1-D motion with friction

Q: When there is no friction, the flight time for an object thrown directly upward is $\tau_0 = 2v_0/g$.

Assume now that there is friction of the form

$$\begin{array}{ll}(a) & \mathbf{F} = -k^*\mathbf{v}, \quad \text{or} \\ (b) & \mathbf{F} = -k^* v^2,\end{array}$$

where k^* is a constant. If the launch speed v_0 is the same w/ and w/o friction, how does the flight time τ with friction compare with τ_0?

(a) *"Linear" friction case*

With Oy oriented upward the equation of time evolution for the speed is

$$\frac{dv}{dt} \equiv \dot{v} = -kv - g, \quad v(t=0) = v_0, \qquad (1)$$

where $k = k^*/m$ (m is the mass of the object). The solution of (1) is given by

$$v = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right) \exp^{-kt}. \qquad (2)$$

For the y co-ordinate we obtain

$$y = -\frac{g}{k}t + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)(1 - \exp^{-kt}). \qquad (3)$$

For proper mathematical handling let us denote $\frac{kv_0}{g} = \varepsilon$ and introduce $t^* = t/\tau_0$ – the non-dimensional time. In terms of these notations (3) becomes

$$y = \frac{2v_0^2}{g}\frac{1}{\varepsilon}\left[-t + \frac{1}{2}\left(1 + \frac{1}{\varepsilon}\right)(1 - \exp^{-2\varepsilon t})\right], \qquad (4)$$

where we relabeled t^* back to t for simplicity.

By Taylor expansion in the limit $\epsilon \ll 1$ – the "weak friction" approximation – (4) becomes

$$y \simeq \frac{2v_0^2}{g}(t - t^2 - \varepsilon t^2). \qquad (5)$$

We see that in this case the equation $y = 0$ gives for the total flight time

$$t = \frac{1}{1+\varepsilon} < 1, \qquad (6)$$

which in terms of dimensional time means that if (weak) friction is present, the flight time τ is always smaller than the flight time without friction τ_0.

An interesting result is obtained if in (4) we take the limit $\varepsilon \to \infty$ (strong friction). One observes that in this case $y \to 0$ while for the flight time we get $t \searrow \frac{1}{2}$. Of course, one may question whether the linear in v friction is an appropriate approximation in this limit.

By graphical analysis (see Fig. 1) it can be seen that for $0 < \varepsilon < \infty$ the solution $t = t(\varepsilon)$ to the equation $y = 0$ in (4) is a monotonic function of ε, with $t \to 1$ as $\varepsilon \to 0$, and $t \to 1/2$ as $\varepsilon \to \infty$. Hence, we conclude that the (dimensional) flight time obeys

$$\frac{v_0}{g} < \tau \leq \frac{2v_0}{g}, \qquad (7)$$

with equality only when there is no friction.

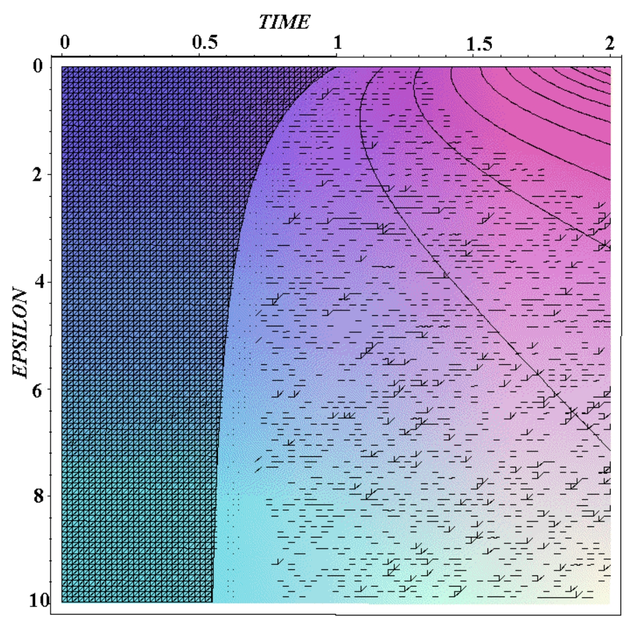

Figure 1: Plot of $y(t,\varepsilon) = (1/\varepsilon)[-t + (1/2)(1 + 1/\varepsilon)(1 - \exp(-2\varepsilon t))]$ and $y = 0$ surfaces in (ε, t) co-ordinates. For $\varepsilon \to 0$ the intersection curve of the two surfaces approaches the line $t = 1$, while for large ε it goes close to the line $t = 0.5$.

(b) *"Quadratic" friction case*

In this case the friction $F \propto v^2$ and the equations of motion are as follows:

(i) Motion *upward* (axis Oy directed upward, origin at the launch point)

$$\dot{v} = -kv^2 - g, \quad v(t=0) = v_0, \qquad (8)$$

and the solution is

$$v=\sqrt{\frac{g}{k}}\tan\left(\arctan\sqrt{\frac{k}{g}}v_0-\sqrt{kg}t\right). \quad (9)$$

The time to reach the highest point of the trajectory is

$$t_u=\frac{1}{\sqrt{kg}}\arctan\sqrt{\frac{k}{g}}v_0 \quad (10)$$

and the maximum height is

$$y_{max}=\frac{1}{2k}\ln\left(1+\frac{k}{g}v_0^2\right) \quad (11)$$

(ii) Motion *downward*: we choose the origin of the reference frame at the highest point (Oy directed downward). The equation of motion is

$$\dot{v}=-kv^2+g, \quad v(t=0)=0, \quad (12)$$

with the solution

$$v=\sqrt{\frac{g}{k}}\frac{\exp^{(2\sqrt{kg}t)}-1}{\exp^{(2\sqrt{kg}t)}+1} \quad (13)$$

which in turn gives for the vertical coordinate

$$y=\frac{1}{2k}\left[2\ln(1+\exp^{(2\sqrt{kg}t)})-2\sqrt{kg}t-2\ln 2\right] \quad (14)$$

By equating the above y with y_{max} we get the time of descent

$$t_d=\frac{1}{\sqrt{kg}}\ln\sqrt{1+2v_0^2\frac{k}{g}\left(1+\sqrt{1+\frac{g}{kv_0^2}}\right)} \quad (15)$$

We now denote $\epsilon=v_0\sqrt{k/g}$ and get the total time of flight $\tau=t_u+t_d$ as

$$\tau = \frac{v_0}{g}\left\{\frac{1}{\epsilon}\arctan\epsilon+ +\frac{1}{2\epsilon}\ln\left[1+2\epsilon^2\left(1+\frac{\sqrt{1+\epsilon^2}}{\epsilon}\right)\right]\right\} \quad (16)$$

After a bit of calculus it can be seen that in (16) the multiplier between the curly brackets is bounded to be in the $(0,2)$ interval as ϵ scans the $(\infty,0)$ interval.

Comparison with the time of flight for the frictionless motion, τ_0, shows that $0<\tau<\tau_0$.

Projectile motion with friction

Q: It is known that for a given launch speed v_0, the maximal range is achieved by shooting at $\alpha_0=45°$ (no friction case). If weak friction is present, to get the maximal range for a given v_0 should one aim higher or lower than α_0?

When there is no friction the dynamic equations are $a_x=0$, $a_y=-g$, from which we get

$$x=v_0t\cos\alpha, \quad y=v_0t\sin\alpha-(g/2)t^2. \quad (17)$$

Eliminating t between x and y expressions we get

$$y=x\tan\alpha-\frac{g}{2}\frac{x^2}{v_0^2\cos^2\alpha}. \quad (18)$$

For $y=0$ we get two solutions for x, namely the trivial $x=0$ and

$$x=\frac{v_0^2\sin 2\alpha}{g}. \quad (19)$$

For a given v_0 the range $x_{\max}$ is achieved for $\sin 2\alpha_0=1$, and hence $\alpha_0=45°$.

Now assume that there is friction of the form $\mathbf{F}_f=-k^*\mathbf{v}$. The equations of motion become

$$a_x=-kv_x,\; a_y=-g-kv_y, \quad (20)$$

with $k=k^*/m$. Since $\dot{v}_x=a_x$, $\dot{v}_y=a_y$, and the initial conditions are $v_x(t=0)=v_0\cos\alpha$, and $v_y(t=0)=v_0\sin\alpha$, from (20) we get

$$v_x = v_0\exp^{-kt}\cos\alpha, \quad (21)$$
$$v_y = \left(v_0\sin\alpha+\frac{g}{k}\right)\exp^{-kt}-\frac{g}{k}. \quad (22)$$

By integrating (21) and (22) one obtains

$$x=\frac{v_0\cos\alpha}{k}\left(1-\exp^{-kt}\right), \quad (23)$$
$$y=\frac{1}{k}\left(v_0\sin\alpha+\frac{g}{k}\right)\left(1-\exp^{-kt}\right)-\frac{g}{k}t, \quad (24)$$

which gives the equation of trajectory

$$y = \frac{g}{k^2}\left[\frac{kx}{v_0\cos\alpha}\left(1+\frac{kv_0\sin\alpha}{g}\right)+ +\ln\left(1-\frac{kx}{v_0\cos\alpha}\right)\right] \quad (25)$$

N.B. In the limit $k\to 0$, (25) recovers (18).

We consider the case of a small enough k, such that $\eta \equiv (kx/v_0 \cos\alpha) \ll 1$. We expand the logarithm up to the 3-rd order according to the formula $\ln(1-\eta) \simeq -\eta - \eta^2/2 - \eta^3/3$. We set then $y = 0$ and obtain that the resulting equation in x has the nontrivial solution

$$x = \frac{3v_0 \cos\alpha}{4k}\left(\sqrt{1 + \frac{16}{3}\frac{kv_0}{g}\sin\alpha} - 1\right) \quad (26)$$

N.B. In the limit $k \to 0$, (26) recovers (19).

To get the angle α which provides the extremal value of the range we request that $\partial x/\partial\alpha = 0$ (simple reasoning convinces us that the extremum is indeed a maximum). Denoting for shortness $16kv_0/3g = p$, one gets

$$1 - \sin^2\alpha = \frac{2}{p}\sin\alpha\left(1 + p\sin\alpha - \sqrt{1 + p\sin\alpha}\right) \quad (27)$$

Assuming for now the parameter p to be small, one may expand the square root in (27) and obtain the equation

$$f(z,p) \equiv \frac{p}{4}z^3 + 2z^2 - 1 = 0, \quad (28)$$

where $\sin\alpha = z$ and we look for solutions around $z_0 = \sqrt{2}/2 = \sin(\pi/4)$. One can easily see that the behaviour of $f(z,p)$ around z_0 is as depicted in Fig.2, whatever $0 < p << 1$ we choose. In particular, $f(z_0,p) > 0$, which means that the solution z_f for (28) obeys $z_f < z_0$, which in turn means that $\alpha_f < \pi/4$ (since $\sin\alpha$ is a monotonically increasing function for $\alpha \in (0,\pi/2)$). Therefore, in the "weak friction" approximation we have shown that for a given initial speed v_0, the maximal range is achieved by shooting at an angle α_f less that 45°.

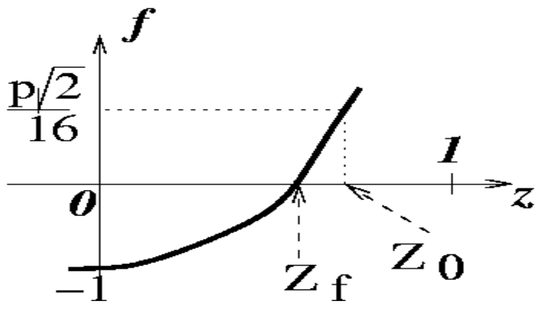

Figure 2: Behaviour of $f(z,p)$ in the $[0,1]$ interval.

Note 1: It can be seen that the smallness of p and η as defined above is related to the smallness of $k\tau_0$ (where $\tau_0 = \frac{2v_0 \sin\alpha_0}{g}$ is the time of flight in the frictionless case).

Indeed, $k\tau_0 \ll 1 \Leftrightarrow p \ll 1$, since $2\sin\alpha_0$ and $16/3$ are both $O(1)$. On the other hand we have

$$\eta = \frac{kx}{2v_0\cos\alpha} = k\tau_0\left(\frac{x}{x_{\max}}\right)\frac{\cos\alpha_0}{2\cos\alpha},$$

where $x_{\max}$ is given by (19) with $\alpha_0 = 45°$. Since we look for solutions with weak friction we expect that $x \le x_{\max}$ and $\frac{\cos\alpha_0}{2\cos\alpha}$ to be $O(1)$. It is true that $\eta \ll 1 \not\Rightarrow k\tau_0 \ll 1$, but this is not an issue here.

Note 2: One may set $\sin\alpha = 1$ in (24) and obtain the same results that we got in the first part (1-dimensional motion, case (a)).

As an exercise the reader is asked to try solving the 2-D projectile motion with "quadratic" friction, although the resulting system of ODEs is more complicated (there is no decoupling).
