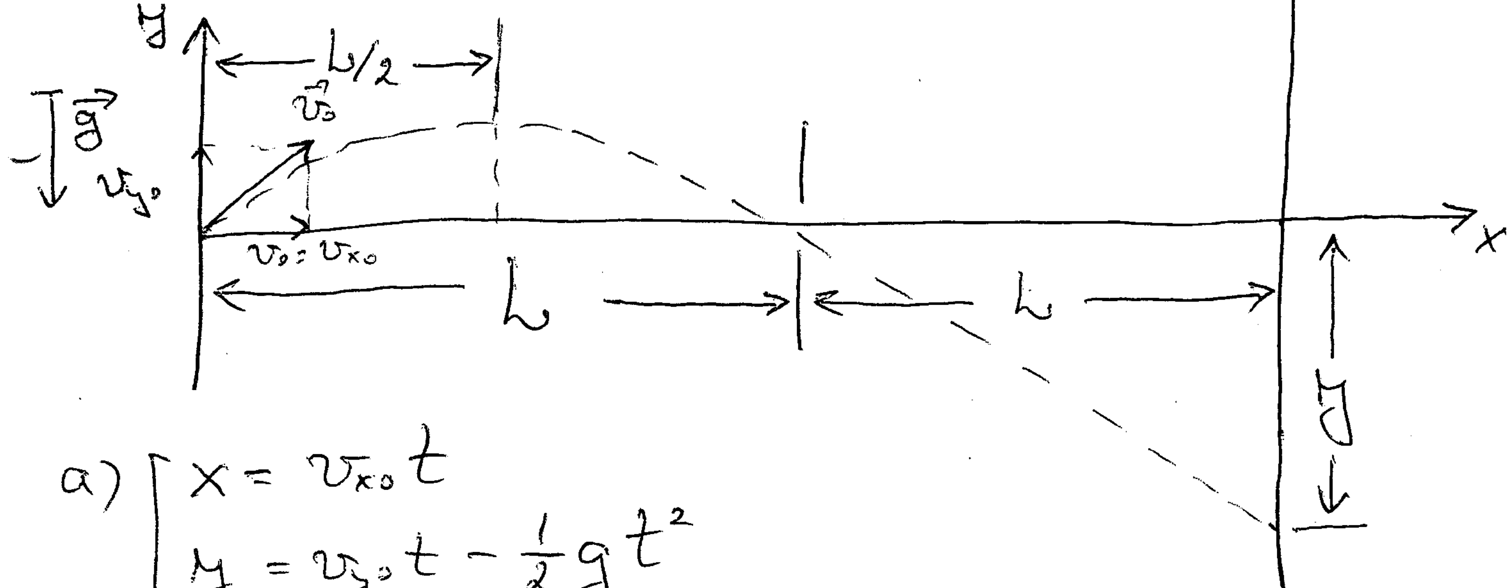
Tutorial # 3: Sept. 25/29, 2006 (PHY180F) P.

Projectile Motion — 2D

Problem # 1 The figure shows the trajectory of a beam of particles moving in a vacuum under the influence of gravity and passing through narrow slits on the same horizontal level as shown. If v_{0x} is the horizontal component of the initial velocity and the particles arrive at the detector a vertical distance y below the slits, what is:

a) the initial vertical component v_{0y} of the velocity

b) the vertical displacement ← both in terms of v_{x0}, L.



a) $$\begin{cases} x = v_{x0} t \\ y = v_{y0} t - \frac{1}{2} g t^2 \end{cases}$$

Form the equation for the vertical displacement in terms of the horizontal displacement

$$y(x) = v_{y0}\left(\frac{x}{v_{x0}}\right) - \frac{1}{2} g \left(\frac{x}{v_{x0}}\right)^2$$

$$y(x) = \left(\frac{v_{y0}}{v_{x0}}\right) x - \frac{1}{2}\left(\frac{g}{v_{x0}^2}\right) x^2$$

P.2 From the given information, the maximum of vertical displacement occurs at $x = L/2$

$$\therefore \left.\frac{dy}{dx}\right|_{x=L/2} = 0$$

$$\frac{dy}{dx} = \left(\frac{v_{y0}}{v_{x0}}\right) - \left(\frac{g}{v_{x0}^2}\right)x = 0 \;\Big|\; \text{at } x = L/2$$

$$\left(\frac{v_{y0}}{v_{x0}}\right) - \left(\frac{g}{v_{x0}^2}\right)\cdot\frac{L}{2} = 0$$

$$\therefore \boxed{v_{y0} = \frac{1}{2}\left(\frac{g}{v_{x0}}\right)L}$$

b) Using the vertical displacement equation developed in (a), we know the detector is located at $x = 2L$:

$$y(2L) = \left(\frac{v_{y0}}{v_{x0}}\right)(2L) - \frac{1}{2}\left(\frac{g}{v_{x0}^2}\right)(2L)^2 =$$

$$= \left(\frac{\frac{1}{\cancel{2}}\left(\frac{g}{v_{x0}}\right)L}{v_{x0}}\right)(\cancel{2}L) - \frac{1}{\cancel{2}}\left(\frac{g}{v_{x0}^2}\right)\cancel{4}^{2}L^2 =$$

$$= \left(\frac{g}{v_{x0}^2}\right)L^2 - 2\left(\frac{g}{v_{x0}^2}\right)L^2 = -g\left(\frac{L}{v_{x0}}\right)^2$$

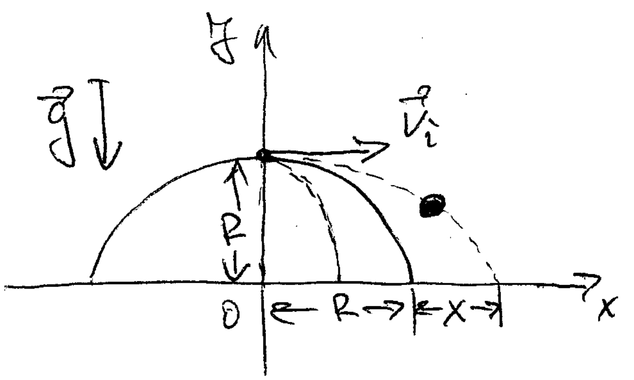
$$\therefore \boxed{y(2L) = -g\left(\frac{L}{v_{x0}}\right)^2}$$

Problem # 4.62

A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\vec{v}_i$.

(a) What must be its minimum initial speed if the ball is never to hit the rock after it is kicked?

(b) With this initial speed, how far from the base of the rock does the ball hit the ground?



Measure heights above the level ground. The elevation y_b of the ball follows

$$\begin{cases} y_b = y_0 - \frac{1}{2}gt^2 = R - \frac{1}{2}gt^2 \\ x_b = v_i t \end{cases}$$

$$\therefore \quad y_b = R - \frac{1}{2}\left(\frac{g}{v_i^2}\right)x^2$$

(a) The elevation y_r of points on the rock is described by $y_r^2 + x^2 = R^2$, We will have $y_b = y_r$, at $x = 0$, but for all other x we require the ball to be above the rock surface as in $\boxed{y_b > y_r}$. Then

$$y_b^2 + x^2 > R^2$$

$$\left[R - \frac{1}{2}\left(\frac{g}{v_i^2}\right)x^2\right]^2 + x^2 > R^2$$

$$\cancel{R^2} - \left(\frac{gR}{v_i^2}\right)x^2 + \frac{1}{4}\left(\frac{g}{v_i^2}\right)^2 x^4 + x^2 > \cancel{R^2}$$

$$\frac{g^2 \cancel{x^4}x^2}{4v_i^4} + \cancel{x^2} > \frac{g\cancel{x^2}R}{v_i^2}$$

$$\frac{g^2x^2}{4v_i^4} + 1 > \frac{gR}{v_i^2}$$

We get the strictest requirement for $x \to 0$. If the ball's parabolic trajectory has large enough radius of curvature at the start, the ball will clear the whole rock:

$$\therefore 1 \geq \frac{gR}{v_i^2} \Rightarrow \boxed{v_i \geq \sqrt{gR}}$$

(b) With $v_i = \sqrt{gR}$ and $y_b = 0$, we have

$$0 = R - \frac{1}{2}\left(\frac{g}{v_i^2}\right)x^2 = R - \frac{1}{2}\left(\frac{\cancel{g}}{\cancel{g}R}\right)x^2$$

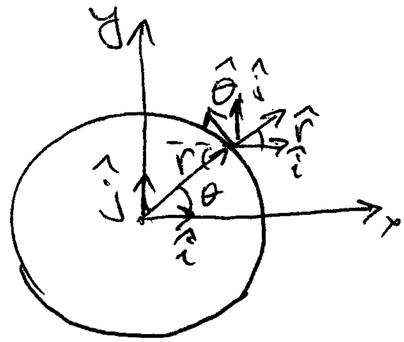
$$\therefore x^2 = 2R^2 \Rightarrow x = \sqrt{2}R$$

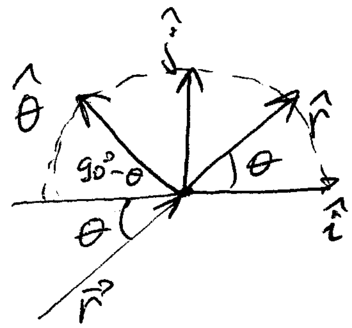
and the distance from the rock's base is

$$x - R = (\sqrt{2} - 1)R = 0.4142R$$

Circular motion

Problem #2 (a) For a point located in the first quadrant, express the unit vectors $\hat{r}$ and $\hat{\theta}$ (polar coordinates) in terms of $\hat{i}$ and $\hat{j}$ and the angle θ.
(b) Consider a particle undergoing a uniform circular motion. Starting with the position vector and using the information in (a), derive the equation $\vec{v} = v\,\hat{\theta}$





(a) $$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{\theta} = -\cos(90°-\theta)\,\hat{i} + \sin(90°-\theta)\,\hat{j} =$$
$$\sin(90°-\theta) = \sin 90\cos\theta - \cancel{\cos 90}^{\,0}\sin\theta = \cos\theta$$
$$\cos(90-\theta) = \cancel{\cos 90}^{\,0}\cos\theta + \sin 90 \sin\theta = \sin\theta$$

$$\therefore\ \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\left[\begin{array}{l} \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{array}\right.$$

(b) To show $\vec{v} = v\,\hat{\theta}$ ← for uniform circular motion

P.6 $\vec{r} = r\hat{r}$; $\theta = 2\pi \dfrac{t}{T}$, where T is the period of rotation

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} =$$

$$= r\frac{d}{dt}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) =$$

$$= r\left(-\sin\theta \cdot \frac{d\theta}{dt}\cdot\hat{i} + \cos\theta\cdot\frac{d\theta}{dt}\cdot\hat{j}\right)$$

since $\theta = 2\pi \dfrac{t}{T} \Rightarrow \dfrac{d\theta}{dt} = \dfrac{2\pi}{T}$

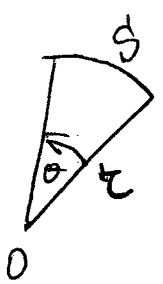
$$\therefore \vec{v} = \frac{2\pi r}{T}\underbrace{\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)}_{\hat{\theta}} = \underline{\frac{2\pi r}{T}\hat{\theta}}$$

But $s = vt$ ← motion with constant speed

where $s = 2\pi r$, and $t \equiv T$

$$\therefore v = \frac{s}{T} = \frac{2\pi r}{T}$$

$$\therefore \boxed{\vec{v} = v\hat{\theta}}$$



$$s = r\theta \Rightarrow \frac{ds}{d\theta} = r$$

and $v = \dfrac{ds}{dt} = r\dfrac{d\theta}{dt} = \underline{\omega r}$