On the projectile motion

These notes discuss a problem started but not finished in the tutorial :*(

Recall the problem: assume one has a setup for projectile motion as depicted on Fig. 1. The questions are:

(a) What is the *minimal* initial velocity $\mathbf{v_0}$ such that the projectile falls beyond the cliff's edge.

(b) Assuming that the initial velocity $\mathbf{v_0}$ is such that the projectile falls beyond the edge (situation depicted in Fig. (1)) find the distance d from the foot of the cliff at which the impact takes place.

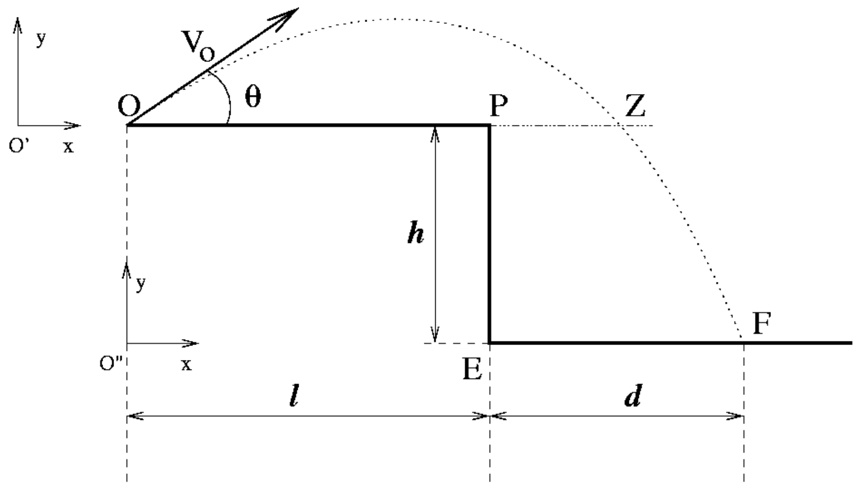

Figure 1: The setup.

(a) This question is related with the one raised by you in the tutorial: whether or not one may use the "common" projectile motion results for situations as the one depicted on Fig. 1. The answer is *"yes"*: as long as you look at what happens above the level of the launch line (OZ) the formulae for the (max) range, time of flight a.s.o. apply without change.

Therefore, if one wants to know which is the minimum value of velocity (to be more precise: the magnitude $|\mathbf{v}_0|$ and the angle θ) which still gets the projectile beyond the edge (point P), there is no need to solve the problem from scratch.

We solve this part of the problem in the frame (xO'y) and also take $O' \equiv O$. The range in the horizontal is given by

$$x_{max} = \frac{v_0^2 \sin 2\theta}{g}. \tag{1}$$

For a given $|\mathbf{v}_0|$ the maximum range is reached for $\theta = 45^o$, and therefore by requiring $x_{max} \geq \ell$ we obtain the minimal value of v_0; it is found that $(v_0)_{min} = \sqrt{g\ell}$ and for this value of v_0 we need to shoot at $\theta = 45^o$.

At this point one may think of a related problem: assume we are given $v_0 > \sqrt{g\ell}$. What is the range of launch angles which allow the projectile to fall over the edge? We do not follow this question – it's only math beyond this point....

(b) To solve this part of the problem it is convenient to work in the frame (xO"y), for which the x-axis is at the level of the baseline EF.

The equations of motion are

$$\begin{aligned} x &= (v_0 \cos\theta)t, & (2) \\ y &= h + (v_0 \sin\theta)t - \frac{g}{2}t^2. & (3) \end{aligned}$$

At this point we have two choices.

The step-by-step method: set $x = \ell + d$, express time t from (2), replace this t into (3) and then impose $y = 0$ (since point F –which sits on the trajectory– has the co-ordinates $(\ell + d, 0)$) and solve for d.

The other method[1] would be to express y as a function of x (by eliminating time t using (2) and (3)) and then set both conditions at once ($x = (\ell+d), y = 0$) in the resulting equation. The result is

$$y = 0 = h + (\ell + d)\tan\theta - \frac{g}{2}\frac{(\ell + d)^2}{v_0^2 \cos^2\theta}. \tag{4}$$

For convenience we let ℓ and d together (do not expand the brackets!) and solve the quadratic equation (4). The solutions are

$$(\ell + d)_{1,2} = \frac{v_0^2}{2g}\sin 2\theta \left(1 \pm \sqrt{1 + \frac{2gh}{v_0^2 \sin^2\theta}}\right). \tag{5}$$

We choose the solution with "+" (the other one is negative and doesn't have physical meaning in our setup). Please, note that to get the "cosmetic" view of the solution I had to manipulate a bit the formula (advice - try to do it yourself). On this stage one needs to acknowledge that certain values of θ may cause problems, in the sense that if $\sin\theta = 0$ we get into troubles. Nevertheless, for our setup these values do not enter the game.

Finally, the answer for d is

$$d = \frac{v_0^2}{2g}\sin 2\theta \left(1 + \sqrt{1 + \frac{2gh}{v_0^2 \sin^2\theta}}\right) - \ell. \tag{6}$$

Discussion: What happens if $v_0 = \sqrt{g\ell}$ and $\theta = 45^o$? In this case (assume we do not hit P...)

$$d = \frac{v_0^2}{2g}\left(1 + \sqrt{1 + \frac{4gh}{v_0^2}}\right) - \ell.$$

Is this the minimum distance from the foot of the cliff that one can ever achieve with *any* combinations of v_0 and θ ? NO, it is not! One may think of the situation when we shoot at a very steep angle (close to 90^o) with a large v_0. In that case the projectile may fall closer to the foothill. In fact, there is nothing special with this (v_0, θ) combination from the point of view of d.

However, it is important to keep in mind that formula for d (6) makes sense only if the projectile can make it over the edge of the cliff :-), and that depends on (v_0, θ).

[1] I prefer this second method, although the first one may seem more "physical".