Interference

These notes present the solution of the problem #**9** from Ch. 14 of Serway&Jewett.

"Two speakers area driven in phase by the same oscillator of frequency f. They are located a distance d from each other on a vertical pole. A man walks straight toward the lower speaker in a direction perpendicular to the pole as shown on figure. (a) How many times will he hear a minimum in sound intensity, and (b) how far is he from the pole at these moments? Let v be the speed of sound and assume the ground doesn't reflect sound."

(a) For the beginning let us note that the path difference Δ between the signals received by **M** (see Fig. 1) is given by the formula

$$\Delta = x_2 - x_1 = \sqrt{d^2 + x_1^2} - x_1. \qquad (1)$$

We know (see also Example **14.1**, pg. 476) that at point **M** we get a minimum when the signals are out of phase by an odd multiple of $\pi\,(= 180°)$, or in other words, when Δ is either $\frac{\lambda}{2}$ or $\frac{3\lambda}{2}$ or $\frac{5\lambda}{2}$ and so on.

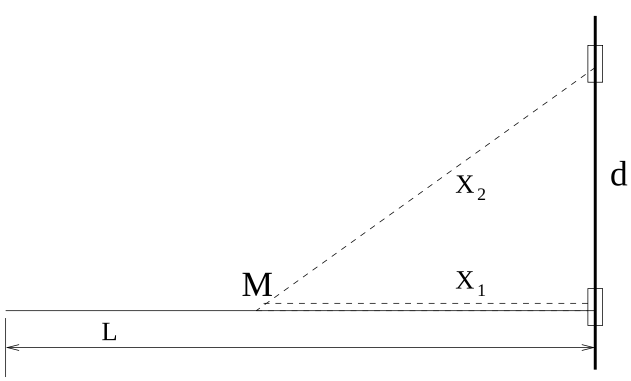

Figure 1: The setup.

There is a minimum in sound intensity whenever

$$\Delta = (2n-1)\frac{\lambda}{2}, \qquad n = 1, 2\ldots \qquad (2)$$

From (1) we notice that $0 < \Delta < d$, $(\forall)\, x_1 > 0$. This and (2) tells us that there is a maximum value of n for which we observe minima. To get that n we solve the inequality

$$(2n-1)\frac{\lambda}{2} \le d \qquad (3)$$

over integer numbers. This gives

$$n \le \frac{d}{\lambda} + \frac{1}{2} = \frac{df}{v} + \frac{1}{2}, \qquad (4)$$

and therefore $n_{\max}$ (the maximum number of minima the man can hear while approaching the pole) is the greatest integer equal or less than $\frac{df}{v} + \frac{1}{2}$.

N.B. On **Fig**2 the lines of constant path (phase) difference between the signals sent by the speakers are depicted by dotted lines. One may see that those curves are hyperbolas since the equation describing them is $\delta = x_2 - x_1 = \text{const}$, (we consider the 2-dimensional problem; only the region below the midline and to the left of the pole is shown). Indeed, a hyperbola is the locus of points for which the difference between the distances to two fixed points named focuses – in our case the focuses are the speakers – is a constant. By varying that constant (i.e. δ) we get different hyperbolas. As one can see, for the middle line $\delta = 0$ (both signals travel the same distance to reach points on that line) while the hyperbolas corresponding to first ($n = 1$), second ($n = 2$) minima have $\delta = \lambda/2$, $3\lambda/2$ a.s.o.

The hyperbola for the n-th destructive path difference (for which $\delta = (2n-1)\lambda/2$) crosses the pole at some point which is at distance z_n from the bottom speaker. In terms of distances x_2, x_1 (now taken along the pole) we get

$$\delta_n = x_2 - x_1 = (d - z_n) - z_n = (2n-1)\lambda/2. \qquad (5)$$

From the geometry of the problem one can see that the number of minima on the ground (denoted as $n = 1, 2\ldots$) is the same as the number of intersection points of the hyperbolas with the vertical pole. You may think of this as being the conservation of the number of "field lines". Therefore, the number of minima one can hear on the ground is not larger than the number of hyperbolas crossing the bottom half of the pole - as shown on Fig.2. On the figure we see that $0 < z_n < d/2$. From $z_n > 0$ we get that

$$n - \frac{1}{2} < \frac{fd}{v}, \qquad (6)$$

which gives the same $n_{\max}$ as the one found from (4).

(b) We can solve this part of the problem by either using the results of the previous part or by starting from scratch, with an alternative approach. We discuss first the alternative approach, but, for reasons to become clear shortly, we'll solve the problem using the method used in (a).

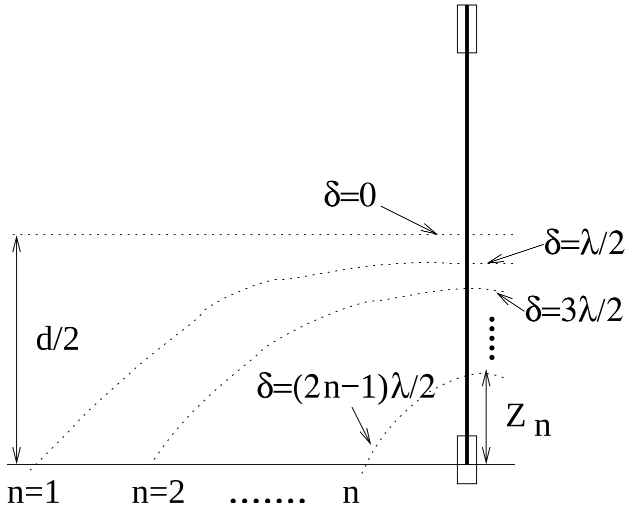

Figure 2: Diagram for the alternative way of determining the number of minima.

<u>"Wrong" solution:</u> Let us assume that the equations describing the disturbances propagating from the lower and the upper speakers are

$$y_1 = A\sin(kx_1 - \omega t),\ \ y_2 = A\sin(kx_2 - \omega t) \quad (7)$$

where we accounted for the fact that the waves are driven in phase ($\phi = 0$). The amplitude of the signal at some point $\mathbf{M}$ is given by

$$\begin{aligned} y &= y_1 + y_2 \\ &= 2A\cos\left(\frac{k(x_2 - x_1)}{2}\right)\sin\left(\frac{k(x_1 + x_2)}{2} - \omega t\right). \end{aligned}$$

To get a minimum sound amplitude the argument of cos() has to be of the form $(2n-1)\dfrac{\pi}{2}$, which makes the cos to be 0.

On a second look at the setup we observe that there are a few important caveats: the amplitude A is NOT constant – it depends on the distances x_1, x_2 from the sources. This is because the amplitude of the wave has to decrease when the distance from the source increases! (think of ripples on water surface). Indeed, at some radius r the energy is the integral of the power P over a period, the integral being also taken along the length of the circle. Since $P \propto \omega^2 A^2 v$ (see pg. 451 in S& J) and ω, v are constants, the only way to get the same energy for different r is to have an A decreasing with r. Specifically, we expect the product $A^2 r$ to be a constant, and hence $A(r) \propto 1/\sqrt{r}$.

We conclude that (7) with constant A is appropriate for 1-dimensional but not for 2-dimensional motion.

Besides, the amplitude of y_1 and y_2 of the waves on the line L are different (because $x_1 \neq x_2$ and because A is a function of x, as argued above). Therefore we do not pursue this approach, although it may give us the correct answer with some more accurate mathematical modeling of the waves.

To solve the problem correctly we return to the path difference derived in part (a). The condition for the minima is again that

$$\Delta = (2n-1)\frac{\lambda}{2}, \quad n = 1, 2\ldots \quad (8)$$

which solved for x_1 gives

$$(x_1)_n \equiv L_n = \frac{d^2 - \left[(2n-1)\dfrac{\lambda}{2}\right]^2}{\lambda(2n-1)} \quad (9)$$

Using the relation $\lambda = vT = v/f$ we finally get

$$L_n = \frac{d^2 - \left(n - \dfrac{1}{2}\right)^2 \dfrac{v^2}{f^2}}{\dfrac{2v}{f}\left(n - \dfrac{1}{2}\right)} \quad (10)$$

N.B. It is of interest to understand why do we get a correct answer even with the wrong formula for amplitude (see the notes above, and finish the calculations regarding the argument of cos()). My take is that although the amplitude A is a function of distance from the source, the minimal amplitude is still determined by the condition (8) in both cases (in the "wrong" case cos() = 0 gives exactly the same equation for $(x_1)_n$ as for the correct solution!).