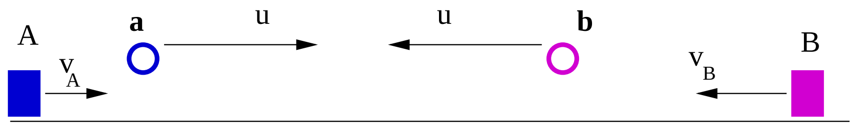

Figure 1:

A geometric solution to a kinematics problem

A few days ago I learned about a variation of a classic problem – the one with a bird flying between the trains (again, I can't find a reference - I guess the problem is part of physics folklore by now).

In its canonic form the problem goes along the line: two trains move towards each other, starting from some initial relative distance D. The speeds of the trains are constant and known. In between the trains there is a bird flying back and forth, with constant speed u. The question is then "what is the total path the bird covers before the trains finally collide?"

The new version I've mentioned has been apparently circulated to high-school students in a Toronto school, circa 2008.

Anyway - here is the complication: imagine you have not one but *two* birds that fly in between the trains. Say, bird **a** starts from train A (flying towards B), and bird **b** starts flying from train B (towards A). The birds fly in a straight line until they meet each other, at which point they make sharp turns and fly back to "their" trains. The speeds of the two birds are the same (u), while the speeds of the trains are v_A and v_B, respectively. For numerical application consider $D = 100\text{km}, u = 50\text{km/h}, v_A = 20\text{km/h}, v_B = 30\text{km/h}$. See Figure 1 for the setup.

Now, here is the novelty: besides asking the total path each bird covers (which can easily be solved the same way as for the classic 1-bird problem), you are now asked to find the distance covered by each bird while flying to the right ($d_{aR},\ d_{bR}$) and to the left ($d_{aL},\ d_{bL}$). Got the picture?

It turns out there is a simple (algebraic) solution (thanks Dmitry), although I for one totally missed it.... Shame! For each bird you end up with a system of two linear equations for the distances ($d_{aR},\ d_{aL}$) and ($d_{bR},\ d_{bL}$) respectively, and the solution is really straightforward. Nevertheless, I managed to solve the problem, but in a convoluted, geometric way. I think this sort of solution might have been the "way to go" in Newton's era or perhaps earlier, in antiquity. And it works beautifully – see below.

Somehow, I think looking for this sort of non-standard solutions is a cool thing to do (for a similar un-orthodox approach see another problem [on my web-page], in which I used the center of mass concept to show that the location of the pirates' treasure doesn't depend on the order one counts the trees).

Now let's get back to the birdie problem. In Figure 2 I plotted the paths of the trains and birds as a function of time. To simplify things (and keep the mathematics clean) I non-dimensionalized both the spatial and the time coordinates (hence both x/D and t/T go from 0 to 1). Here T is the total time we have the action going - that is 2h that it takes the trains to collide.

The thin lines 1E and 0E depict the paths of the trains A and B, respectively. The paths of the birds (thick lines) are of course bounded by the trains, as can be seen. The first stretch of the birds' flights lasts until $t/T = 0.5$, at which point the birds turn around

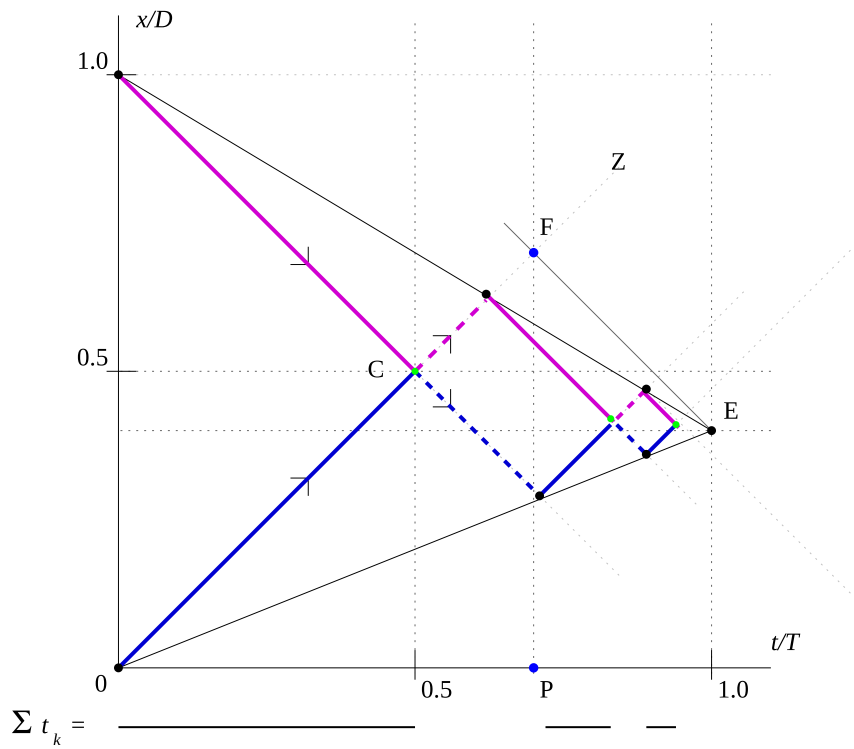

Figure 2:

(dashed solid lines) and fly towards their trains, then turn back again, etc, etc. I guess the figure is self-explanatory. Note however, that the angle $\angle 1C0$ is $\pi/2$ by happy coincidence (see also the note at the end). We are going to use the $\pi/2$!

Since the speed of the birds is constant, finding the distance travelled to the right (left) amounts to finding the "time projection" of the stretches depicted with solid (dashed) thick lines – I am talking here about the bird **a**, which starts flying from point 0 and has it's path shown with blue lines (alternating solid and dashed). Similar arguments apply to bird **b**, with obvious changes.

The distance bird **a** flies "to the right" will then be $d_{aR} = u \cdot \sum_k t_{aR\,k}$. Well, this is clearly easier said than done - what we have is a fill-gap-fill-gap-… infinite sequence of segments to sum up!

Now look diagonally across the picture from the bottom-right corner. Do you see something?

If you choose the viewpoint just right, the *solid* blue lines (depicting the bird **a** flying to the right) will cast their shadows on the 0Z line in such a way that there is no gap between them (and the blue *dashed* lines corresponding to bird **a** do not cast a shadow at all on the OZ line !!).

Evrika!

Summing up the infinite sequence of fill-gap-fill-gap-… segments reduces to calculating the (rectangular) projection of the segment OE onto 0Z. That gives 0F - which is clearly the sum of the shadows of the solid blue line segments that represent the bird **a** flight to the right.

We then project 0F onto the time axis and voilá – we get the total time spent by the bird in its flight "to the right".

Of course there are some circumstances that helped (e.g., constant u, the $\pi/2$ angle between the paths of the birds, and $t/T = 0.5$ for the first time of the birds encounters – to name a few), but the idea is clearly workable.

The nitty-gritty details, involving a bit of trigonometry (playing around with sin and arccos) are left to the reader.

The final answer is $d_{aR} = 70\text{km},\ d_{aL} = 30\text{km},\ d_{bR} = 20\text{km},\ d_{bL} = 80\text{km}.$

Note: if the speeds of the birds are not equal then we may switch to the "birds center of mass" reference frame, where their speeds are equal (of course, assuming birds have equal masses).

If angle $\angle 1C0$ is not $\pi/2$ then we can try stretching the time axis, etc, etc... or perhaps perform an "oblique" projection of the relevant segments. It becomes ugly, I guess – perhaps it then makes more sense to stick with the algebraic solution, which is really simple and robust. At least we had a pleasant detour into the geometry of the kinematics...

Last updated December 3, 2008; Solution by Sorin Codoban