Angle between the products of collision

These notes deal with the problem that we discussed in the tutorial. I decided to write down the solution, so that you can think it over in detail, while preparing for the test.

The problem: Suppose we have two bodies of equal masses m (the original problem, solved in the class by Dr. Harrison, made use of two protons). One of them is at rest, the other approaches it with a velocity $\vec{v}_0$. After the collision (*not* perfectly frontal), the two bodies leave the scene on trajectories defined by the angles α and β, relative to the direction of $\vec{v}_0$.

At the lecture, Dr. Harrison showed that if the collision is *perfectly elastic* (i.e. the kinetic energy of the whole system is conserved through the collision) then the angle between the trajectories of the products is $\theta = 90°$. That is, $\theta = \alpha + \beta = \pi/2$. See Fig 1 for the setup.

The question I asked in the tutorial was: **"if the collision is *not perfectly elastic*, is the angle θ smaller, equal to, or bigger than $90°$?"**.

By "not ...", I meant that the collision is not perfectly *inelastic* either — the bodies still separate and fly away after collision. The point is that during the collision some kinetic energy (KE) is being lost to heat, sound waves, etc., so that the total kinetic energy of the system after is less than what we had before the collision.

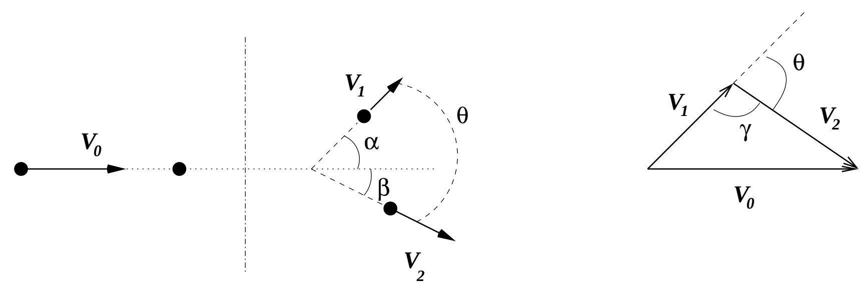

Figure 1: The setup: masses are equal (m); the velocity vectors are represented by bold-face symbols. On the right, because of the equality of masses the conservation of momentum reduces to the addition of velocity vectors.

During the collision, the momentum of the *whole* system is conserved. Therefore,

$$\vec{P}_i = m\vec{v}_0 + 0 = m\vec{v}_1 + m\vec{v}_2 = \vec{P}_f. \tag{1}$$

Since the masses are equal, the above equation becomes

$$\vec{v}_0 = \vec{v}_1 + \vec{v}_2. \tag{2}$$

On the other hand, we know that during the collision some kinetic energy is lost, therefore

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + Q, \tag{3}$$

where by Q I denote the lost kinetic energy (by the way I am defining it I assume that $Q > 0!$). Equation (3) can be rewritten as

$$v_0^2 = v_1^2 + v_2^2 + \frac{2Q}{m}. \tag{4}$$

Now, if you attended the lecture, you may recall that at this point, with KE conserved, we would have $v_0^2 = v_1^2 + v_2^2$. By using eq. (2) and a diagram like the one I draw in Fig. 1 (right) the lecturer came to the conclusion that the only way this could happen (the sum of the squares of two sides of the triangle equals the square of the third side) is for the angle γ to be 90°. Therefore, if KE is conserved, the angle between the outgoing trajectories is also 90°, since it is clear that $\theta = 180° - \gamma$.

In our case, from (4) we find instead that

$$v_0^2 > v_1^2 + v_2^2. \tag{5}$$

But in a triangle we always have

$$v_0^2 = v_1^2 + v_2^2 - 2v_1v_2\cos\gamma. \tag{6}$$

Given inequality (5), from (6) we conclude that $v_1v_2\cos\gamma < 0$. This points to the conclusion that γ is obtuse ($\cos\gamma < 0$), hence, its complementary angle θ satisfies $\theta < 90°$. See Fig. 1 for geometrical details.

The problem can be solved in a more formal way, with less geometry reasoning, as follows.

We know that the magnitude of any vector $\vec{A}$ and its dot-product with itself satisfy

$$A^2 = |\vec{A}|^2 = \vec{A}\cdot\vec{A}. \tag{7}$$

Hence, in our case

$$\begin{aligned} v_0^2 = |\vec{v}_0|^2 = \vec{v}_0\cdot\vec{v}_0 &= (\vec{v}_1 + \vec{v}_2)\cdot(\vec{v}_1 + \vec{v}_2) \\ &= \vec{v}_1\cdot\vec{v}_1 + 2\vec{v}_1\cdot\vec{v}_2 + \vec{v}_2\cdot\vec{v}_2 = v_1^2 + v_2^2 + 2\vec{v}_1\cdot\vec{v}_2. \end{aligned} \tag{8}$$

But from the *non*-conservation of KE we have the inequality (5), therefore, the above equation gives

$$2\vec{v}_1\cdot\vec{v}_2 > 0. \tag{9}$$

Hence, $\vec{v}_1\cdot\vec{v}_2 = v_1v_2\cos\theta > 0$ which means that $\cos\theta > 0$, therefore the angle $\theta = \angle(\vec{v}_1, \vec{v}_2) < \pi/2$.

Last revised: November 2, 2004; solution by Sorin Codoban.