Chapter 15

The Addition of Angular Momentum

Suppose we have two electrons;

$$[S_{1i}, S_{1j}] = \epsilon_{ijk}\, i\hbar\, S_{1k} \qquad (1)$$
$$[S_{2i}, S_{2j}] = \epsilon_{ijk}\, i\hbar\, S_{2k}$$

Also $\quad [\bar{S}_1, \bar{S}_2] = 0 \qquad (2)$

Let us now define $\quad \bar{S} = \bar{S}_1 + \bar{S}_2 \quad (3) \qquad$ Total spin

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}] = [S_{1x}, S_{1y}] + [S_{2x}, S_{2y}] + 0 + 0$$
$$= i\hbar(S_{1z} + S_{2z}) = i\hbar S_z \qquad (4)$$

In general $\quad [S_i, S_j] = \epsilon_{ijk}\, i\hbar\, S_k \quad (5)$

We are therefore justified in calling $\bar{S}$ the <u>total</u> spin.

The eigenvalues and eigenfuncs. of S^2 and S_z:

If $\chi_{\pm}^{(1)}$: the spinors of the first electron

and $\chi_{\pm}^{(2)}$: " " " " second "

$$S_1^2\,\chi_\pm^{(1)} = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hbar^2\,\chi_\pm^{(1)}$$

$$S_{1z}\,\chi_\pm^{(1)} = \pm\tfrac{1}{2}\hbar\,\chi_\pm^{(1)} \qquad (6)$$

and

$$S_2^2\,\chi_\pm^{(2)} = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hbar^2\,\chi_\pm^{(2)}$$

$$S_{2z}\,\chi_\pm^{(2)} = \pm\tfrac{1}{2}\hbar^2\,\chi_\pm^{(2)} \qquad (7)$$

The two-spin system actually has <u>4-states</u>, they are

$$\chi_+^{(1)}\chi_+^{(2)}, \quad \chi_+^{(1)}\chi_-^{(2)}, \quad \chi_-^{(1)}\chi_+^{(2)}, \quad \chi_-^{(1)}\chi_-^{(2)} \qquad (8)$$

$$S_z\left(\chi_\pm^{(1)}\chi_\pm^{(2)}\right) = \left(S_{1z}+S_{2z}\right)\left(\chi_\pm^{(1)}\chi_\pm^{(2)}\right)$$

$$= \left(S_{1z}\,\chi_\pm^{(1)}\right)\chi_\pm^{(2)} + \chi_\pm^{(1)}\left(S_{2z}\,\chi_\pm^{(2)}\right)$$

$$= \underbrace{\left(\pm\frac{\hbar}{2}\pm\frac{\hbar}{2}\right)}_{\text{eigenvalue}}\underbrace{\chi_\pm^{(1)}\chi_\pm^{(2)}}_{\text{eigenfunc.}} \qquad (9)$$

that is

$$S_z\left(\chi_+^{(1)}\chi_+^{(2)}\right) = \hbar\left(\chi_+^{(1)}\chi_+^{(2)}\right)$$

$$S_z\left(\chi_+^{(1)}\chi_-^{(2)}\right) = 0$$

$$S_z\left(\chi_-^{(1)}\chi_+^{(2)}\right) = 0 \qquad (10)$$

$$S_z\left(\chi_-^{(1)}\chi_-^{(2)}\right) = -\hbar\left(\chi_-^{(1)}\chi_-^{(2)}\right)$$

One might expect that one linear combination of them will form an $S=1$ state, to form a triplet with $m=1$ and $m=-1$ states, and the orthogonal combination will form a singlet $S=0$ state.

To check this expectation, let us construct the lowering op.

$$S_- = S_{1-} + S_{2-} \qquad (11)$$

and apply this to $m=1$ state

$$S_- \chi_+^{(1)} \chi_+^{(2)} = (S_{1-}\chi_+^{(1)})\chi_+^{(2)} + \chi_+^{(1)}(S_{2-}\chi_+^{(2)})$$

$$= \hbar \chi_-^{(1)} \chi_+^{(2)} + \hbar \chi_+^{(1)} \chi_-^{(2)} = \sqrt{2}\hbar \, \frac{\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}}{\sqrt{2}} \qquad (12)$$

where we have used $S_-^{(i)} \chi_+^{(i)} = \hbar \chi_-^{(i)}$ (13)

which can be established by noting that:

$$\tfrac{1}{2}\hbar \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (14)$$

The combination in (12) has been normalized, and the factor $\sqrt{2}\hbar$ agrees with what one would expect from;

$$L_\pm Y_\ell^m = \sqrt{\ell(\ell+1) - m(m\pm 1)}\ \hbar\, Y_\ell^{m\pm 1}$$

$$\xrightarrow[\text{by}]{\text{replaced}} \quad S_\pm \chi_s^{m_s} = \sqrt{s(s+1) - m_s(m_s \pm 1)}\ \hbar\, \chi_s^{m_s \pm 1} \qquad (15)$$

with $s=1$, $m_s=1$

Then $\chi_1^0 = \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}\right) \qquad (16)$

Now using $S_-^{(1)}\chi_-^{(1)} = 0 \qquad (17)$

we get ; $S_-\chi_1^0 = \frac{\hbar}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}\right) = \sqrt{2}\hbar\, \chi_-^{(1)}\chi_-^{(2)}$

(18)

$$\chi_1^{-1} = \chi_-^{(1)}\chi_-^{(2)} \qquad (19)$$

Also $\chi_1^1 = \chi_+^{(1)}\chi_+^{(2)} \qquad (20)$

The remaining state (orthogonal to (16)) ;

$$\frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}\right) \qquad (21)$$

Because it has no patterns, we conjecture that it is an $S=0$ state.

$$\chi_0^0 = \frac{1}{\sqrt{2}}\left(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}\right) \qquad (22)$$

Let us check it;

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2\, S_1 \cdot S_2$$

$$= S_1^2 + S_2^2 + \left(2 S_{1z} S_{2z} + S_{1+} S_{2-} + S_{1-} S_{2+}\right) \qquad (23)$$

$$S_1^2 \chi_1^0 = \frac{1}{\sqrt{2}} \left(S_1^2 \chi_+^{(1)} \chi_-^{(2)} + S_1^2 \chi_-^{(1)} \chi_+^{(2)} \right) = \frac{3}{4}\hbar^2 \chi_1^0$$

$$S_1^2 \chi_0^0 = \frac{1}{\sqrt{2}} \left(\quad \text{''} \quad - \quad \text{''} \quad \right) = \frac{3}{4}\hbar^2 \chi_0^0 \qquad (24)$$

Similarly:

$$S_2^2 \chi_1^0 = \frac{3}{4}\hbar^2 \chi_1^0$$

$$S_2^2 \chi_0^0 = \text{''}\, \chi_0^0 \qquad (25)$$

Next;

$$2 S_{1z} S_{2z} \chi_1^0 = -\frac{1}{2}\hbar^2 \chi_1^0$$

$$2 S_{1z} S_{2z} \chi_0^0 = -\frac{1}{2}\hbar^2 \chi_0^0 \qquad (26)$$

Finally; $(S_{1+} S_{2-} + S_{1-} S_{2+}) \chi_1^0 =$

$$= \frac{1}{\sqrt{2}} \Big(S_{1+} \chi_+^{(1)} S_{2-} \chi_-^{(2)} + S_{1-} \chi_+^{(1)} S_{2+} \chi_-^{(2)}$$

$$+ S_{1+} \chi_-^{(1)} S_{2-} \chi_+^{(2)} + S_{1-} \chi_-^{(1)} S_{2+} \chi_+^{(2)} \Big) = +\hbar^2 \chi_1^0$$

Similarly $(S_{1+} S_{2-} + S_{1-} S_{2+}) \chi_0^0 = -\hbar^2 \chi_0^0 \qquad (27)$

$$\rightarrow \quad S^2 \chi_1^0 = \hbar^2 [1(1+1)]\, \chi_1^0$$

$$S^2 \chi_0^0 = \hbar^2 \underbrace{[0(0+1)]}_{s(s+1)}\, \chi_1^0 = 0 \qquad (28)$$

Then we have singlet state $\left\{ \chi_0^0 \right.$ triplet states $\left\{ \begin{array}{l} \chi_1^1 \\ \chi_1^0 \\ \chi_1^{-1} \end{array} \right.$ (29)

Spin-dep. forces:

$$V = V(\bar{r}, \bar{s}) \qquad (30)$$

In this case the eigenfuncs of individual spins are no longer simultaneous eigenfuncs of

$$\left\{ \begin{array}{l} H \\ \text{and say } S_1^2,\ S_{1z},\ S_2^2,\ S_{2z} \end{array} \right. \qquad (31)$$

but they may be simultaneous eigenfuncs. of

$$\left\{ \begin{array}{l} H \\ \text{and } S^2,\ S_z,\ S_1^2,\ S_2^2 \end{array} \right. \qquad (32)$$

Ex.

$$V(r) = V_1(r) + \frac{1}{\hbar^2} S_1 \cdot S_2 V_2(r) \qquad (33)$$

Obviously $[S_{1z}, S_1 \cdot S_2] \neq 0 \quad [S_{2z}, S_1 \cdot S_2] \neq 0 \qquad (34)$

Note that $S_1 \cdot S_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \qquad (35)$

$\rightarrow$ $\{H, S_{1z}, S_{2z}\}$ do not form a complete set of observables.

$\rightarrow$ The eigenstates of H cannot be simple product of $\chi_\pm^{(1)}$ and $\chi_\pm^{(2)}$.

However; $$S_1 \cdot S_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2) \qquad (36)$$

Since $[S^2, S_1^2] = 0 \quad [S^2, S_2^2] = 0 \quad [S_1^2, S_2^2] = 0 \quad (37)$

$\rightarrow [H, S^2] = 0 \quad [H, S_1^2] = 0 \quad [H, S_2^2] = 0 \qquad (38)$

$\rightarrow$ $\{H, S^2, S_1^2, S_2^2\}$ form a complete set of observables

$\rightarrow$ They have simultaneous eigenfunc.

Remark: $[S^2, S_1^2] = [S_1^2 + S_2^2 + 2S_1 \cdot S_2, S_1^2]$
$= 0 + 0 + 2[S_1, S_1^2] \cdot S_2 = 0$

$$\left[V_1(r) + \frac{1}{\hbar^2} S_1 \cdot S_2 V_2(r)\right] \Psi = \left[V_1(r) + \frac{1}{2} V_2(r)\left[S(S+1) - \frac{3}{2}\right]\right]\Psi \quad (39)$$

eigenfnc. op. op. eigenvalue

$$\rightarrow \quad V(r) = V_1(r) + \frac{1}{4} V_2(r) \times \begin{cases} 1 & \text{for } S=1 \\ -3 & , \ S=0 \end{cases} \quad (40)$$

Such a spin-dep. pot. is actually observed in the n-p system.

The bound state : $S=1$ (deuteron)

Unbound : $S=0$ (only possible if $V_2(r) \neq 0$)

Addition of Ang. Mom. :

$$[S, L] = 0 \quad (41)$$

$$\bar{J} = \bar{L} + \bar{S} \quad (42)$$

$$(41)(42) \longrightarrow \quad [J_i, J_j] = \epsilon_{ijk} i\hbar J_k \quad (43)$$

Question: What is the linear combination of Y_ℓ^m and $\chi_\pm$ that form eigenstates of $J_z = L_z + S_z$ (44)

Let us consider the linear combination;

$$\mathcal{Y}_j^{m+\frac{1}{2}} = \Psi_{j,m+\frac{1}{2}} = \alpha Y_\ell^m \chi_+ + \beta Y_\ell^{m+1} \chi_- \qquad (45)$$

It is by <u>construction</u>, an eigenfunc. of J_z with eigenvalue $(m+\frac{1}{2})\hbar$.

We now determine α and β <u>such that</u> it is also eigenfunc. of J^2.

We know;
$$L_+ Y_\ell^m = \sqrt{\ell(\ell+1) - m(m+1)}\ \hbar Y_\ell^{m+1}$$
$$= \sqrt{(\ell+m+1)(\ell-m)}\ \hbar Y_\ell^{m+1} \qquad (46)$$

$$L_- Y_\ell^m = \sqrt{(\ell-m+1)(\ell+m)}\ \hbar Y_\ell^{m-1} \qquad (47)$$

Also
$$S_+\chi_+ = 0 \qquad S_-\chi_- = 0 \qquad (48)$$
$$S_+\chi_- = \hbar\chi_+$$
$$S_-\chi_+ = \hbar\chi_- \qquad (49)$$

Since $J^2 = L^2 + S^2 + 2L\cdot S = L^2 + S^2 + 2L_zS_z + L_+S_- + L_-S_+$

$$\rightarrow J^2\Psi_j^{m+1/2} = \alpha\hbar^2\left\{\ell(\ell+1)Y_\ell^m\chi_+ + \tfrac{3}{4}Y_\ell^m\chi_+ + 2m\left(\tfrac{1}{2}\right)Y_\ell^m\chi_+ + \sqrt{(\ell-m)(\ell+m+1)}\ Y_\ell^{m+1}\chi_-\right\} + \beta\hbar^2\left\{\ell(\ell+1)Y_\ell^{m+1}\chi_- + \tfrac{3}{4}Y_\ell^{m+1}\chi_- + 2(m+1)\left(-\tfrac{1}{2}\right)Y_\ell^{m+1}\chi_- + \sqrt{(\ell-m)(\ell+m+1)}\ Y_\ell^m\chi_+\right\} \qquad (50)$$

This will be of the form

$$\hbar^2 j(j+1)\, \mathcal{Y}_j^{m+\frac{1}{2}} = \hbar^2 j(j+1) \left(\alpha Y_l^m \chi_+ + \beta Y_l^{m+1} \chi_- \right) \quad (51)$$

provided that:

$$\alpha \left[l(l+1) + \frac{3}{4} + m \right] + \beta \sqrt{(l-m)(l+m+1)} = j(j+1)\, \alpha$$

$$\beta \left[l(l+1) + \frac{3}{4} - m - 1 \right] + \alpha \sqrt{(l-m)(l+m+1)} = j(j+1)\, \beta \quad (52)$$

This requires that; det = 0

$$(l-m)(l+m+1) = \left[j(j+1) - l(l+1) - \frac{3}{4} - m \right] \cdot$$

$$\cdot \left[j(j+1) - l(l+1) - \frac{3}{4} + m + 1 \right] \quad (53)$$

$$\rightarrow \quad j(j+1) - l(l+1) - \frac{3}{4} = l \quad (54)$$

Now let $l \rightarrow -l-1$ in (53) (a check)

$$(-l-1-m)(-l-1+m+1) = \left[j(j+1) - l(l+1) - \frac{3}{4} - m \right] \cdot$$

$$\cdot \left[j(j+1) - l(l+1) - \frac{3}{4} + m + 1 \right] \quad (55)$$

$$\rightarrow \quad j(j+1) - l(l+1) - \frac{3}{4} = -l-1 \quad (56)$$

$$\begin{array}{l} (54) \\ (56) \end{array} \rightarrow j = \begin{cases} l - \frac{1}{2} \\ l + \frac{1}{2} \end{cases} \quad (57)$$

For $j = l + \frac{1}{2}$

$$(52) \rightarrow \quad \alpha = \sqrt{\frac{l+m+1}{2l+1}} \qquad \beta = \sqrt{\frac{l-m}{2l+1}} \qquad (58)$$

Actually we just get the ratio; these are already the normalized forms

$$\text{i.e. } \alpha^2 + \beta^2 = 1 \qquad (59)$$

$$\rightarrow \mathcal{Y}_{l+\frac{1}{2}}^{m+\frac{1}{2}} = \sqrt{\frac{l+m+1}{2l+1}}\, Y_l^m \chi_+ + \sqrt{\frac{l-m}{2l+1}}\, Y_l^{m+1} \chi_- \qquad (60)$$

We can guess that for $j = l - \frac{1}{2}$:

$$\mathcal{Y}_{l-\frac{1}{2}}^{m+\frac{1}{2}} = \sqrt{\frac{l-m}{2l+1}}\, Y_l^m \chi_+ - \sqrt{\frac{l+m+1}{2l+1}}\, Y_l^{m+1} \chi_- \qquad (61)$$

in order to be <u>orthogonal</u> to the $j = l + \frac{1}{2}$ sol.

General feature:

If we have the eigenstates

$$\begin{aligned} &Y_{l_1 m_1}^{(1)} \quad \text{of } L_1^2 \text{ and } L_{1z} \\ &Y_{l_2 m_2}^{(2)} \quad \text{"} \quad L_2^2 \text{ " } L_{2z} \end{aligned} \qquad (62)$$

We can form:

$$Y^{(1)}_{\ell_1 m_1} \, Y^{(2)}_{\ell_2 m_2} \qquad (2\ell_1+1)(2\ell_2+1) - \text{in number} \qquad (63)$$

where $\quad -\ell_1 \leq m_1 \leq \ell_1 \qquad -\ell_2 \leq m_2 \leq \ell_2 \qquad (64)$

These may be classified by the eigenvalue of

$$J_z = L_{1z} + L_{2z} \qquad (65)$$

which is $\quad m_1 + m_2 \equiv m$

$$-(\ell_1+\ell_2) \leq m \leq \ell_1 + \ell_2 \qquad (66)$$

Now for $\quad m_1 = \ell_1 \,, \; m_2 = \ell_2$

$$J^2 \, Y^{(1)}_{\ell_1 \ell_1} Y^{(2)}_{\ell_2 \ell_2} = \left(L_1^2 + L_2^2 + 2 L_{1z} L_{2z} + L_{1+} L_{2-} + L_{1-} L_{2+} \right) Y^{(1)}_{\ell_1 \ell_1} Y^{(2)}_{\ell_2 \ell_2} \qquad (67)$$

$$= \hbar^2 \left[\ell_1(\ell_1+1) + \ell_2(\ell_2+1) + 2\ell_1 \ell_2 \right] Y^{(1)}_{\ell_1 \ell_1} Y^{(2)}_{\ell_2 \ell_2}$$

$$= \hbar^2 \underbrace{(\ell_1+\ell_2)}_{j} \underbrace{(\ell_1+\ell_2}_{j}+1) \, Y^{(1)}_{\ell_1 \ell_1} Y^{(2)}_{\ell_2 \ell_2} \qquad (68)$$

Remark: $Y^{(1)}_{\ell, m} \equiv |\ell, m\rangle_1$

Successive application of

$$J_- = L_{1-} + L_{2-} \qquad (69)$$

will produce a linear combination of $Y^{(1)}_{\ell_1}, Y^{(2)}_{\ell_2}$ with different m_1 and m_2.

Some Results for decomposition of Ang. Mom. Ops.:

a) The products $Y^{(1)}_{l_1 m_1} Y^{(2)}_{l_2 m_2}$ can be decomposed into eigenstates of $\vec{J}^2$ with eigenvalues $j(j+1)\hbar^2$, where

$$|l_1 - l_2| \leq j \leq l_1 + l_2 \qquad (70)$$

b) In general

$$|jm\rangle = \mathcal{Y}_{jm} = \sum_{m_1, m_2} \langle l_1 m_1, l_2 m_2 | jm \rangle Y^{(1)}_{l_1 m_1} Y^{(2)}_{l_2 m_2} \qquad (71)$$

Clebsch-Gordan series

$$m_1 + m_2 = m \qquad (72)$$

For each j we have $(2j+1)$-states

(70) $\rightarrow$ The total number of states $= [2(l_1+l_2)+1] + [2(l_1+l_2-1)+1] + \cdots + [2(l_1-l_2)+1]$

$$= \sum_{n=0}^{2l_2} [2(l_1 - l_2 + n) + 1]$$

$$= (2l_1+1)(2l_2+1) \qquad (73) \qquad \text{for } l_1 \geq l_2$$

where we have used, $\displaystyle\sum_{n=0}^{2k} n = \frac{2k}{2}(0+2k) + \underbrace{\frac{2k}{2}}_{\text{mid term}} = k(2k+1)$

$$\sum_{n=0}^{2k} (n+\alpha) = \frac{2k}{2}\left[(0+\alpha) + (2k+\alpha)\right] + \underbrace{\left(\frac{2k}{2}+\alpha\right)}_{\text{mid term}} = (k+\alpha)(2k+1)$$

A final comment is in order.

For fermions: $\psi(\bar{x}_1, \bar{\sigma}_1, \dots \bar{x}_n, \bar{\sigma}_n)$ is antisymmetric under the interchange of two particles.

For a system of two identical spin $\frac{1}{2}$ particles;

The $S=1$ triplet states $\begin{cases} \chi_1^1 \\ \chi_1^0 \\ \chi_1^{-1} \end{cases}$ symmetric under spin label interchange

The $S=0$ singlet state $\{ \chi_0^0$ antisymmetric under spin label interchange

$\rightarrow$ The spacial wave func. $\begin{cases} \text{antisymmetric for } \underline{\text{triplet}} \text{ state} \\ \text{symmetric " } \underline{\text{singlet}} \text{ state} \end{cases}$

The spacial wave func of 2-particle system in C.M. system:

$$u(r) = R_{n\ell m}(r)\, Y_{\ell m}(\theta, \varphi) \qquad (74)$$

An interchange of the coordinates of two particles is equivalent to the change;

$$r \longrightarrow r$$
$$\theta \longrightarrow \pi - \theta \qquad (75)$$
$$\varphi \longrightarrow \varphi + \pi$$

$$\rightarrow Y_{\ell m}(\theta,\varphi) \longrightarrow Y_{\ell m}(\pi-\theta, \varphi+\pi) = (-1)^{\ell} Y_{\ell m}(\theta,\varphi) \qquad (76)$$

but $R_{n\ell m}(r) \rightarrow R_{n\ell m}(r)$ (77)

Thus for <u>triplet</u> states : ℓ odd
and " <u>singlet</u> " : ℓ even (78)

The Parity of Pion:

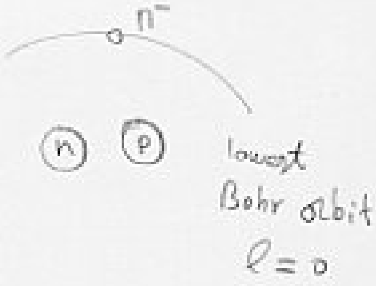
We Know spin of pion $S_p = 0$

and also intrinsic parity of proton and neutron = +

consider a capture of pion by deutron (n-p).

A slow pion in liquid deuterium (atom) loses energy by a variety of mechanism, till it finally ends up in the <u>lowest</u> Bohr orbit about the (p-n) nucleus and it is captured through the <u>action</u> of the nuclear forces.

$$\pi^- + d \longrightarrow n + n \qquad (79)$$



In this reaction the ang. mom. is $j = 1$

$$\left.\begin{array}{l} S_\pi = 0 \quad l_\pi = 0 \\ J_d = 1 \end{array}\right\} \longrightarrow j = 1 \quad \text{(left hand side)} \qquad (80)$$

a)

If $S_{nn} = 0$ the spin of two neutrons

$$\longrightarrow l_{nn} = 1 \qquad (81)$$

b)

If $S_{nn} = 1 \longrightarrow l_{nn} = 0, 1, 2 \qquad (82)$

However, (78) $\longrightarrow l =$ even for singlet state

$\longrightarrow$ Case (a) is excluded.

(78) $\rightarrow l =$ odd for triplet state

$\longrightarrow l_{nn} = 1$ (in case (b)) (odd parity)

$\longrightarrow$ Pion must have odd parity.

In terms of spectroscopic notation,

$$^{2S+1}L_j \qquad (83)$$

the two neutron state, form the total class of states

$^1S_0$, $^1P_1$, $^1D_2$, $^1F_3$, ... $^3S_1$, $^3P_2$, $^3P_1$, $^3P_0$, $^3D_3$, $^3D_2$, $^3D_1$, $^3F_4$, $^3F_3$, $^3F_2$, ... are restricted to

$^1S_0$, $^1D_2$, ... $^3P_{2,1,0}$, $^3F_{4,3,2}$ ... by the Fermi-Dirac statistic argument, and of these there is only one state, the $^3P_1$, state that has angular momentum 1.