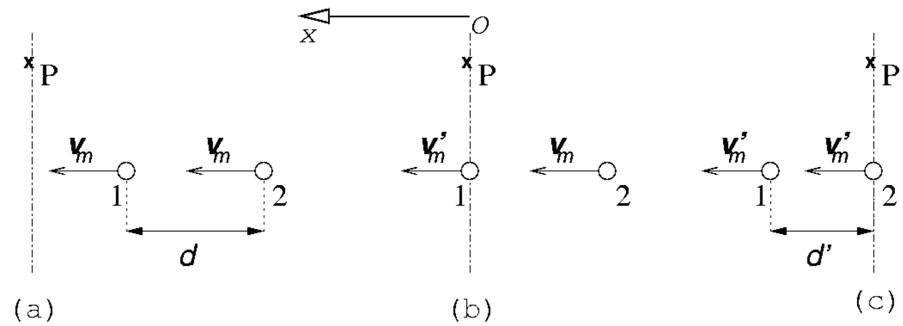

Figure 2: The setup in the moving reference frame.

The situation is depicted in Fig. 2. One can easily find the velocity of the cars **1** and **2** in the MRF by using the relationship

$$\mathbf{v} = \mathbf{v_m} + \mathbf{u}. \tag{8}$$

The projection of velocity $\mathbf{v_m}$ on the Ox axis (see Fig. 2) is

$$v_m = |\mathbf{v} - \mathbf{u}| = v - (-u) = u + v. \tag{9}$$

As car **1** passes by P it's velocity changes to $\mathbf{v_m}'$. The same happens to car **2** after some time interval τ. We note that d' will be the distance between **1** and **2** starting with the moment when **2** passes by P (indeed, after that moment **1** and **2** have the same velocity, so the distance between them doesn't change anymore). We are left with the task of finding the time τ it takes **2** to reach P after **1** just passed by P. One finds

$$\tau = \frac{d}{v_m} = \frac{d}{u+v}. \tag{10}$$

Distance d' is then given by

$$d' = v_m'\tau = (u+v')\frac{d}{u+v}. \tag{11}$$

Note: One may ask the same question (find d') in a relativistic setup (v or/and u close to c). This case is left as an exercise to the reader. :-)

Last revised: September 29, 2003