Battery & resistor games

These notes are a wrapped-up version of the problem solved in the tutorial. It is a well known problem in Physics and Engineering (although I don't have a precise reference for it).

The problem: Assume we have the DC circuit as sketched in Figure 1. The battery has the EMF $\mathcal{E}$ and internal resistance r. The external resistor has resistance R which can be varied (from 0 to ∞). The task: find the value of R for which the power P_R delivered to/dissipated on R reaches its maximum.

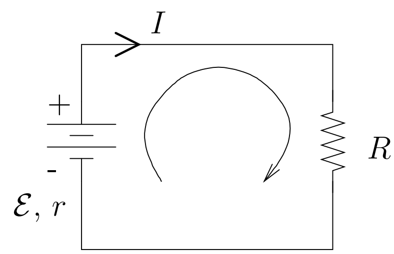

Figure 1: Circuit diagram for one battery + resistor

To solve the problem we apply Kirchhoff's law for the loop. We have

$$\mathcal{E} = Ir + IR, \tag{1}$$

where I stands for the current through the circuit. The power delivered to R is given by

$$P_R = P(R) = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}. \tag{2}$$

At this point we can just simply take the derivative of $P(R)$ with respect to R, set it to zero and solve for R. The function $P(R)$ at the solution will have an extremum; to check that we have a maximum we would need to take the second derivative, $P''(R)$ and see if it is negative when evaluate at the "suspect" point.

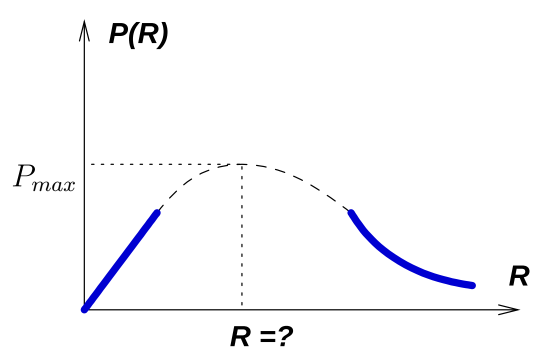

Figure 2: A sketch of the P(R) behaviour

It would be a bit more educating to proceed at a slower pace, with a bit less mathematics (at least at the beginning). Let us first stare a bit at equation (2), which is now rewritten in the form

$$P(R) = \frac{\mathcal{E}^2}{r^2} \frac{R}{\left(1 + \frac{R}{r}\right)^2}. \tag{3}$$

Now, it becomes obvious that for $R \to 0$ the expression for P_R also goes to 0, and almost linearly (indeed, in the denominator we have $(1+R/r)^2 \approx 1$, hence $P_R \propto R$ in this regime). For $R \to \infty$ the picture is different, but $P_R \to 0$ as well. In this case the denominator may be approximated by $(R/r)^2$; therefore, the $P(R)$ behaves like $R/(R/r)^2 \propto 1/R$. The behaviour of $P(R)$ in both extreme cases is sketched in Fig. 2. Clearly, there must be a value(s) of R for which $P(R)$ reaches maximum.

To find the value of R for which P_R reaches the extremum we still need to do the math. The first derivative on "the top of the hill" has to vanish:

$$\frac{dP(R)}{dR} = \mathcal{E}^2 \frac{r-R}{(r+R)^3} = 0. \tag{4}$$

The unique solution of the above equation is $R = r$. To convince yourself that this value provides a maximum for P_R you have to take the second derivative of $P(R)$ and check it is negative when evaluated at $R = r$. This task is left to the reader.

For completeness, let us calculate the value of P_{max}. We have

$$P_R\Big|_{R=r} = \frac{1}{4}\frac{\mathcal{E}^2}{r} = \frac{1}{4} I_{short}^2 r. \tag{5}$$

Here I_{short} stands for the shortcircuit current – this is the maximum current the battery can deliver.

One may complicate the problem by considering (see Fig. 3) two batteries connected in parallel and delivering power to a (variable) resistor R. The reader is again advised to perform the same task: find the condition R should obey for the power dissipated on it to reach its maximum. As an incentive here is the answer for the power:

$$P_R = \frac{\left(\frac{\mathcal{E}_1}{r_1} + \frac{\mathcal{E}_2}{r_2}\right)^2}{\left(1 + R\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right)^2} R. \tag{6}$$

The power is maximum when

$$R = r_e = \frac{r_1 r_2}{r_1 + r_2} \tag{7}$$

and the expression for the maximum power is similar to (5), with r being replaced by r_e and I_{short} being now the sum of the shortcircuit currents.

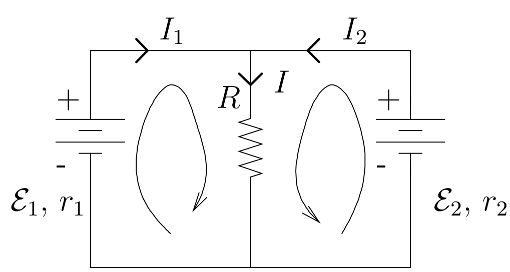

Figure 3: The circuit for the 2 batteries + resistor case