Capacitor “games”

How long it takes to charge a capacitor?

This problem is well-known in the Physics folklore. I heard it myself from one of my high school teachers (I. Bartos-Elekes).

Let us consider a circuit like the one depicted in Fig. 1. The capacitors have equal capacitances ($C_1 = C_2 = C$), and at the beginning there is no charge on capacitors. The voltage provided by the battery is U. For numerical application later on we shall use $C = 1\mu F$ and $U = 10V$. One sets the switch to position **1** and waits some time for C_1 to get fully charged. Obviously, the final charge on the capacitor is

$$q_0 = UC. \tag{1}$$

The switch is then set to position **2** and the charge on C_1 gets redistributed on both capacitors. After some time[1] the charges on C_1 and C_2 reach their equilibrium values q_1 and q_2, respectively. We then set back the switch to **1**, and charge C_1 again, up to q_0, and repeat the above steps again and again.

Question: how many times does it make sense to continue the back-and-forth switching? Let us see what happens after a few steps of switching, and hope for the best, that the reason for which we should be aware of “makes sense” will arise in our way.

Let us denote by $q_1^{(1)}$ and $q_2^{(1)}$ the charges on C_1 and C_2 after the first cycle. We have, by charge conservation

$$q_1^{(1)} + q_2^{(1)} = q_0, \tag{2}$$

and from the fact that the capacitors (for the switch in position **2**) are in parallel, the potential difference between their plates should

[1] since we are looking at an idealized problem with no time constants, we may assume the redistribution process to be almost instant

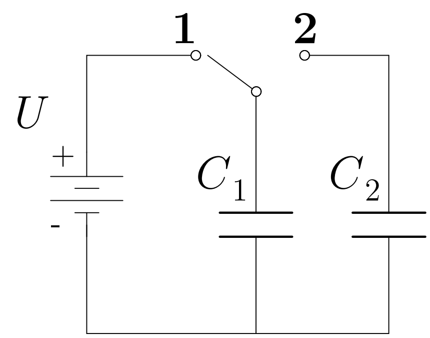

Figure 1: The setup.

be the same, therefore

$$\frac{q_1^{(1)}}{C_1} = \frac{q_2^{(1)}}{C_2}. \tag{3}$$

Solving (2) and (3) for $q_1^{(1)}$ and $q_2^{(1)}$ one gets

$$q_1^{(1)} = q_2^{(1)} = \frac{1}{2}q_0. \tag{4}$$

Next, the switch is placed again in position **1** and the charging of C_1 resumes — the final charge on it is again q_0.

Pitfall prevention: when I say “the charge on the capacitor is q_0” I mean that the upper plate has charge $+q_0$ on it, while the bottom plate has $-q_0$. Therefore, the above and the forthcoming calculations deal, *rigorously speaking* with e.g., the charges on the upper plates of the capacitors.

The next round of “charge transfer” between capacitors starts by setting the switch to position **2**. Now, the total charge on the upper plates is $q_0 + q_2^{(1)}$; therefore, the new charge balance equation, at the end of the second cycle reads

$$q_1^{(2)} + q_2^{(2)} = q_0 + q_2^{(1)} = q_0 + \frac{1}{2}q_0 = \frac{3}{2}q_0. \tag{5}$$

Besides (5), the equation (3) is still valid here, but with the superscript (1) replaced by (2).

Since the capacitances are equal, at the end of the day we get

$$q_1^{(2)} = q_2^{(2)} = \frac{3}{4} q_0 = \frac{2^2 - 1}{2^2} q_0. \tag{6}$$

One may continue this "game" and observe that the charge on capacitor C_2 builds up continuously. It can be shown (please, check this for your own good) that after the n–th switching, the charge on C_2 is

$$q_2^{(n)} = \frac{2^n - 1}{2^n} q_0. \tag{7}$$

At this point it becomes clear (hopefully) what do they mean by "makes sense". The difference between the charge q_0 which C_1 brings for sharing with C_2 and the charge already on C_2 from previous steps decreases continuously. The difference is

$$\Delta q^{(n)} \equiv q_0 - q_2^{(n)} = \frac{1}{2^n} q_0. \tag{8}$$

One can observe that $\Delta q^{(n)}$ never vanishes, although in the limit $n \to \infty$ we get $\Delta q^{(n)} \to 0$. But the difference does not have to vanish for the switching to be stopped — at some point the $\Delta q^{(n)}$ becomes less than the charge of an electron $|e|$. And you cannot cut an electron in pieces, to continue the process! That is the moment when one should cease the switching.

What is left is pure math: one should find n for which $\Delta q^{(n)} = |e|$. We have

$$\Delta q^{(n)} = \frac{1}{2^n} q_0 = |e| \Rightarrow n = \frac{1}{\ln 2} \ln \frac{q_0}{|e|}. \tag{9}$$

With the numerical values for U and C given at the beginning the final result is

$$n \approx 46.$$

Inspired by the above results one may think of a related problem. *In real–life capacitor charging, when shall we consider the charging/discharging to end: at $t \to \infty$ or earlier?*

Let us consider the discharge of a capacitor through a resistor (R). The charging problem is similar. The charge on capacitor obeys

$$q(t) = q_0 \times \exp\left(-\frac{t}{\tau}\right), \tag{10}$$

where $\tau = RC$, and q_0 is the initial charge on capacitor ($q_0 = UC$) at time $t = 0$.

Again, with the same reasoning, the discharging is "done" when

$$q_0 \exp^{-t/\tau} \approx |e|. \tag{11}$$

Solving for t in terms of τ one gets

$$t \approx \tau \ln \frac{q_0}{e} \approx 32\tau,$$

where the numerical estimate is based on the values quoted at the beginning of these notes.

The same result applies to the charging process. It doesn't make sense to wait longer than a few tens of time constants, and this is because the difference between the asymptotic target value q_0 and the charge on capacitor $q(t) = q_0(1 - \exp(-t/\tau))$ becomes less than an electron charge after that time interval. The real–life case is a bit more complex: there will be stray electrons "floating" around (say, created by photoelectric effect) – so that the problem is not so clear-cut as here. Nevertheless, the conclusion is the same — it does not make sense to wait an infinite time to get the job done.

01/26/06, rev. 2a Written by Sorin Codoban