Calculating g from a cubic fit

The data analysed in these notes are for the *plastic ball* (the Free fall exp.).

The theory says that if the friction is accounted for, the dependence $s = s(t)$ has the form

$$s = v_0 t + \frac{1}{2}(g - \alpha v_0^2)t^2 - \frac{1}{3}\alpha v_0 (g - \alpha v_0^2)t^3 + \ldots, \tag{1}$$

where the dots stand for the terms[1] with higher powers of t. To check this prediction we fit our data (s, t) to a polynomial of the form

$$s = A_1 t^1 + A_2 t^2 + A_3 t^3. \tag{2}$$

One has to compare the coefficients of t's in (1) and (2), and solve for g, v_0, α. We obtain

$$\begin{aligned} g &= 2A_2 - \frac{3}{2}\frac{A_1 A_3}{A_2}, & (3)\\ v_0 &= A_1, & (4)\\ \alpha &= -\frac{3}{2}\frac{A_3}{A_1 A_2}. & (5) \end{aligned}$$

The cubic fit (real data!) gave me the following values for the coefficients:

$$\begin{aligned} A_1 &= (1.7902 \pm 0.0066)\, mm/ms, & (6)\\ A_2 &= (0.004822 \pm 0.000033)\, mm/ms^2, & (7)\\ A_3 &= (-1.09 \pm 0.4) \cdot 10^{-7}\, mm/ms^3, & (8) \end{aligned}$$

with $\chi^2 = 3.6$ for 7 d.o.f. It is not a "perfect" cubic fit, but still OK.

First step would be to calculate the relative error for each coefficient – this will give as a tool to perform the error propagation in a "smart" way!

$$\frac{\delta A_1}{A_1} = 0.0037 \approx 0.004 (= 0.4\%), \quad \frac{\delta A_2}{A_2} = 0.0068 \approx 0.007 (= 0.7\%), \quad \frac{\delta A_3}{A_3} = 0.37 \approx 0.4 (= 40\%). \tag{9}$$

Well, looking at the numbers above we can clearly see that everywhere A_3 is involved as a factor (in our case in the 2-nd term on r.h.s. of (3) and in the expression (5) for α) the leading relative error is (by far!) the one in A_3.

So, right away you may say that $\frac{\delta\alpha}{\alpha} \simeq \frac{\delta A_3}{A_3}$, without doing the "full" error propagation for α.

Let us now calculate the mean value of α:

$$\bar{\alpha} = -\frac{3}{2}\frac{-1.09}{1.7902 \cdot 0.004822}\frac{mm \cdot ms^{-3}}{mm \cdot ms^{-1} \cdot mm \cdot ms^{-2}} = 189.4 \cdot 10^{-7} mm^{-1}. \tag{10}$$

By using the above result on the error we get $\delta\alpha \simeq 0.37\alpha = 70 \cdot 10^{-7} mm^{-1}$, so the final answer for α would be[2]

$$\alpha = (190 \pm 70) \cdot 10^{-7} mm^{-1} = (19 \pm 7) \cdot 10^{-3} m^{-1}. \tag{11}$$

[1] In our case those terms are assumed to be negligible. Under which conditions this holds true, and which is the impact of this approximation (i.e. accuracy!) error on our results is another story; we won't go into details here.

[2] Because the error in the mean value $\bar{\alpha}$ is so big we can safely round $189.4 \approx 190$, the rounding error being here truly insignificant.

Let us now calculate v_0. This is straightforward, since $v_0 = A_1$, so that

$$v_0 = (1.790 \pm 0.007)\, mm/ms = (1.790 \pm 0.007)\, m/s. \tag{12}$$

Please observe that we rounded up both the error and the mean value. Both roundings are acceptable in my opinion. In the error, "66" works as fine as "70" does (and "70" looks better, right?). The "02" at the end of the mean value in (6) can be dropped, given the big error (66 or, if you prefer, 70). Once you agree to have "70" for the error, the rounding of the mean and letting aside of the end 0s is straightforward.

To summarize, the reasoning for the rounding process was as follows:

- we start with the full, "untouched" value $v_0 = (1.7902 \pm 0.0066)\, m/s$.
- the "02" in the mean is negligible compared to the error "66". So, rounding it up introduces a negligible rounding error in the mean [3].
- once we have "1.7900" for the mean, the next step would be to review the error so that the reading is more human readable.
- we decide to "play conservative", so we increase the value of the error a bit (by about 6% - that's reasonable and allowable) from "66" to "70".
- we get rid of the ending 0s and present the final result as $v_0 = (1.790 \pm 0.007)\, m/s$. Neat!

Let us now calculate "g". The mean value of g is (see (3))

$$g = 2A_2 - \frac{3}{2}\frac{A_1 A_3}{A_2} = 0.009644 + 607 \cdot 10^{-7} = 0.0097047\, mm/ms^2 = 9.7047 m/s^2. \tag{13}$$

According to our observation (9) on the error in A_3, the error in the $(A_1 A_3 / A_2)$ term is

$$\epsilon_2 \equiv \delta\left(-\frac{3A_1A_3}{2A_2}\right) = 0.4 \times 607 \cdot 10^{-7} mm/ms^2 = 240 \cdot 10^{-7} mm/ms^2 = 0.024 m/s^2. \tag{14}$$

The error in the $(2A_2)$ term is

$$\epsilon_1 \equiv \delta(2A_2) = 2 \times 0.000033 mm/ms^2 = 0.066 m/s^2. \tag{15}$$

Hence,

$$\delta g = \sqrt{(\epsilon_1)^2 + (\epsilon_2)^2} = \sqrt{(0.066)^2 + (0.024)^2} \simeq 0.07 m/s^2. \tag{16}$$

The final result[4] for g is therefore

$$g = (9.70 \pm 0.07) m/s^2. \tag{17}$$

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[3] The rounding in the mean still leaves the "true" value of the velocity in the same 68% confidence interval (with the center slightly shifted). Besides, the mean values and the errors are based on our guess of reading errors. This allows us to play a bit with the results — within reasonable limits!

[4] **Don't hurry** with the rounding of the mean value of g given by (13). Get the error calculus done before taking any decision on roundings! Indeed, let us suppose that the error in g is found to be $0.0007\, m/s^2$. Then, the final answer for g should have been quoted as $g = (9.7047 \pm 0.0007)\, m/s^2$. Any rounding in the error or/and the mean value would be a bit too nasty now. It would introduce rounding errors bigger than 15–20%!
Observation: The last result's relative error $\delta g/g \approx 0.007\%$ (!) seems to be **too small** to be true. Here "too small" has two aspects: the first is that with the devices you have at hand in Free Fall exp. it is unlikely to get such a small relative error. The second issue is that although the error calculus may give you such results, the method of the cubic fit and the underlying hypothesis of friction depending on v^2 are (perhaps) introducing *accuracy errors* bigger than those 0.007% ! On the other hand, the value of g as quoted in (17) has a relative error $\delta g/g \approx 0.7\%$ which looks more realistic.