The RC circuit

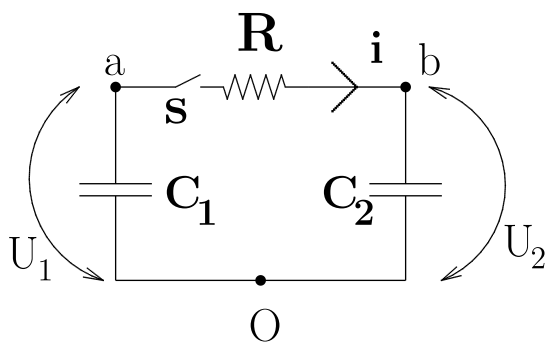
Let us consider a simple RC circuit, as depicted in the diagram below. Assume for the beginning, that the switch **s** is open, $\mathbf{C_2}$ has no charge and that $\mathbf{C_1}$ has charge q_0 ($+q_0$ on the top plate, $-q_0$ on the bottom one). Let E_0 denote the energy on $\mathbf{C_1}$ in this initial setup. Let us close the switch.

Our task is to find:

i) the time constant τ of this circuit.

ii) which are the energies (E_1, E_2) "deposited" on each capacitor after a very long time ($t \to \infty$).

iii) how does the energy loss $\Delta E = (E_1 + E_2) - E_0$ depend on the value of $\mathbf{R}$?



1. Low-tech solution

We don't want to get into complicated math. To solve the problem we use the conservation laws in the asymptotic regime ($t \to \infty$).

i) The time constant is that of a circuit with a resistor R connected in series with an equivalent capacitor C_e. Obviously,[1] C_1 and C_2 are connected in series, so that

$$\frac{1}{C_e} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_e = \frac{C_1 C_2}{C_1 + C_2}.$$

The time constant is therefore

$$\tau = RC_e = \frac{RC_1C_2}{C_1 + C_2}. \tag{1}$$

ii) The charging/discharging of the capacitors ends when there is no current flowing through the circuit. In particular, this means that when the current vanishes the potential drop across the resistor R is 0, i.e. $U_R = V_b - V_a = 0$. This means that $V_a = V_b$, where we calculate the potential relative to a reference point (say O, as depicted). But $U_1 = V_a - V_O$, $U_2 = V_b - V_O$. The implication is that the transient process stops when the voltage drops on C_1 (U_1) and C_2 (U_2) are equal. This allows us to calculate the state (i.e. charge, voltage) of each capacitor.

Indeed, the total charge is conserved

$$q_0 = q_1 + q_2, \tag{2}$$

and, as noted above

$$\frac{q_1}{C_1} = U_1 = U_2 = \frac{q_2}{C_2}. \tag{3}$$

One may solve (2) and (3) for q_1, q_2 and calculate the energy on each capacitor

$$E_1 = \frac{1}{2}\frac{q_1^2}{C_1} = \frac{1}{2}\frac{C_1 q_0^2}{(C_1 + C_2)^2}, \tag{4}$$

$$E_2 = \frac{1}{2}\frac{q_2^2}{C_2} = \frac{1}{2}\frac{C_2 q_0^2}{(C_1 + C_2)^2}. \tag{5}$$

It is readily seen that

$$E_0 = \frac{1}{2}\frac{q_0^2}{C_1} > \frac{1}{2}\frac{q_0^2}{(C_1 + C_2)} = E_1 + E_2. \tag{6}$$

iii) As can easily be found from the above expression, the energy loss is

$$\Delta E = -\frac{q_0^2}{2C_1}\frac{C_2}{C_1 + C_2} < 0, \tag{7}$$

and does not depend on R at all! The energy dissipated during the discharging of C_1 and charging of C_2 does not depend on the value of the resistor on which the dissipation (by heat generation) takes place. Puzzling, isn't it?

2. High-tech solution

i) The Kirchhoffs's 2nd rule for the RC circuit reads

$$U_1 = iR + U_2, \tag{8}$$

where

$$U_1 = \frac{q_1}{C_1},\ U_2 = \frac{q_2}{C_2},\ i = -\frac{dq_1}{dt} = +\frac{dq_2}{dt}, \tag{9}$$

[1] Perhaps it's a bit premature to say "obvious". The "high-tech", mathematically based solution will show that the capacitors behave as if connected in series.

with "$-$" in front of dq_1/dt because q_1 is decreasing ($dq_1 < 0$), but according to (8) we have $i > 0$ (the clockwise direction is the positive one).

The circuit is isolated from the point of view of charge (i.e. the charge is conserved), which means that at any time t we have $q_1(t)+q_2(t) = q_0$. However, the system it is not isolated from an energetic point of view - as time goes by, there is energy dissipated (to heat) on R.

Using (9) one may rewrite (8) as an ordinary differential equation (ODE) for q_1:

$$\frac{dq_1}{dt} + \frac{q_1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{q_0}{RC_2}. \tag{10}$$

Right away we may claim, by carefully inspecting (10), that the time constant of the circuit is the inverse[2] of the coefficient in front of q_1. Hence,

$$\tau = R\frac{C_1C_2}{C_1 + C_2}. \tag{11}$$

Indeed, equation (10) is of the form

$$\frac{dx}{dt} + ax = b, \tag{12}$$

with a, b constants. For $b = 0$ it has a solution of the form

$$x(t) = x_0e^{-at}. \tag{13}$$

Since we know that the charge/discharge of the capacitors is described by an exponential (relaxational) law — of the sort $\exp(-t/\tau)$ — by comparison we get (11).

It is true that in our case $b \neq 0$, but it is also true that this doesn't matter for the way we find τ (b is not a function of time, and unlike a it is not coupled to x, hence one may expect that its value doesn't impact the time constant of the relaxational process). Indeed, b is given by the asymptotic condition on the problem — actually $b = ax$ when $dx/dt = 0$; this situation occurs when the charge doesn't change anymore, i.e. when the current $i = dx/dt = 0$. In our case this happens for $t \to \infty$.

[2]The reader may recall a similar situation in the "simple harmonic oscillator" problem. In that case one rearranges the equation of motion in such a way that it reads $\frac{d^2x}{dt^2} + \omega^2x = 0$, and right away finds the period of oscillation via $T = 2\pi/\omega$.

The reader familiar with the theory of ODE's will recognize in the above reasoning a mere rederivation of the general rule that the solution to a (non-homogeneous, like (12)) ODE is the solution of the homogeneous ODE (with $b = 0$) plus a particular solution of the non-homogeneous ODE (which we find by an educated guess, or by other methods). One can easily verify that the solution of (10) is

$$q_1(t) = \frac{q_0C_2}{C_1 + C_2}e^{-t/\tau} + \frac{q_0C_1}{C_1 + C_2}, \tag{14}$$

where we accounted for the initial condition $q_1(t = 0) = q_0$ to determine the coefficient in front of the exponent; τ is given by (11).

ii) We now reproduce the results of the "low-tech" section. Finding the (final) charges on the capacitors proves to be simple. For q_1 we find the answer from (14) by taking $t \to \infty$, while q_2 is obtained from the conservation law (2).

$$q_1 = \frac{C_1}{C_1 + C_2}q_0, \quad q_2 = \frac{C_2}{C_1 + C_2}q_0. \tag{15}$$

The values for E_1 and E_2 are derived as in (4) and (5), and we get the same ΔE as in (7).

iii) Let us see why one gets the somewhat puzzling result that the energy loss *does not* depend on R. From (14) one may derive the expression for the current

$$i = -\frac{dq_1}{dt} = \frac{q_0}{RC_1}e^{-t/\tau}, \tag{16}$$

and then calculate the energy loss by integrating w.r.t. time the power dissipated on R

$$\begin{aligned} |\Delta E| &= \int_0^\infty i^2R\,dt = \frac{q_0^2}{RC_1^2}\int_0^\infty e^{-2t/\tau}dt \\ &= \frac{q_0^2}{RC_1^2}\left(-\frac{\tau}{2}e^{-2t/\tau}\right)\Big|_0^\infty = \frac{q_0^2C_2}{2C_1(C_1 + C_2)}. \end{aligned} \tag{17}$$

Indeed, R cancels out in the final result!

What happens if $R = 0$? Do we still have energy loss? Well, the answer is *yes*, because there is no such thing as $R \equiv 0$ for the real-life circuits (save for, maybe, the superconducting version of it). If one takes the limit $R \to 0$ then (17) holds all the way to 0.

For more on RC circuits read Ch. 21 in Serway & Jewett "Principles of Physics" (3rd ed.).
