Chapter 14

Operators, Matrices, and Spin

The spin has no classical analog, and it must be treated by somehow abstract methods.

For angular momentum we discussed by the op. method;

$$L^2 Y_\ell^m = \hbar^2 \ell(\ell+1) Y_\ell^m \quad (1)$$

$$L_z Y_\ell^m = \hbar m Y_\ell^m \quad (2)$$

And for the Harmonic Osc;

$$u_n = \frac{1}{\sqrt{n!\,\hbar^n}} (A^+)^n u_0 \quad (3)$$

For which $\quad H u_n = \hbar\omega(n+\frac{1}{2}) u_n \quad (4)$

And we had $\quad A^+ u_n = \sqrt{(n+1)\hbar}\; u_{n+1} \quad (5)$

$$A u_n = \sqrt{n\hbar}\; u_{n-1} \quad (6)$$

$$\langle u_m | u_n \rangle = \delta_{mn} \quad (7)$$

Also $\quad \langle u_m | H u_n \rangle \equiv \langle u_m | H | u_n \rangle = (n+\frac{1}{2})\hbar\omega\,\delta_{mn} \quad (8)$

$$\langle u_m | A^+ u_n \rangle \equiv \langle u_m | A^+ | u_n \rangle = \sqrt{(n+1)\hbar}\;\delta_{m,n+1} \quad (9)$$

$$\langle u_m | A u_n \rangle \equiv \langle u_m | A | u_n \rangle = \sqrt{n\hbar}\;\delta_{m,n-1} \quad (10)$$

These quantities may be arranged in arrays called matrices.

$$H = \hbar\omega \begin{pmatrix} 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & 0 & \cdots \\ 0 & 0 & 5/2 & 0 & \cdots \\ 0 & 0 & 0 & 7/2 & \cdots \\ \cdots & & & & \end{pmatrix} \qquad (11)$$

$$A^+ = \sqrt{\hbar} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & \cdots \\ & & \cdots & & \end{pmatrix} \qquad A = \sqrt{\hbar} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} \cdots \\ & & \cdots & & \end{pmatrix} \qquad (12)$$

Now, the product of two matrices;

$$(FG)_{ij} = \sum_n (F)_{in} (G)_{nj} \qquad (13)$$

Remark: $|\alpha\rangle = \sum_n c_n |u_n\rangle$ (completeness) $\rightarrow c_n = \langle u_n|\alpha\rangle$

$\rightarrow |\alpha\rangle = \sum_n |u_n\rangle\langle u_n|\alpha\rangle \rightarrow \sum_n |u_n\rangle\langle u_n| = I \qquad (14)$

Note also that, $c_n = \langle u_n|\alpha\rangle \sim c_n = \int u_n^* \psi_\alpha \, dx^3$

Now $\underbrace{G|u_j\rangle}_{|\alpha\rangle} = \sum_n c_n |u_n\rangle \rightarrow c_n = \langle u_n|G|u_j\rangle \quad (15)$

$$\rightarrow \langle u_i|FG|u_j\rangle = \langle u_i|F(\sum_n c_n|u_n\rangle) = \sum_n \langle u_i|F|u_n\rangle\langle u_n|G|u_j\rangle \qquad (16)$$

Equ. (16) is the same as (13), provided we write;

$$\langle u_i | F | u_n \rangle = F_{in} \qquad (17)$$

and so on.

Also, $\langle u_m | F | u_n \rangle^* = \langle F u_n | u_m \rangle = \langle u_n | F^+ | u_m \rangle$ (18)

i.e. $\left(\int u_m^* F u_n \right)^* d\tau = \int (F u_n)^* u_m \, d\tau = \int u_n^* F^+ u_m \, d\tau$ (19)

Remark: def. of inner product $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$

because $F | u_n \rangle \longleftrightarrow \langle u_n | F^+$ (20)

This shows that if the op. F is represented by a matrix, then the hermitian conjugate op F^+ will be represented by the hermitian conjugate matrix, since the latter is defined by:

$$(F^+)_{nm} = F_{mn}^* \qquad (21)$$

$\{ Y_l^m \}$ diagonlize L^2 and L_z simultaneously.

$$\langle l'm' | L_z | l m \rangle = \hbar m \, \delta_{mm'} \delta_{ll'} \qquad (22)$$

$$\langle l'm' | L^2 | l m \rangle = l(l+1)\hbar^2 \, \delta_{mm'} \delta_{ll'} \qquad (23)$$

$$\langle l'm' | L_\pm | l m \rangle = \hbar \sqrt{l(l+1) - m(m\pm1)} \, \delta_{m', m\pm1} \delta_{ll'} \qquad (24)$$

For $l=1$

$$
L_z=\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\qquad L_+=\hbar\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}\qquad L_-=\hbar\begin{array}{c}\begin{array}{ccc}m=1&0&-1\end{array}\\\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}\end{array}\begin{array}{c}1\\0\\-1\end{array}
$$

(25)

One may easily obtain:

$$
[L_+,L_-]=2\hbar^2\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}=2\hbar L_z \qquad (26)
$$

General relations between states in matrix representation:

Consider a relation like:

$$
\Psi=A\varphi \qquad (27)
$$

$$
\langle u_i|\Psi\rangle=\langle u_i|A\varphi\rangle \qquad (28)
$$

$$
\rightarrow \langle u_i|\Psi\rangle=\sum_n\langle u_i|A|u_n\rangle\langle u_n|\varphi\rangle \qquad (29)
$$

If $$\langle u_n|\varphi\rangle\rightarrow\begin{pmatrix}\langle u_1|\varphi\rangle\\\langle u_2|\varphi\rangle\\\vdots\\ \\\end{pmatrix}=\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\ \\\end{pmatrix}=|\varphi\rangle \qquad (30)$$

and $$\langle u_n|\Psi\rangle\rightarrow\begin{pmatrix}\langle u_1|\Psi\rangle\\\langle u_2|\Psi\rangle\\\vdots\\ \\\end{pmatrix}=\begin{pmatrix}\beta_1\\\beta_2\\\vdots\\ \\\end{pmatrix}=|\Psi\rangle \qquad (31)$$

The matrix representation of (27) is:

$$\beta_i = \sum_n A_{in} \alpha_n \qquad (32)$$

and $\langle \varphi | u_n \rangle = \langle u_n | \varphi \rangle^*$

$$\langle \varphi | u_n \rangle \to (\alpha_1^*, \alpha_2^*, \cdots) \qquad (33)$$

also
$$\langle \varphi | \psi \rangle = \sum_n \langle \varphi | u_n \rangle \langle u_n | \psi \rangle$$
$$= \sum_n \alpha_n^* \beta_n \qquad (34)$$

An eigenvalue equ.; $A\varphi = a\varphi \qquad (35)$

$$\sum_n A_{in} \alpha_n = a \alpha_i \qquad (36)$$

Or
$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ & & & \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} = a \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix}$$

$$\to \begin{pmatrix} A_{11}-a & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22}-a & A_{23} & \cdots \\ & & & \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} = 0 \qquad (37)$$

For nontrivial sol.;

$$\det |A_{in} - a\delta_{in}| = 0 \qquad (38)$$

$$\text{or} \quad \det |A - aI| = 0$$

The matrix representation is a good way of finding eigenvalues and eigenvector when the dim. is <u>finite.</u>

Ex. Ang. mom. $l = \frac{1}{2}$

$$Y_{1/2}^{\pm 1/2} = C_{\pm} \sqrt{\sin\theta}\; e^{\pm i\varphi/2} \qquad (39)$$

Now using the functional form of L_- (i.e. $L_- = L_-(\theta, \varphi)$, chap. 10) $\rightarrow$

$$L_- Y_{1/2}^{1/2} \sim \frac{\cos\theta}{\sqrt{\sin\theta}}\, e^{-i\varphi/2} \qquad (40)$$

However

$$\frac{\cos\theta}{\sqrt{\sin\theta}}\, e^{-i\varphi/2} \neq Y_{1/2}^{-1/2} \qquad (41)$$

Furthermore it is singular at $\theta = 0, \pi$

Thus for $l = \frac{1}{2}$ there are troubles and we must turn to matrix <u>representations</u>.

Remark: Def. $-$

$$\underbrace{\int (F\psi_1)^* \psi_2 \, d\tau = \int \psi_1^* F^\dagger \psi_2 \, d\tau}_{\text{Def. of Hermitian adjoint op}} \underbrace{= \int \psi_1^* F \psi_2 \, d\tau}_{\text{Def. of Hermitian op.}}$$

If A is Hermitian;

$A\psi_1 = A_1' \psi_1 \quad \rightarrow \psi_2^* A \psi_1 = A_1' \psi_2^* \psi_1$

$A\psi_2 = A_2' \psi_2 \quad \rightarrow (A\psi_2)^* = (A_2' \psi_2)^* \rightarrow \psi_2^* A = A_2'^* \psi_2^* \rightarrow \psi_2^* A \psi_1 = A_2'^* \psi_2^* \psi_1$

subtraction $\rightarrow$ $(A_1' - A_2'^*) \int \psi_2^* \psi_1 \, d\tau = 0$ $\quad$ if $\psi_1 = \psi_2 \rightarrow A_1' = A_1'^*$ (A': real)

If $\psi_1 \neq \psi_2 \rightarrow \int \psi_2^* \psi_1 \, d\tau = 0$

The Spin Op.

The spin ops. are S_x, S_y, S_z and they are defined by

$$[S_x, S_y] = i\hbar S_z \qquad (42)$$

or $$[S_i, S_j] = \epsilon_{ijk} \, i\hbar S_k$$

Note that $S = \frac{1}{2}$

While $\bar{L} = \bar{r} \times \bar{p}$, $\bar{S}$ is something different.

$$(22) \rightarrow \quad \langle S=\tfrac{1}{2}, m_s' | S_z | S=\tfrac{1}{2}, m_s \rangle = \hbar m_s \delta_{m_s m_s'} \qquad (43)$$

$$(24) \rightarrow \quad \langle S\, m_s' | L_\pm | S\, m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s \pm 1)} \; \delta_{m_s', m_s \pm 1} \qquad (44)$$

$$\longrightarrow S_z = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad (45)$$

We may write: $$\bar{S} = \tfrac{1}{2}\hbar \bar{\sigma} \qquad (46)$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (47)$$

Pauli matrices

They satisfy

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

$$(48)$$

or $$[\sigma_i, \sigma_j] = \epsilon_{ijk} \, 2i\sigma_k$$

Also, $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \underline{I}$ (49)

and
$$\sigma_x \sigma_y = -\sigma_y \sigma_x$$
$$\sigma_z \sigma_x = -\sigma_x \sigma_z \qquad (50)$$
$$\sigma_y \sigma_z = -\sigma_z \sigma_y$$

or $\{\sigma_i, \sigma_j\} = 0$ for $i \neq j$

These are relations peculiar to the spin $\frac{1}{2}$ representation and do not hold for the $l=1$ matrices, for example.

Eigenspinors:

$$S_z \begin{pmatrix} u \\ v \end{pmatrix} = \pm \hbar \begin{pmatrix} u \\ v \end{pmatrix} \qquad (51)$$

$$\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix}$$

For + sign : $v = 0$

For − " : $u = 0$

$$\rightarrow \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (52)$$

spin up, $s_z = +\frac{\hbar}{2}$ spin down, $s_z = -\frac{\hbar}{2}$

$$\{ \chi_+, \chi_- \} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{complete set} \tag{53}$$

An arbitrary spinor;

$$\chi = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \alpha_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or $$\chi = \sum_{i=\pm} \alpha_i \chi_i = \alpha_+ \chi_+ + \alpha_- \chi_- \tag{54}$$

$\langle \chi | \chi \rangle = 1$ normalization

$$\rightarrow |\alpha_+|^2 + |\alpha_-|^2 = 1 \tag{55}$$

$|\alpha_+|^2$: The probability that a measurement of S_z on $\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$ yields $\frac{\hbar}{2}$

$|\alpha_-|^2$: " " " " " " " " " " $-\frac{\hbar}{2}$

The Eigenstates of the op. $S_x \cos\varphi + S_y \sin\varphi$:

$$(S_x \cos\varphi + S_y \sin\varphi) \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\hbar}{2} \lambda \begin{pmatrix} u \\ v \end{pmatrix} \tag{56}$$

$$\begin{pmatrix} 0 & \cos\varphi - i \sin\varphi \\ \cos\varphi + i \sin\varphi & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\hbar}{2} \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rightarrow \begin{cases} v e^{-i\varphi} = \lambda u \\ u e^{i\varphi} = \lambda v \end{cases} \tag{57} \quad (\det |A - \lambda I| = 0)$$

$$\rightarrow \quad uv = \lambda^2 uv \quad \rightarrow \quad \lambda = \pm 1 \qquad (58)$$

For $\lambda = +1$ $\begin{cases} u \sim e^{-i\varphi/2} \\ v \sim e^{+i\varphi/2} \end{cases}$ For $\lambda = -1$ $\begin{cases} u \sim e^{-i\varphi/2} \\ v \sim -e^{+i\varphi/2} \end{cases}$ (59)

Using the normalization $(u^*, v^*)\begin{pmatrix} u \\ v \end{pmatrix} = 1$ (60)

the normalized eigenvectors are

$$\chi_{+1} = \begin{pmatrix} u \\ v \end{pmatrix}_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi/2} \\ e^{i\varphi/2} \end{pmatrix} \quad \text{for } \lambda = 1$$

$$\chi_{-1} = \begin{pmatrix} u \\ v \end{pmatrix}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi/2} \\ -e^{i\varphi/2} \end{pmatrix} \quad " \quad \lambda = -1 \qquad (61)$$

We observe that if $\varphi \longrightarrow \varphi + 2\pi$

$$\longrightarrow \quad \chi_{\pm 1}(\varphi + \pi) = -\chi_{\pm 1}(\varphi) \qquad (62)$$

This is characteristic of odd-half-integer spin wave fncs. (fermion states), although this does not violate quantum mechanics, since -1 is just a phase factor, it does mean that no classical macroscopic wave packet can be constructed that has odd half-integral angular momentum.

Expectation value of $\bar{S}$ for an arbitrary state $|\alpha\rangle$

$$\langle\alpha|\bar{S}|\alpha\rangle = \sum_i \sum_j \langle\alpha|i\rangle\langle i|S|j\rangle\langle j|\alpha\rangle \quad (63)$$

where $|1\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|2\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ (64)

Or, equivalently;

$$\langle\alpha|\bar{S}|\alpha\rangle = (\alpha_+^*, \alpha_-^*)\, \bar{S} \begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} \quad (65)$$

Thus:

$$\langle S_x\rangle = \langle\alpha|S_x|\alpha\rangle = (\alpha_+^*, \alpha_-^*)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} \quad 66$$

$$\langle S_x\rangle = \frac{\hbar}{2}(\alpha_+^*, \alpha_-^*)\begin{pmatrix}\alpha_-\\ \alpha_+\end{pmatrix} = \frac{\hbar}{2}(\alpha_+^*\alpha_- + \alpha_-^*\alpha_+)$$

$$\langle S_y\rangle = \frac{\hbar}{2}(\alpha_+^*, \alpha_-^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = \frac{\hbar}{2}(\alpha_+^*, \alpha_-^*)\begin{pmatrix}-i\alpha_-\\ i\alpha_+\end{pmatrix}$$

$$= -\frac{i\hbar}{2}(\alpha_+^*\alpha_- - \alpha_-^*\alpha_+) \quad (67)$$

$$\langle S_z\rangle = \frac{\hbar}{2}(\alpha_+^*, \alpha_-^*)\begin{pmatrix}\alpha_+\\ -\alpha_-\end{pmatrix} = \frac{\hbar}{2}\left(|\alpha_+|^2 - |\alpha_-|^2\right) \quad (68)$$

Note that all of these are <u>real</u>, as expected for <u>hermitian ops</u>. . .

We shall see later that the spin of an electron appears in the Hamiltonian for the hydrogen atom, for example, coupled to the orbital ang. mom.

Ex.

When an electron is localized at a crystal lattice site, for example, it is often possible to treat the spin as the only degree of freedom that the electron possesses.

Now; $\bar{M} = -\frac{eg}{2mc}\bar{S}$ (69) intrinsic mag. dipole moment of the electron by virtue of its spin

where $g = 2\left(1+\frac{\alpha}{2\pi}+\cdots\right) = 2.0023192$ gyromagnetic ratio

In the presence of an external mag. field;

$$H = -\bar{M}\cdot\bar{B} = \frac{eg\hbar}{4mc}\,\sigma\cdot B \qquad (70)$$

The schrödinger equ. for $\Psi(t) = \begin{pmatrix}\alpha_+(t)\\ \alpha_-(t)\end{pmatrix}$ is;

$$i\hbar\frac{d\Psi(t)}{dt} = \frac{eg\hbar}{4mc}\,\sigma\cdot B\,\Psi(t) \qquad (71)$$

If $\bar{B} = B\hat{z}$

and if we write $\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = e^{i\omega t} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$ (72)

$$\rightarrow \hbar\omega \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \frac{eg\hbar B}{4mc} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} \quad (73)$$

$$\rightarrow \omega = \pm \frac{egB}{4mc} \quad (74)$$

$$\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \omega = + \frac{egB}{4mc}$$

$$\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{"} \quad \omega = - \frac{egB}{4mc}$$

(75)

$$\left\{ \begin{array}{l} \hbar\omega \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \frac{egB}{4mc} \begin{pmatrix} \alpha_+ \\ -\alpha_- \end{pmatrix} \\ \rightarrow \hbar\omega \alpha_+ = \frac{egB}{4mc} \alpha_+ \\ \quad \hbar\omega \alpha_- = -\frac{egB}{4mc} \alpha_- \\ \rightarrow \begin{cases} \alpha_+ = 1 \\ \alpha_- = 0 \end{cases} \text{ or } \begin{cases} \alpha_+ = 0 \\ \alpha_- = 1 \end{cases} \end{array} \right.$$

Thus if the initial state is;

$$\psi(0) = \begin{pmatrix} a \\ b \end{pmatrix} \quad (76)$$

$$\rightarrow \psi(t) = \begin{pmatrix} a\, e^{-i\omega t} \\ b\, e^{i\omega t} \end{pmatrix} \qquad \omega = \frac{geB}{4mc} \quad (77)$$

Suppose at $t=0$ the state is an eigestate of S_x with eigenvalue $\frac{\hbar}{2}$ (i.e. spin points in the x-dir.)

i.e. $$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \qquad (78)$$

$$\rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (79)$$

At a later time;

$$\langle S_x \rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(e^{i\omega t}, e^{-i\omega t}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\omega t}, e^{-i\omega t}\right)\begin{pmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{pmatrix} = \frac{\hbar}{2}\cos 2\omega t \qquad (80)$$

Similarly:

$$\langle S_y \rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(e^{i\omega t}, e^{-i\omega t}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(-i\,e^{2i\omega t} + i\,e^{-2i\omega t}\right) = \frac{\hbar}{2}\sin 2\omega t \qquad (81)$$

Thus, the spin precesses about z-axis, the dir. of B with the frequency,

$$2\omega = \frac{egB}{2mc} \simeq \frac{eB}{mc} \qquad (82)$$

Paramagnetic Resonance Method:

In a solid the gyromagnetic factor g of an electron is affected by the nature of the forces acting in the solid. A knowledge of g provides very useful constraints on what these forces could be, and it is therefore important to be able to measure g.

This can be done by the paramagnetic resonance method; which we now describe;

Consider an electron, whose only degrees of freedom are spin states, also consider;

$\bar{B}_0 = B_0 \hat{z}$: a large const. mag. field in the z-dir.

$B_1 \cos\omega t\, \hat{x}$: a small oscillating " " " x-dir.

i.e. $$\bar{B} = B_0 \hat{z} + B_1 \cos\omega t\, \hat{x} \qquad (83)$$

$$(71)\,(83) \Rightarrow i\hbar \frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{eg\hbar}{4mc}\begin{pmatrix} B_0 & B_1 \cos\omega t \\ B_1 \cos\omega t & -B_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \qquad (84)$$

with $$\omega_0 = \frac{egB_0}{4mc}, \quad \omega_1 = \frac{egB_1}{4mc} \qquad (85)$$

$$\rightarrow \begin{cases} i\dfrac{da(t)}{dt} = \omega_0\, a(t) + \omega_1 \cos\omega t\; b(t) \\[2ex] i\dfrac{db(t)}{dt} = \omega_1 \cos\omega t\; a(t) - \omega_0\, b(t) \end{cases} \qquad (86)$$

Let
$$\begin{aligned} A(t) &= a(t)\, e^{i\omega_0 t} \\ B(t) &= b(t)\, e^{-i\omega_0 t} \end{aligned} \qquad (87)$$

These satisfy the equ.;

$$i\frac{dA(t)}{dt} = \omega_1 \cos\omega t\; B(t)\, e^{2i\omega_0 t} \approx \frac{1}{2}\omega_1\, e^{i(2\omega_0-\omega)t}\, B(t)$$

$$i\frac{dB(t)}{dt} = \omega_1 \cos\omega t\; A(t)\, e^{-2i\omega_0 t} \approx \frac{1}{2}\omega_1\, e^{-i(2\omega_0-\omega)t}\, A(t)$$

(88)

where we have used the following approx.;

$$\cos\omega t\; e^{2i\omega_0 t} = \frac{1}{2}\left[e^{i(2\omega_0+\omega)t} + e^{i(2\omega_0-\omega)t} \right] \approx \frac{1}{2} e^{i(2\omega_0-\omega)t} \qquad (89)$$

Note that, here we have droped the first term, because we are interested in values of $\omega = 2\omega_0$ when both of them are large, then the first term oscillates rapidly and we may expect its contribution averages to zero.

First equ. of (88) $\rightarrow$ $$B(t) = \frac{2i}{\omega_1} \frac{dA(t)}{dt} e^{-i(2\omega_0-\omega)t}$$ (90)

Differentiating this and using the second equ. of (88);

$$\rightarrow \quad \frac{d^2A(t)}{dt^2} - i(\omega_0-\omega)\frac{dA(t)}{dt} + \frac{\omega_1^2}{4}A(t) = 0 \qquad (91)$$

A trial sol. $$A(t) = A(0)\, e^{i\lambda t} \qquad (92)$$

(92) in (91) $\rightarrow$ $$-\lambda^2 + (2\omega_0-\omega)\lambda + \frac{\omega_1^2}{4} = 0 \qquad (93)$$

$$\lambda_\pm = \frac{2\omega_0 - \omega \pm \sqrt{(2\omega_0-\omega)^2+\omega_1^2}}{2} \qquad (94)$$

The most general sol.;

$$A(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t} \qquad (95)$$

$$\rightarrow B(t) = -\frac{2}{\omega_1} e^{-i(2\omega_0-\omega)t}\left(\lambda_+ A_+ e^{i\lambda_+ t} + \lambda_- A_- e^{i\lambda_- t}\right) \qquad (96)$$

This finally yields;

$$a(t) = e^{-i\omega_0 t}\left(A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t}\right) \qquad (97)$$

$$b(t) = -\frac{2}{\omega_1} e^{-i(\omega_0-\omega)t}\left(\lambda_+ A_+ e^{i\lambda_+ t} + \lambda_- A_- e^{i\lambda_- t}\right) \qquad (98)$$

If at $t=0$ $\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (99)

i.e. the electron spin points in the z-dir.

$$(97)(98)(99) \longrightarrow \begin{cases} A_+ + A_- = 1 \\ \lambda_+ A_+ + \lambda_- A_- = 0 \end{cases} \qquad (100)$$

$$\rightarrow \begin{cases} A_+ = \dfrac{\lambda_-}{\lambda_- - \lambda_+} \\ A_- = -\dfrac{\lambda_+}{\lambda_- - \lambda_+} \end{cases} \qquad (101)$$

The probability that at some later time t the spin points in the negative z-dir. is $|b(t)|^2$;

$$|b(t)|^2 = \frac{4}{\omega_1^2} \left| \frac{\lambda_+\lambda_-}{\lambda_- - \lambda_+} e^{i\lambda_+ t} - \frac{\lambda_+\lambda_-}{\lambda_- - \lambda_+} e^{i\lambda_- t} \right|^2$$

$$= \frac{\omega_1^2}{(2\omega_0 - \omega)^2 + \omega_1^2} \left| 1 - e^{-i(\lambda_+ - \lambda_-)t} \right|^2$$

$$= \frac{\omega_1^2}{(2\omega_0 - \omega)^2 + \omega_1^2} \cdot \frac{1 - \cos\sqrt{(2\omega_0 - \omega)^2 + \omega_1^2}\; t}{2} \qquad (102)$$

Since $\omega_1 \ll \omega, \omega_0 \rightarrow |b(t)|^2 \approx$ small

When $\omega = 2\omega_0 \rightarrow |b(t)|^2 \rightarrow \dfrac{1 - \cos\omega_1 t}{2}$ (103)

Since the energy of the up state is different from that of the down state, such an energy difference, absorbed from the external field signals the resonance frequency; so that ω_0 and hence g can be measured with great precision.