Racing balls – which one is faster?

These notes deal with the problem demonstrated during the lecture by Dr. Harrison.

Here I present a solution to the question "which ball gets to the end faster"? It's rather sketchy; I hope however that you'll get to think over the details on your own.

The setup is presented in Fig 1. The initial speeds are the same (the balls are identical). The red ball travels along a horizontal track **1**. The blue ball travels along track **2** which has a portion at a *lower* level than track **1**.

In the lecture, Dr. Harrison showed that the ball on track **2** reaches the end of the track faster than the ball on track **1**.

The question is: does this result (2 faster than 1) depend on the particular shape of the track? Or is it a generic result (as long as track 2 has a portion which is lower than the common level, and no portions higher)?

I will sketch below a quick-and-dirty way of convincing yourself that the result is generic.

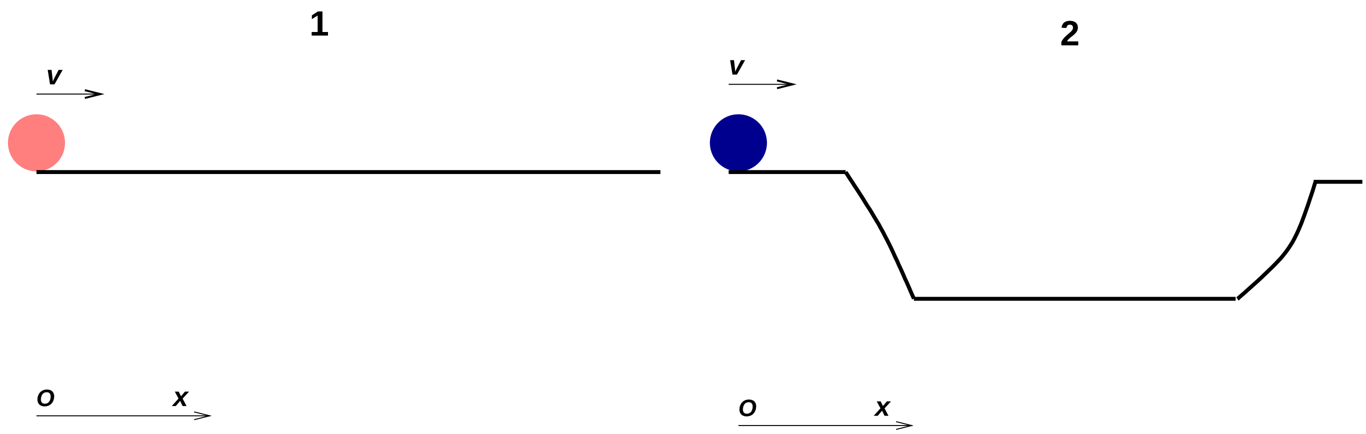

Figure 1: The racing balls experiment.

The time it takes the red ball (on track **1**) to reach the end of the track is obviously

$$\tau_1 = \frac{d}{v}. \tag{1}$$

For the blue ball, the travel time is a sum

$$\tau_2 = \tau_{h1} + \tau_d + \tau_h + \tau_u + \tau_{h2}, \tag{2}$$

where $\tau_{h1}, \tau_d, \tau_h, \tau_u, \tau_{h2}$ denote the travel time on the initial horizontal stretch, down the incline, on the lower horizontal stretch, up the incline and on the final horizontal stretch, respectively.

In a generic way, without being too rigorous, each time interval depends on the *horizontal* distance traveled and the *horizontal component* of the velocity:

$$\tau = \frac{\Delta x}{v_x} = \frac{1}{v_x}\Delta x. \tag{3}$$

Of course, this would require the v's to be constant for each interval – which they are not. But in the infinitesimal limit, as in the case with approximating the integrals as a sum of areas of rectangles ... this works out well.

I hope it is (intuitively) clear that τ is the area under the plot of $\frac{1}{v}$ versus x – see Fig. 2. The horizontal dotted line corresponds to the ball on track **1**. The solid magenta line corresponds to the ball on track **2**.

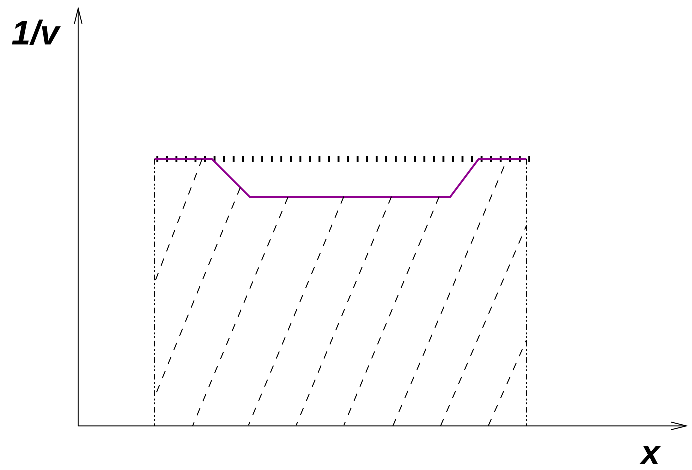

Figure 2: The travel time as an area.

Since the horizontal projection of the velocity of the blue ball is larger than the one for the red ball (on the horizontal track), the magenta line is *lower* than the dotted line, for some portion. The hashed area is the time interval it takes the ball on track **2** to reach the end of the track. Obviously, it is smaller than the rectangular area delimited by the dotted lines, which corresponds to the ball on the horizontal track **1**.

There is one remaining issue – what if the plot for the blue ball (track **2**) has the spikes depicted on Fig 3? Such a situation could invalidate the area comparison argument.

So, we need to make sure such a situation is not possible. Let us assume it is possible – then at some point (say, the top of the spikes) the horizontal component of the blue ball velocity (track **2**) is *lower* than the (horizontal) velocity of the red ball on track **1**. Therefore, some negative acceleration on the x axis did this to the blue ball.

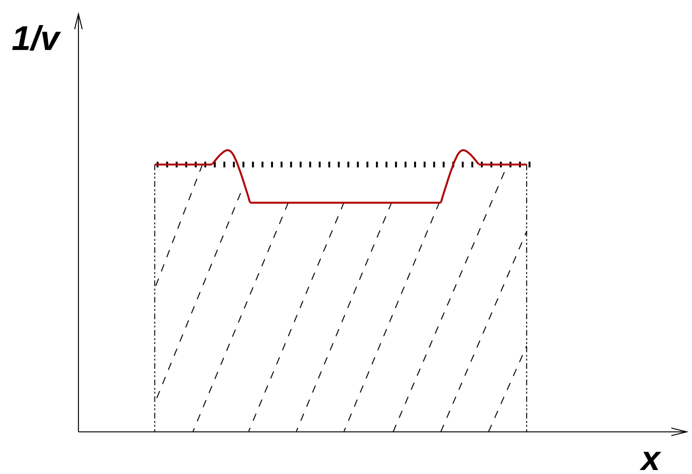

Figure 3: A hypothetical situation.

But this never happens to the blue ball, because there is no force acting "backward" – the projection of the reaction force (the normal) is always positive on the first incline (the one on the left). On the incline on the right the projection of the normal on the x axis is *always* negative, but a spike is not allowed as well, because positive horizontal acceleration would then appear (when the spike goes down).

Last revised: October 28, 2006; solution by Sorin Codoban.