Chapter 13

Interaction of Electrons with Electromagnetic Field

Classical Theory:

Maxwell's equs. in Gaussian units in the vacuum

$$\nabla\cdot B(\bar{r},t) = 0 \qquad (a)$$
$$\nabla\times E(\bar{r},t) + \frac{1}{c}\frac{\partial B(\bar{r},t)}{\partial t} = 0 \qquad (b)$$
$$\nabla\cdot E(\bar{r},t) = 4\pi\rho(\bar{r},t) \qquad (c) \qquad (1)$$
$$\nabla\times B(\bar{r},t) - \frac{1}{c}\frac{\partial E(\bar{r},t)}{\partial t} = \frac{4\pi}{c}J(\bar{r},t) \qquad (d)$$

$\rho(\bar{r},t)$: charge density
$J(\bar{r},t)$: current "
: The sources of electromag. fields $E(\bar{r},t)$, $B(\bar{r},t)$

$$(1) \Rightarrow \frac{\partial\rho(\bar{r},t)}{\partial t} + \nabla\cdot J(\bar{r},t) = 0 \qquad (2) \quad \text{Charge conservation}$$

In terms of scalar and vector pot.:

$$B(\bar{r},t) = \nabla\times A(\bar{r},t)$$
$$E(\bar{r},t) = -\frac{1}{c}\frac{\partial A(\bar{r},t)}{\partial t} - \nabla\varphi(\bar{r},t) \qquad (3)$$

In order to satisfy the first two of (1).

Remark:
$\nabla\cdot\nabla\times F = 0$
$\nabla\times\nabla\varphi = 0$

The fields E and B <u>do not</u> determine φ and A <u>uniquely</u>

New pots. given by

$$\begin{aligned} A'(\bar{r},t) &= A(r,t) - \nabla f(\bar{r},t) \\ \varphi'(\bar{r},t) &= \varphi(\bar{r},t) + \frac{1}{c}\frac{\partial f(\bar{r},t)}{\partial t} \end{aligned} \qquad (4)$$

yield the <u>same</u> E and B.

$$\begin{cases} A \\ \varphi \end{cases} \xrightarrow{\text{Gauge tr.}} \begin{cases} A' \\ \varphi' \end{cases}$$

$$\begin{cases} E \\ B \end{cases} \xrightarrow[\text{invariance}]{\text{Gauge}} \begin{cases} E \\ B \end{cases} \qquad (5)$$

$\longrightarrow$ we can choose $f(\bar{r},t)$ in the most convenient way.

(1-c)(1-d)(3) $\longrightarrow$ the source-dep. equs.

$$\begin{cases} -\nabla^2\varphi(\bar{r},t) - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot A) = 4\pi\rho(\bar{r},t) & (a) \\ \nabla\times(\nabla\times A) + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\nabla\varphi = \frac{4\pi}{c}J(\bar{r},t) & (b) \end{cases} \qquad (6)$$

Since $\nabla\times(\nabla\times A) = -\nabla^2 A + \nabla(\nabla\cdot A)$ (only valid in Cartesian coord.)

$$(6\text{-}b) \rightarrow \quad -\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla\left(\nabla\cdot A + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) = \frac{4\pi}{c}J \qquad (7)$$

If $\frac{\partial\rho}{\partial t} = 0$ (static charge dist.), it is convenient to choose the gauge such that

$$\nabla\cdot A(\bar{r},t) = 0 \qquad (8) \qquad \text{Coulomb gauge}$$

(with a suitable choice of $f(\bar{r},t)$ (equ.(4))), In that case; $\begin{matrix}(3)\\(1\text{-}c)\\(8)\end{matrix} \longrightarrow \quad -\nabla^2 \varphi(\bar{r}) = 4\pi\rho(\bar{r}) \qquad (9)$

t-indep.

$$(7)(9) \rightarrow \quad -\nabla^2 A(\bar{r},t) + \frac{1}{c^2}\frac{\partial^2 A(\bar{r},t)}{\partial t} = \frac{4\pi}{c} J(\bar{r},t) \qquad (10)$$

$\left(\text{Since } \frac{\partial\varphi}{\partial t} = 0\right)$

When $\frac{\partial\rho}{\partial t} \neq 0$, it is convenient to choose Lorentz gauge;

$$\nabla \cdot A(\bar{r},t) + \frac{1}{c}\frac{\partial\varphi(\bar{r},t)}{\partial t} = 0 \qquad (11)$$

This leaves the equ. (6-b) unaltered,

$$(11)(7) \rightarrow \quad -\nabla^2 A(\bar{r},t) + \frac{1}{c^2}\frac{\partial^2 A(\bar{r},t)}{\partial t^2} = \frac{4\pi}{c} J(\bar{r},t) \qquad (12)$$

$$(11)(6\text{-}a) \rightarrow \quad -\nabla^2 \varphi(\bar{r},t) + \frac{1}{c^2}\frac{\partial^2\varphi(\bar{r},t)}{\partial t^2} = 4\pi\rho(\bar{r},t) \qquad (13)$$

Now, the int. of a point electron of mass μ and charge $(-e)$ with an elmag. field is the classical Lorentz force equ.;

$$\mu \frac{d^2\bar{r}}{dt^2} = -e\left[\bar{E}(\bar{r},t) + \frac{\bar{V}}{c} \times \bar{B}(\bar{r},t)\right] \qquad (14)$$

Remark: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (15)$

$$P_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \qquad (16)$$

$$L = T - U = \frac{1}{2}mv^2 - (q\varphi - \frac{q}{c}A\cdot v) \quad (17) \quad \text{(Goldstein)}$$

U
generalized
pot

$(16)(17) \rightarrow P_i = m\dot{x}_i + \frac{q}{c}A_i \quad (18)$

$\rightarrow \quad m\dot{x}_i = P_i - \frac{q}{c}A_i$

$$H = \frac{(m\dot{x}_i)^2}{2m} + q\varphi = \frac{1}{2m}\left(\bar{P} - \frac{q}{c}\bar{A}\right)^2 + q\varphi \quad (19)$$

Remarks; $\dot{q}_i = \frac{\partial H}{\partial P_i}$, $-\dot{P}_i = \frac{\partial H}{\partial q_i}$, $\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$ (20)

Using (20) and (19), the equ. (16) can be proved for each component.

So in the presence of the vector and the scalar fields $A(\bar{r},t)$ and $\varphi(\bar{r},t)$; the Hamiltonian for an electron having charge $(-e)$ and mass μ is given by:

$$H = \frac{1}{2\mu}\left[P + \frac{e}{c}A(\bar{r},t)\right]^2 - e\varphi(\bar{r}) \qquad (21)$$

we are dealing with static scalar pots.

The corresponding Schrödinger equ:

with, $$P \longrightarrow P - \frac{q}{c}A(r,t) \qquad (22)$$

and $$V = q\varphi(\bar{r}) \qquad (23)$$

$$-\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla - \frac{-e}{c}A\right)^2 \psi(\bar{r},t) + (-e)\varphi(\bar{r})\psi(\bar{r},t) = E\,\psi(\bar{r},t) \qquad (24)$$

Now;

$$\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}A\right)\cdot\left(\frac{\hbar}{i}\nabla\psi + \frac{e}{c}A\psi\right) =$$

$$= -\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{ie\hbar}{\mu c}A\cdot\nabla\psi - \frac{ie\hbar}{2\mu c}(\nabla\cdot A)\psi + \frac{e^2}{2\mu c^2}A^2\psi$$

$$= -\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{ie\hbar}{\mu c}A\cdot\nabla\psi + \frac{e^2}{2\mu c^2}A^2\psi \qquad (25)$$

where we have used;

$$\nabla\cdot(A\psi) = (\nabla\cdot A)\psi + A\cdot\nabla\psi$$

and $$\nabla\cdot A = 0 \qquad (26)$$

For a const. uniform mag. field, $\bar{B}$, we may take

$$A = -\tfrac{1}{2} r \times B \qquad (27)$$

Note that this choice is not <u>unique</u>, because

$$A = -\tfrac{1}{2} r \times B + \nabla g \qquad (28) \qquad g: \text{arbitrary func.}$$

gives the same $\bar{B}$.

Now $A = -\frac{1}{2}(yB_z - zB_y,\ zB_x - xB_z,\ xB_y - yB_x)$

$$\nabla \times A = (\tfrac{1}{2} B_x + \tfrac{1}{2} B_x,\ B_y,\ B_z) = \bar{B} \qquad (29)$$

The second term in (25)

$$+\frac{ie\hbar}{2\mu c} r \times B \cdot \nabla \psi = -\frac{ie\hbar}{2\mu c} B \cdot r \times \nabla \psi = \frac{e}{2\mu c} B \cdot r \times \left(\frac{\hbar}{i} \nabla \psi\right)$$
$$= \frac{e}{2\mu c} B \cdot L \psi \qquad (30)$$

where we have used

$$(A \times B) \cdot C = B \cdot (A \times C)$$

and the third term, taking <u>$\bar{B} = B\hat{z}$</u>

$$\frac{e^2}{8\mu c^2} (r \times B)^2 \psi = \frac{e^2}{8\mu c^2} \left[r^2 B^2 - (r \cdot B)^2 \right] \psi = \frac{e^2 B^2}{8\mu c^2} (x^2 + y^2) \psi \qquad (31)$$

similar to two-dim. Harmonic osc pot.

where we have used

$$(A\times B)\cdot(A\times B) = A^2B^2 - (A\cdot B)^2$$

Let us compare the magnitude of the two terms:

$$\frac{\left\langle \frac{e^2B^2}{8\mu c^2}(x^2+y^2)\right\rangle}{\left\langle \frac{e}{2\mu c} B\cdot L\right\rangle} = \frac{\frac{e^2B^2}{8\mu c^2}\langle(x^2+y^2)\rangle}{\frac{e}{2\mu c}B\cdot\langle L\rangle} = \frac{\frac{e^2B^2}{8\mu c^2}\langle(x^2+y^2)\rangle}{\frac{e}{2\mu c}B\langle L_z\rangle}$$

$$\approx \frac{\frac{e^2B^2}{8\mu c^2}a_0^2}{\frac{e}{2\mu c}B(\hbar)} \approx \frac{1}{4}\frac{e^2}{\hbar c}\frac{B}{e/a_0^2} \approx \frac{1}{548}\frac{B}{e/a_0^2}$$

$$\simeq \frac{B}{548(4.8\times10^{-10})/(0.5\times10^{-8})^2} = \frac{B}{9\times10^{9}\,\text{gauss}} \qquad (32)$$

Then for atomic systems, with $B \leqslant 10^4$ gauss (available in the laboratory) the quadratic term is negligible.

Also

$$\frac{\text{linear term}}{\text{Coulomb pot. energy}} = \frac{\langle\frac{e}{2\mu c}B\cdot L\rangle}{\frac{e^2}{r}} \simeq \frac{\frac{e}{2\mu c}\hbar B}{\frac{e^2}{r}} \simeq \frac{1}{2}\frac{\hbar/\mu c}{e/a_0}B$$

$$\simeq \frac{1}{274}\frac{B}{e/a_0^2} \simeq \frac{B}{5\times10^9\,\text{gauss}} \qquad (33)$$

So that the linear term will only <u>slightly</u> perturb the atomic energy levels.

The quadratic is important in two conditions:

i- $B \to$ large (on the surface of neutron stars $B \sim 10^{12}$ gauss)

ii- macroscopic motion of an electron (out of the atom $\to (x^2+y^2) \simeq$ large) in an external field.

The linear term;

$$H_1 = \frac{e}{2\mu c} B L_z = \omega_L L_z \qquad (34)$$

where $\omega_L = \dfrac{eB}{2\mu c}$ Larmor frequency

If the energy eigenstates are simultaneous eigenstates of L^2 and L_z;

i.e $\quad H\Psi = E\Psi, \quad L^2\Psi = \ell(\ell+1)\hbar^2\Psi, \quad L_z\Psi = m\hbar\Psi \qquad (35)$

$\longrightarrow \quad H_1\Psi = \omega_L(m\hbar)\Psi$

$$\text{or} \quad H_1 U_{n\ell m}(\bar{r}) = \hbar\omega_L m\, U_{n\ell m}(\bar{r}) \qquad (36)$$

$$-\ell \leq m \leq \ell$$

$$E = -\frac{1}{2}\mu c^2 \left(\frac{Z\alpha}{n^2}\right)^2 + \hbar\omega_L m \qquad (37)$$

$\longrightarrow$ $(2l+1)$-fold degeneracy has been disappeared.

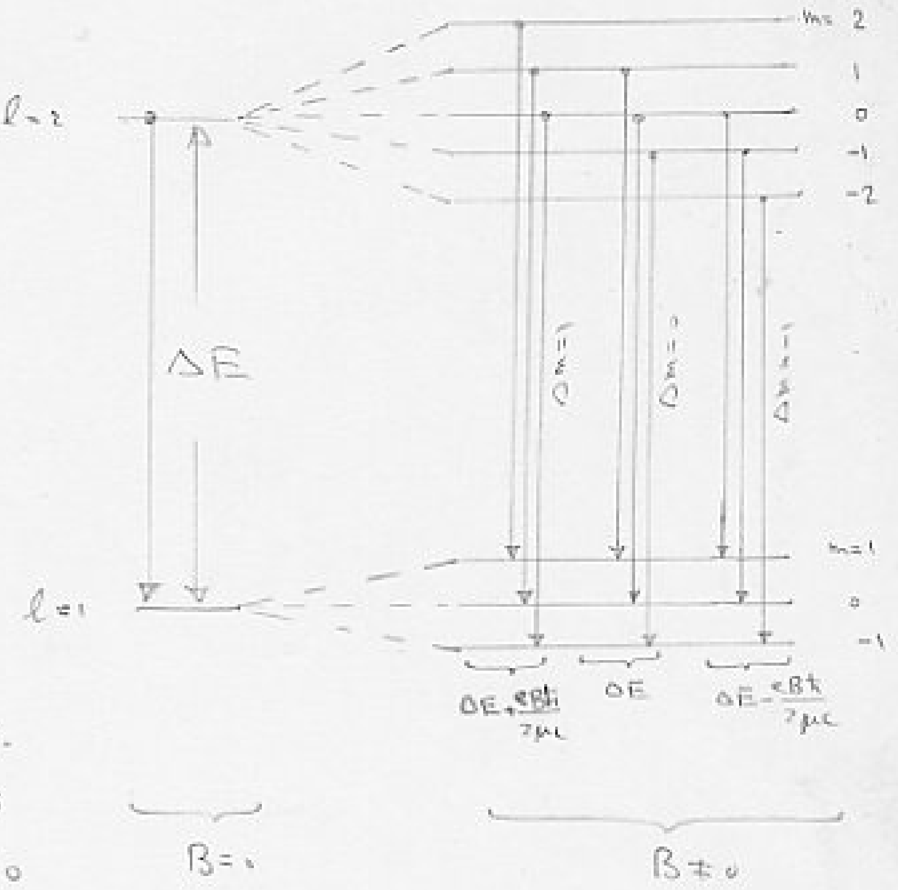
The size of splitting:

$$\hbar\omega_L = \frac{eB\hbar}{2\mu c} = \frac{e\hbar}{2\mu c}\left(\frac{B}{e/a_0^2}\right)\frac{e}{a_0^2} = \frac{e^2\hbar}{2\mu c}\left(\frac{\mu c\alpha}{\hbar}\right)^2\left(\frac{B}{e/a_0^2}\right)$$

$$= \left(\frac{1}{2}\alpha^2\mu c^2\right)\frac{B}{e/a_0^2} = \frac{B}{2.4\times10^9}\,(13.6)\,eV \qquad (38)$$

So, the energy levels split to $2l+1$ distinct levels (in the presence of $B\neq0$); This is the normal Zeeman effect.

The anomalous Zeeman effect takes into account the spin of the electron.



Taking into account the selection rule (to be discussed later) $\Delta m = 0, \pm1$

A single tr. line $\longrightarrow$ 3-tr. lines

$B=0$ $\qquad$ $B\neq0$

Another approximation;

under conditions where $\begin{cases} \text{quadratic term } (\sim B^2) \text{ is not negligible} \\ \text{Coulomb term } (-e\varphi(\bar{r})) \text{ is negligible} \end{cases}$

The Schrödinger equ.;

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \frac{eB}{2\mu c} L_z\psi + \frac{e^2B^2}{8\mu c^2}(x^2+y^2)\psi = E\psi \qquad (39)$$

The presence of the potential $\sim (x^2+y^2)$ suggests the use of cylindrical coordinates for the separation of the variables;

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases} \rightarrow \begin{cases} d\rho = \cos\varphi\, dx + \sin\varphi\, dy \\ d\varphi = \frac{1}{\rho}(-\sin\varphi\, dx + \cos\varphi\, dy) \end{cases} \qquad (40)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x} = \cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial y} = \sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi} \qquad (41)$$

$$\rightarrow \quad \nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} \qquad (42)$$

If we now write $\psi(\bar{r}) = u_m(\rho)\, e^{im\varphi}\, e^{ikz}$

$$\rightarrow \frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} - \frac{m^2}{\rho^2}u - \frac{e^2B^2}{4\hbar^2c^2}\rho^2 u + \left(\frac{2\mu E}{\hbar^2} - \frac{eB\hbar m}{\hbar^2 c} - k^2\right)u = 0$$

No let $$x = \sqrt{\frac{eB}{2\hbar c}}\, \rho \qquad (43)$$

$$\rightarrow \frac{d^2 U}{dx^2} + \frac{1}{x}\frac{dU}{dx} - \frac{m^2}{x^2} U - x^2 U + \lambda U = 0 \qquad (44)$$

where $$\lambda = \frac{4\mu c}{eB\hbar}\left(E - \frac{\hbar^2 k^2}{2\mu}\right) - 2m \qquad (45)$$

i) The behavior of $U(x)$ at infinity;

$$(44) \underset{x\to\infty}{\longrightarrow} \frac{d^2 U}{dx^2} - x^2 U \simeq 0 \qquad (46)$$

$$\longrightarrow U(x) \sim e^{-x^2/2} \qquad (47)$$

ii) The behavior of $U(x)$ near $x \simeq 0$

$$(44) \longrightarrow \frac{d^2 U}{dx^2} + \frac{1}{x}\frac{dU}{dx} - \frac{m^2}{x^2} U \simeq 0 \qquad (48)$$

$$\rightarrow U(x) \sim x^{|m|}. \qquad (49)$$ ($x^{-|m|}$ is singular near $x \simeq 0$)

$$\rightarrow U(x) = x^{|m|} e^{-x^2/2} G(x) \qquad (50)$$

$$(44)(50) \rightarrow \quad \frac{d^2G}{dx^2} + \left(\frac{2|m|+1}{x} - 2x\right)\frac{dG}{dx} + (\lambda - 2 - 2|m|)\,G = 0 \qquad (51)$$

Now let $\quad y \equiv x^2 \qquad (52)$

$$\rightarrow \quad \frac{d^2G}{dy^2} + \left(\frac{|m|+1}{y} - 1\right)\frac{dG}{dy} + \frac{\lambda - 2 - 2|m|}{4y}\,G = 0 \qquad (53)$$

This is of the form of equ. for H discussed in chap. 12, (equ. 12-35);

Comparing (53) with (12-35, p223); we must have;

$$\frac{\lambda}{4} - \frac{1+|m|}{2} = n_r \qquad (54)$$

where $n_r = 0, 1, 2, \ldots \qquad (55)$

$$\begin{matrix}(45)\\(54)\end{matrix} \rightarrow \quad E - \frac{\hbar^2 k^2}{2\mu} = \frac{eB\hbar}{2\mu c}\left(2n_r + 1 + |m| + m\right) \qquad (56)$$

and $\quad G(y) = L_{n_r}^{|m|}(y) \qquad (57)$

Classical limit;

Given equ. (21) for H, without the scalar pot. term;

we have $\quad \mu\dot{X} = P - \left(\frac{-e}{c}A\right) \rightarrow V = \frac{P + \frac{e}{c}A}{\mu} \qquad (58)$

and with $A = -\frac{1}{2} r \times B$

$$\mu(r \times V) = r \times P + \frac{e}{c} r \times \left(-\frac{1}{2} r \times B\right) = \bar{L} - \frac{e}{2c}\left[\bar{r}(r \cdot B) - r^2 \bar{B}\right] \quad (59)$$

where we have used the identity;

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \quad (60)$$

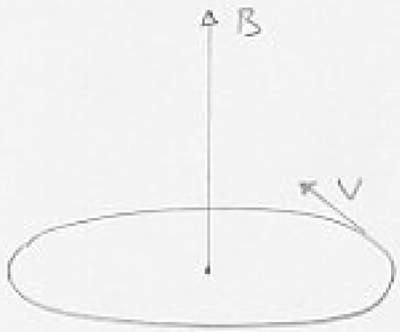
The z-component of (59)

$$\left[\mu(r \times V)\right]_z = L_z + \frac{e}{2c} B (x^2 + y^2) \quad (61)$$

that is; $\mu \rho V = L_z + \frac{eB}{2c} \rho^2$ (62)

The Lorentz force on the electron;

$$\bar{F} = -\frac{e}{c} \bar{V} \times \bar{B} + 0 \quad (63)$$



$$\rightarrow \frac{\mu V^2}{\rho} = \frac{eVB}{c} \quad (64)$$

for circular motion

(62)
(64) $\rightarrow$

$$\begin{cases} \frac{1}{2}\mu V^2 = \frac{eB}{\mu c} L_z & (65) \\ \rho = \left[\frac{2c}{eB} L_z\right]^{\frac{1}{2}} & (66) \end{cases}$$

Now return to the expression for E, (56),

Since $\hbar$ = very small;

E can be of macroscopic size for reasonable B, if

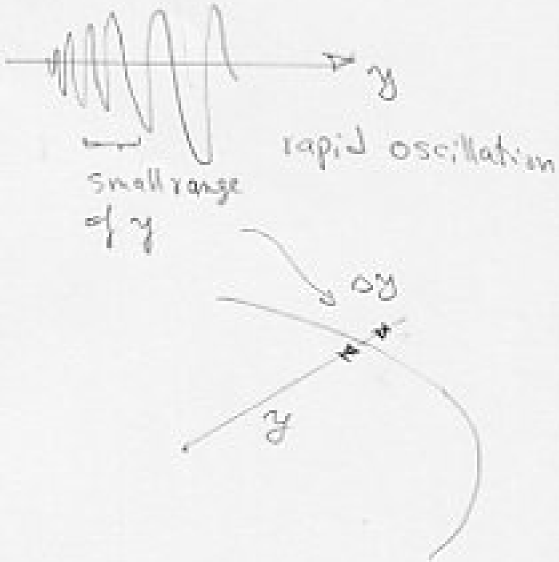
$$(2n_r + 1 + |m| + m) = \text{very large} \qquad (66')$$

We have two cases:

a) $m < 0 \xrightarrow{(66')} n_r$ = very large

n_r : degree of the polynomial $L_{n_r}^{|m|}(y)$ (number of zeros)

Now if n_r = large $\longrightarrow$



$\rightarrow$ the func. cannot be large for some small range of y where the classical orbit would be located

b) $m > 0$ the coeff. $= (2n_r + 1 + 2m)$

This can be large with n_r = small, provided m = large

$$\rightarrow \quad E - \frac{\hbar^2 k^2}{2\mu} \simeq \frac{eB}{\mu c}\hbar m \qquad (67)$$

This is in agreement with the classical result (65)

Note that $$L_z = m\hbar \qquad (68)$$

is positive, as expected.

We can also show that the radius of the orbit, as determined by the peaking of the radial probability dist., corresponds to the classical value.

Let us take $n_r = 0 \longrightarrow L^{|m|}_{n_r=0}(y) = \text{const.}$

$$(50) \rightarrow P(x) = |u(x)|^2 \sim x^{2|m|} e^{-x^2} \qquad (69)$$

$$\frac{dP}{dx} = \left(2|m|\, x^{2|m|-1} - 2x^{2|m|+1}\right) e^{-x^2} = 0 \qquad (70)$$

has a max. at

$$x = \sqrt{|m|} \qquad (71)$$

$$\rho = \left(\frac{2c}{eB}\hbar m\right)^{1/2} \qquad (72)$$

which is in agreement with (66)

This prob. is a beautiful illustration of the correspondence principle.

There are several interesting quantum mechanical effects connected with the interaction with a mag. field that we now turn to.

The Schrödinger equ. (24) appears to violate the principle of gauge invariance, since it is $A(\bar{r},t)$ that appears in the equ., and under the tr.;

$$A \longrightarrow A + \nabla f(\bar{r},t) \qquad (73)$$

the Hamiltonian is changed acc. to;

$$\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}A\right)^2 \longrightarrow \frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}A + \frac{e}{c}\nabla f\right)^2 \qquad (74)$$

It is possible to save gauge invariance by using the fact that a change of the wave func. by a phase factor, which may depend on $\bar{r}$, has no physical consequences.

Thus if we require that (73) must be accompanied by the tr.

$$\Psi(\bar{r},t) \rightarrow e^{i\Lambda(\bar{r},t)}\,\Psi(\bar{r},t) \qquad (75)$$

The first term of the left hand side of (24) becomes;

$$\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}A+\frac{e}{c}\nabla f\right)\cdot\left(\frac{\hbar}{i}\nabla+\frac{e}{c}A+\frac{e}{c}\nabla f\right)e^{i\Lambda}\Psi=$$

$$=\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}A+\frac{e}{c}\nabla f\right)\cdot\left[e^{i\Lambda}\left(\frac{\hbar}{i}\nabla\Psi+\frac{e}{c}A\Psi+\frac{e}{c}\nabla f\,\Psi+\hbar\nabla\Lambda\Psi\right)\right]$$

$$=\frac{1}{2\mu}e^{i\Lambda}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}A+\frac{e}{c}\nabla f+\hbar\nabla\Lambda\right)^2\Psi \qquad (76)$$

Thus with the choice; $\Lambda=-\frac{e}{\hbar c}f$ (77)

That is, with the tr. law,

$$\Psi(\bar{r},t)\longrightarrow e^{-\frac{ie}{\hbar c}f(\bar{r},t)}\,\Psi(\bar{r},t) \qquad (78)$$

gauge invariance is restored.

In the field-free region, $B=0$

$$\text{i.e.}\quad \nabla\times A=0 \qquad (79)$$

$$\rightarrow\quad A=\nabla f \qquad (80)$$

In this case we may describe the motion of an electron in two ways:

i) $$\left[\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla\right)^2+V(\bar{r})\right]\Psi=E\Psi \qquad (81)$$

ii) $$\frac{1}{2\mu}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}A\right)^2\Psi'+V(r)\Psi'=E\Psi' \qquad (82)$$

where A is given by (80)

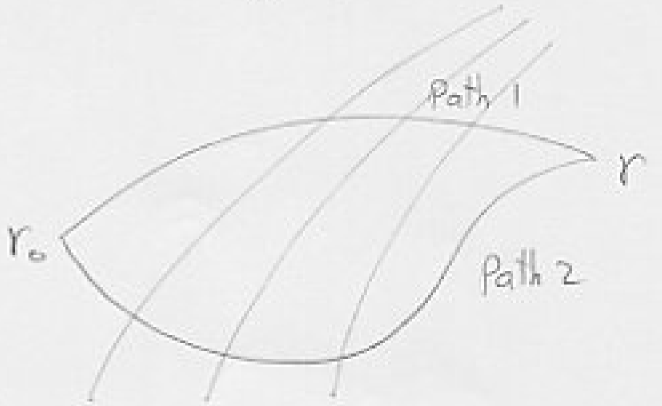
and take $\quad \psi' = e^{-\frac{ie}{\hbar c} f} \psi \qquad (83)$

$$(80) \longrightarrow \quad f(\bar{r},t) = \int_{p}^{r} d\bar{r}' \cdot A(\bar{r}',t) \qquad (84)$$

arbitrary point
the origin or infinity

The integral only makes sence if $B = 0$ (field-free region)

i.e.



$$\int_1 d\bar{r}' \cdot A(r',t) - \int_2 d\bar{r}' \cdot A(r',t)$$

$$= \oint d\bar{r}' \cdot A(\bar{r}',t)$$

$$= \int_S \nabla' \times A(\bar{r}',t) \cdot ds = \int_S B \cdot ds = \Phi \qquad (85)$$

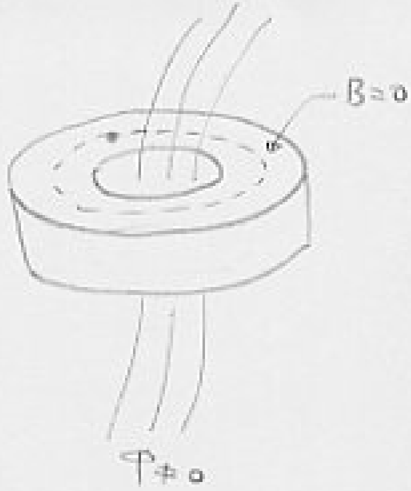
where we have used Stokes' theorem.

$$\longrightarrow \quad \int_1 d\bar{r}' \cdot A(r',t) \neq \int_2 d\bar{r}' \cdot A(r',t) \qquad \text{if } B \neq 0 \text{ in general}$$

Thus if $\Phi = 0$ $\longrightarrow$ the phase factor will be indep. of the path

Such independence is required if we insist that the wave func. be single-valued.

An interesting Consequence:



An electron moves in a field-free region surrounds a hole containing flux φ.

Upon completing a circuit the electron aquires an additional phase factor $e^{ie\varphi/\hbar c}$.

Since the electron wave func. is single-valued;

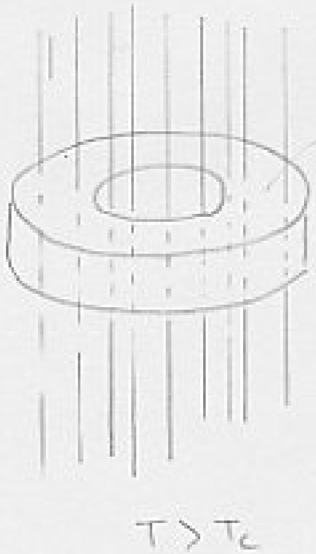
$$\rightarrow \quad e^{ie\varphi/\hbar c} = 1 \quad \rightarrow \quad \varphi = \frac{2\pi\hbar c}{e} n \qquad (86)$$

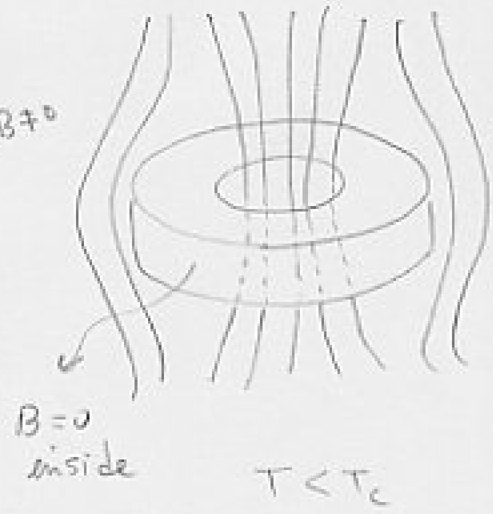
$n = 0, \pm 1, \pm 2 \ldots$

An ingenious measurement of the flux shows that (86) holds with the modification that

$$\varphi = \frac{2\pi\hbar c}{(2e)} n \qquad (87)$$

$\rightarrow$ The electrons move in pair, (correlated states).





A superconductor at $T > T_c$ behaves like any other metal. When $T < T_c$ it becomes superconductor expeling the mag. field

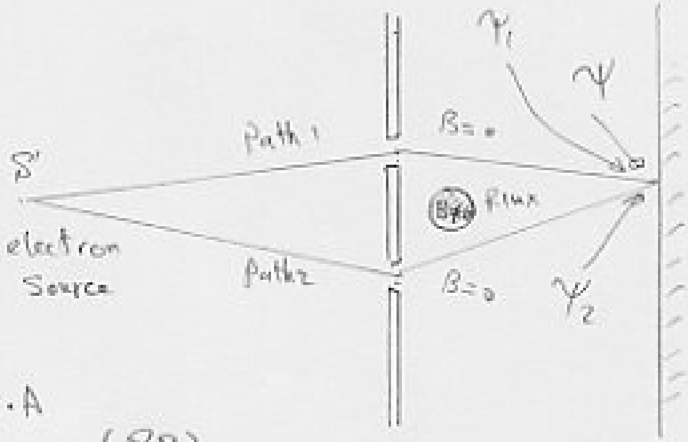
Another Example: Aharanov and Bohm

Interference experiment of the electrons;

In the absence of solenoid;

$$\psi = \psi_1 + \psi_2$$

In the presence of solenoid;



$$\psi = \psi_1 e^{\frac{ie}{\hbar c}\int_1 d\mathbf{r}\cdot\mathbf{A}} + \psi_2 e^{\frac{ie}{\hbar c}\int_2 d\mathbf{r}\cdot\mathbf{A}}$$

$$= \left(\psi_1 e^{\frac{ie\Phi}{\hbar c}} + \psi_2\right) e^{\frac{ie}{\hbar c}\int_2 d\mathbf{r}\cdot\mathbf{A}} \qquad (88)$$

$\longrightarrow$ Thus the flux causes a relative change in phase between ψ_1 and ψ_2 (i.e. $e^{ie\Phi/\hbar c}$)

$\longrightarrow$ This will change the interference pattern.