Projectile motion

These notes present the solution of the problem #43 from Ch. 3 of Serway&Jewett.

A projectile is launched up an incline (incline angle ϕ) with an initial speed v_i at an angle θ_i with respect to the horizontal ($\theta_i > \phi_i$), (see the setup on the right of Fig. 1).

(a) Show that the projectile travels a distance d up the incline, where $d = \dfrac{2v_i^2 \cos\theta_i \sin(\theta_i - \phi)}{g\cos^2\phi}$.

(b) For what value of θ_i is d a maximum, and what is that maximum value?

Review of the task: (a) to check that the distance along the incline $d = |OP|$ is given by a certain formula, and (b) to find the value of θ which, for a fixed ϕ, gives us the maximum value of the range d. To solve the problem we have at least two choices for the co-ordinate system.

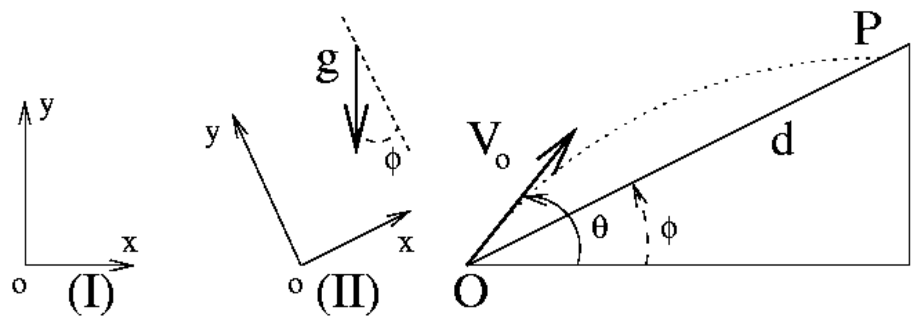

Figure 1: The setup. Some notations have been changed for the sake of clarity.

The choice (I) has the x axis along the baseline of the incline, while the second choice (II) has the x axis along the incline itself (see Fig. 1).

We write the kinematic equations the usual way. For the case (I) we have

$$x = (v_0 \cos\theta)t, \tag{1}$$

$$y = (v_0 \sin\theta)t - \frac{g}{2}t^2. \tag{2}$$

It's obvious that the point of impact with the incline (P) is a point on the trajectory $y = Y(x)$, with Y being the function one gets by eliminating t using (1) and (2). By doing so we obtain

$$y = x\tan\theta - \frac{g}{2}\frac{x^2}{v_0^2\cos\theta} \tag{3}$$

The coordinates of P $(x_P, y_P) = (d\cos\phi, d\sin\phi)$ have to satisfy (3), and hence we get

$$d\sin\theta = d\cos\theta\tan\theta - \frac{g}{2}\frac{d^2\cos^2\theta}{v_0^2\cos^2\theta}. \tag{4}$$

Equation (4) has two solutions. One of them is $d = 0$ as expected. The other one is[1]

$$d = -\frac{2v_0^2\cos^2\theta}{g\cos^2\phi}(\sin\phi - \cos\phi\tan\theta) = \frac{2v_0^2\cos\theta\sin(\theta-\phi)}{g\cos^2\phi}, \tag{5}$$

which coincides with the expression provided.

[1] A bit of trigonometry: $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$, $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$, $\sin\alpha\cos\beta = [\sin(\alpha+\beta) + \sin(\alpha-\beta)]/2$

For the case (II) the equations of motion read

$$
\begin{aligned}
x &= [v_0 \cos(\theta - \phi)]t - \frac{g \sin \phi}{2} t^2, && (6)\\
y &= [v_0 \sin(\theta - \phi)]t - \frac{g \cos \phi}{2} t^2. && (7)
\end{aligned}
$$

Apparently we got a complication, since both x and y are now quadratic in t. Fortunately, in the case (II) the condition which gives us the distance d is particularly simple, and removes the above mentioned obstacle. Namely, one observes that for the choice (II) we have $y = 0$ only at points O (where $x = 0$) and P (where $x = d$). Equation (7) then gives us

$$
0 = t\left(v_0 \sin(\theta - \phi) - \frac{g \cos \phi}{2} t\right), \tag{8}
$$

with two solutions for the flight time. One is the trivial $t = 0$, the other one, corresponding to the impact time at P is

$$
t = \frac{v_0 \sin(\theta - \phi)}{g \cos \phi}. \tag{9}
$$

Substituting (9) into (6) (see also footnote 1) we get the distance d along the incline, namely

$$
d = \frac{2v_0 \sin(\theta - \phi)}{g \cos \phi} \left\{ v_0 \cos(\theta - \phi) - \frac{g}{2} \sin \phi \frac{2v_0 \sin(\theta - \phi)}{g \cos \phi} \right\} = \frac{2v_0^2 \sin(\theta - \phi) \cos \theta}{g \cos^2 \phi}. \tag{10}
$$

Moral: both methods worked just fine.

Note: To get d we didn't divide anywhere by expressions which vanish when $\phi = 0$. Hence, we expect that in the limit $\phi \to 0$ the expression for d recovers a known result for projectile motion (*e.g.* find the range for given initial v_0 and angle θ with horizontal). In our case, for $\phi = 0$ we get $d = v_0^2 \sin 2\theta / g$, so the check for a limiting case is passed.

To solve the part (b) we also have at least two choices.

One choice would be to take the derivative of d with respect to θ, while keeping v_0, ϕ, g constant, and demand $\frac{\partial d}{\partial \theta} = 0$. This will give us θ for which d has an extremum or saddle point. We need then to check that this extremum is indeed a maximum.

The other choice is to rewrite the expression for d in such a way that the answer becomes obvious. We will pursue this path.

Let us note that for given v_0, ϕ, g (and kept constants) the only factor in the expression for d that depends on θ is $\sin(\theta - \phi) \cos \theta$. This can be rewritten as (see footnote 1)

$$
\sin(\theta - \phi) \cos \theta = \frac{1}{2} \{\sin(\theta - \phi + \theta) + \sin(\theta - \phi - \theta)\} = \frac{1}{2} \{\sin(2\theta - \phi) - \sin(\phi)\}. \tag{11}
$$

For a fixed ϕ the above expression reaches its *maximum* value for $\sin(2\theta - \phi) = 1$, which implies that $2\theta - \phi = \pi/2$. One may easily check that for $\theta, \phi \in (0, \pi/2)$ this is the only solution of $\sin(2\theta - \phi) = 1$. Hence, the angle which gives us the maximum d is $\theta = \frac{\pi}{4} + \frac{\phi}{2}$. Using (11), the maximum value of d is found to be

$$
d = \frac{v_0^2}{g \cos^2 \phi}(1 - \sin \phi) \tag{12}
$$

Note: In the limit case $\phi \to 0$, eqn. (12) gives us the well known result $d = v_0^2/g$.

Last revised: September 25, 2003