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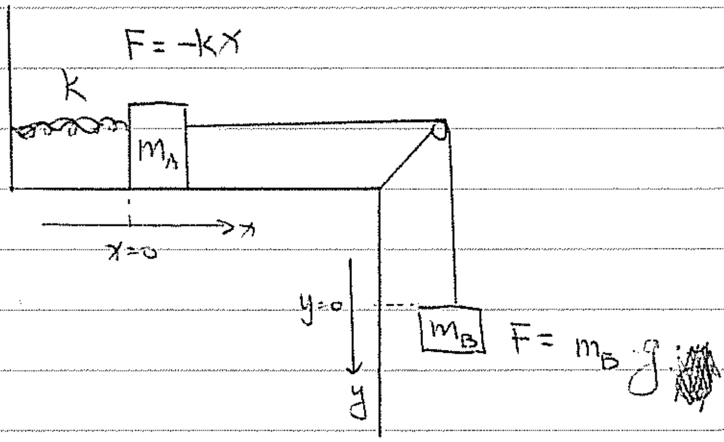
Question ① S.H.O. and Gravity

Two blocks are connected by a light cord over a frictionless pulley. Block m_A rests on a horizontal frictionless surface and is connected to a light spring of force constant k. Initially, block m_A is at $x=0$, the unstretched position of the spring. The apparatus is released from rest.



(a) Prove that m_B exhibits S.H.O. motion.

Solution:

Since the string connecting M_A & M_B is always tight, $x=y$.

$\therefore$ If we take $(m_A + m_B)$ as the system,

there are two forces acting on it:

① the spring: $F = -kx = -ky$

② gravity: $F = m_B g$.

$\therefore$ If we use y to label the position of

the system $(m_A + m_B)$. then:

$$(m_A + m_B)\ddot{y} = \text{sum of forces} = m_B g - k \cdot y$$

$$\therefore \quad (m_A + m_B)\ddot{y} = -k\left(y - \frac{m_B g}{k}\right)$$

$\therefore$ If we define $Q = y - \frac{m_B g}{k}$

$M = m_A + m_B$

then $\ddot{Q} = \ddot{y}$, we have:

$M\ddot{Q} = -kQ$, note Q is simply a re-labeling of the y-coordinate.

The crucial differential eqn governing the S.H.O.

1b) Find the amplitude of this oscillation.

Solution

The solution to $M\ddot{Q} = -kQ$ is

$Q(t) = A\sin(wt) + B\cos(wt)$, $w = \sqrt{\frac{k}{M}}$

This is from class notes

The initial conditions are

$\dot{Q}(0) = \dot{y}(0) = 0$

$Q(0) = y(0) - \frac{m_B g}{k} = -\frac{m_B g}{k}$

$\therefore Q(0) = A \cdot 0 + B \cdot 1 = -\frac{m_B g}{k} \Rightarrow B = -\frac{m_B g}{k}$

$\dot{Q}(t) = A \cdot w \cdot \cos(wt) - B \cdot w \cdot \sin(wt)$

$\therefore \dot{Q}(0) = 0 = A \cdot w \Rightarrow A = 0.$

$\therefore Q(t) = -\frac{m_B g}{k} \cdot \cos(wt)$

$\therefore$ amplitude $= \left| -\frac{m_B g}{k} \right| = \frac{m_B g}{k}$ //

Question 2 S.H.O. & conservation of energy

For a S.H.O., the force is given as $F = m\ddot{x} = -kx$.

(a) If we define

$$H = \frac{m}{2}(\dot{x})^2 + \frac{k}{2}x^2$$

Then show that H is a conserved quantity.
That is, $\frac{dH}{dt} = 0$.

Solution:

$$\frac{dH}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx)$$

Since $m\ddot{x} = -kx$, $m\ddot{x} + kx = 0$

$$\therefore \frac{dH}{dt} = \dot{x} \cdot 0 = 0$$

Hence, H is a conserved quantity.

Remark

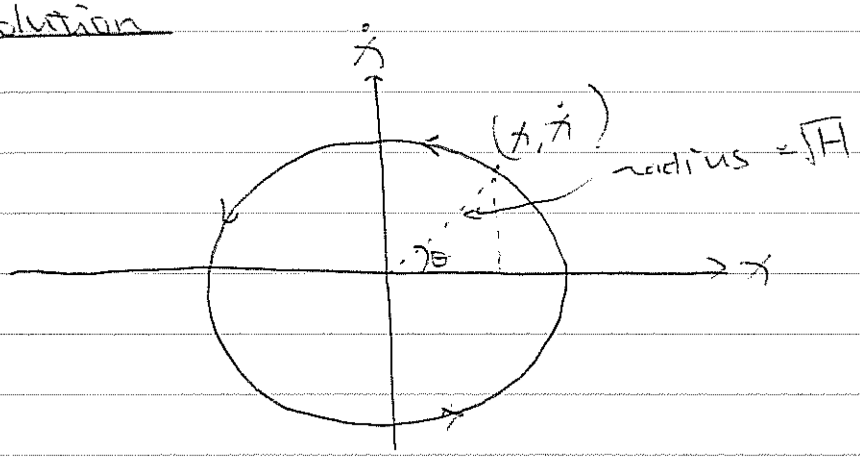
$\left(T = \frac{m}{2}\dot{x}^2\right)$

The first term of H is the kinetic energy. The second term is something called the potential energy $\left(U = \frac{k}{2}x^2\right)$, which corresponds to the energy stored in spring. Hence, H, being the sum of kinetic & potential energy, is the total energy of the system. What we have shown is a specific case of conservation of energy.

(b) Say in some unit system we have $m = 1$, $k = 1$. Then, $H = \frac{1}{2}\left[(\dot{x})^2 + x^2 \right]$. Given initial conditions $(x(t=0), \dot{x}(t=0))$, $H_0 \equiv \frac{1}{2}\left[(\dot{x}(0))^2 + (x(0))^2 \right]$ is a constant throughout the motion. Hence, if we label the state of the oscillator by $(x, \dot{x})$, the ~~m~~ time evolution of $(x, \dot{x})$ is restricted to the circle $H = \frac{1}{2}\left[\dot{x}^2 + x^2 \right]$ on the $(x, \dot{x})$ plane. Draw the circle. Now we define the <u>angle</u> of the system as $\theta \equiv \tan^{-1}\left(\frac{\dot{x}}{x} \right)$, show that $\frac{d\theta}{dt}$ is a constant.

<u>Note</u>: $\frac{d}{dz} \tan^{-1}(z) = \frac{1}{1+z^2}$

Solution



$$\frac{d\theta}{dt} = \frac{1}{1+(\dot{x}/x)^2} \cdot \frac{-\dot{x}\dot{x} + \ddot{x}x}{x^2}$$

but $\ddot{x} = -x$ from $m\ddot{x} = -kx$

$$\therefore \frac{d\theta}{dt} = \frac{1}{x^2+(\dot{x})^2} \cdot \left[-(\dot{x})^2 - x^2\right] = -1$$

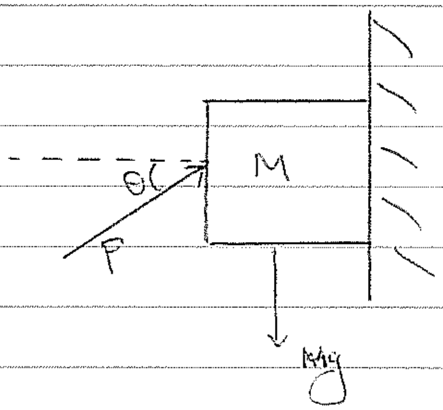
Remark

This means on the $(x, \dot{x})$ space, the system $(x(t), \dot{x}(t))$ moves along the circle with constant angular velocity $\alpha = -1$. Hence, the S.H.O. is mathematically equivalent to that of uniform circular motion.

Infact, in general we would get $\frac{d\theta}{dt} = -\sqrt{\frac{k}{m}}$, the frequency of the S.H.O.

Question ③ Static Friction

A block of mass M is being pushed up against a wall by a force of magnitude P that makes an angle of θ with the horizontal. The coefficient of static friction between the block & the wall is μ_s. Determine the range of P that would allow the block to remain stationary.



$\theta \in [0, \frac{\pi}{2}]$

Solution

The normal force is $P \cdot \cos\theta$
" upward " : $P \sin\theta$
" gravitational " : $M \cdot g$

∴ The magnitude friction force must be $F_r = |Mg - P\sin\theta| > 0$
to remain stationary.

now since

$F_r \leq \mu_s F_N$
└ normal force

$|Mg - P\sin\theta| \leq \mu_s P\cos\theta$

$\therefore\ Mg - P\sin\theta \leq \mu_s P\cos\theta \quad (*)$ or $P\sin\theta - Mg \leq \mu_s P\cos\theta \quad (**)$

inequality (*) is easy:

$Mg \leq P(\mu_s\cos\theta + \sin\theta)$

$P \geq \dfrac{Mg}{\mu_s\cos\theta + \sin\theta}$

whereas inequality (**) needs more work:

$(**) \Rightarrow P(\sin\theta - \mu_s\cos\theta) \leq Mg$

We have 3 cases:

case I: $\sin\theta - \mu_s\cos\theta > 0$

then $P \leq \dfrac{Mg}{\sin\theta - \mu_s\cos\theta}$

case II: $\sin\theta - \mu_s\cos\theta = 0$

This simply says $P \cdot 0 = 0 \leq Mg$
which does not provide constraints on P.

case III: $\sin\theta - \mu_s \cos\theta < 0$

in this case $P \geq \dfrac{Mg}{\sin\theta - \mu_s\cos\theta}$

but $\dfrac{Mg}{\sin\theta - \mu_s\cos\theta} < 0$ and by assumption we already have $P > 0$

$\therefore$ This case does not provide us any additional constraints on P either.

$\therefore$ we get:

$$\begin{cases} \text{if } \sin\theta - \mu_s\cos\theta > 0 \\ \dfrac{Mg}{\mu_s\cos\theta + \sin\theta} \leq P \leq \dfrac{Mg}{\sin\theta - \mu_s\cos\theta} \\ \text{if } \sin\theta - \mu_s\cos\theta \leq 0 \\ \dfrac{Mg}{\mu_s\cos\theta + \sin\theta} \leq P \end{cases}$$

Remark

It might be somewhat confusing why intuitively we have two distinctive cases. Especially, when $\sin\theta - \mu_s\cos\theta \leq 0$, we have no upper bound

~~For~~ for P. Let's convince ourselves that this conclusion is reasonable:

Suppose we have $\theta = 0$

then $\sin(\theta=0) - \mu_s \cos(\theta=0) = -\mu_s < 0$

$\therefore$ We have $P \geq \frac{Mg}{\mu_s \cos\theta} = \frac{Mg}{\mu_s}$

or $P \cdot \mu_s \geq Mg$

This corresponds to the case where P is in fact the normal force, and of course $P \cdot \mu_s$ is just the max. friction force the wall surface can provide.

Then suppose we have $\theta = \pi/2$

then $\sin(\pi/2) - \mu_s \cos(\pi/2) = 1 > 0$

$\therefore \frac{Mg}{\sin(\pi/2)} \leq P \leq \frac{Mg}{\sin(\pi/2)} \Rightarrow P = Mg$ exactly.

Why? This corresponds to us pushing the block upwards, so friction does not come into play. If we push harder than Mg, then the block will move upwards instead of being stationary.

So we see that the upper bond we have for $\sin\theta - \mu_s\cos\theta > 0$ occurs when, if we exceed that amount of force, we no longer need to worry about the block slipping downwards, but the block will slip upwards.

Note that $\sin\theta - \mu_s\cos\theta > 0$ when θ is big. This is when a majority of our force is pointing upwards & only when do we worry about slipping upwards. When θ is small, we have a large normal force that is good enough to provide ~~pr~~ enough friction to prevent any surplus of (upward force - gravity).

Lastly, when $\mu_s > 1$, $\sin\theta - \mu_s\cos\theta \leq 0 \ \forall\ \theta \in [0, \pi/2]$, and we never need to worry about the upper bond of P since the surface causes so much friction we will never slip ~~s~~ upwards.