Angular Momentum

$$\left.\begin{aligned} L_z Y_{\ell m} &= m\hbar\, Y_{\ell m} \\ L^2 Y_{\ell m} &= \underbrace{\ell(\ell+1)}_{\lambda}\hbar^2\, Y_{\ell m} \end{aligned}\right. \qquad (1)$$

Note that ang. mom. has the dim. of $\hbar$.

m, $\ell(\ell+1)$: real numbers

The peculiar way of writing the eigenvalue of L^2 will prove its convenience later.

Now
$$\begin{cases} x = r\,S\theta\,C\varphi \\ y = r\,\sin\theta\,\sin\varphi \\ z = r\cos\theta \end{cases} \qquad (2)$$

$$\begin{cases} dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\theta}d\theta + \frac{\partial x}{\partial\varphi}d\varphi \\ dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\varphi}d\varphi \\ dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial\theta}d\theta + \frac{\partial z}{\partial\varphi}d\varphi \end{cases}$$

$$\rightarrow \begin{cases} dx = S\theta\,C\varphi\,dr + r\,C\theta\,C\varphi\,d\theta - r\,S\theta\,S\varphi\,d\varphi \\ dy = S\theta\,S\varphi\,dr + r\,C\theta\,S\varphi\,d\theta + r\,S\theta\,C\varphi\,d\varphi \\ dz = C\theta\,dr - r\,S\theta\,d\theta \end{cases} \qquad (3)$$

$$\rightarrow \begin{cases} dr = S\theta\, C\varphi\, dx + S\theta\, S\varphi\, dy + C\theta\, dz \\ d\theta = \frac{1}{r}\left(C\theta\, C\varphi\, dx + C\theta\, S\varphi\, dy - S\theta\, dz\right) \\ d\varphi = \frac{1}{rS\theta}\left(-S\varphi\, dx + C\varphi\, dy\right) \end{cases} \quad (4)$$

Remark: Note that: $\begin{cases} \frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} = \ldots = 0 \\ \text{They are indep.} \end{cases}$

$$(4) \Rightarrow \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

$$= S\theta\, C\varphi \frac{\partial}{\partial r} + \frac{1}{r} C\theta\, C\varphi \frac{\partial}{\partial \theta} - \frac{S\varphi}{rS\theta}\frac{\partial}{\partial \varphi}$$

Similarly:

$$\frac{\partial}{\partial y} = S\theta\, S\varphi \frac{\partial}{\partial r} + \frac{1}{r} C\theta\, S\varphi \frac{\partial}{\partial \theta} + \frac{C\varphi}{rS\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = C\theta \frac{\partial}{\partial r} - \frac{S\theta}{r}\frac{\partial}{\partial \theta} \quad (5)$$

$$\rightarrow L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \varphi} \quad (6)$$

We introduce: $L_\pm = L_x \pm i L_y$ (7)

$$L_\pm = \frac{\hbar}{i}\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \pm i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]$$

$$= \frac{\hbar}{i}\left[\pm iz\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) \mp i(x \pm iy)\frac{\partial}{\partial z}\right]$$

$$= \pm \hbar r C\theta \left(S\theta\, e^{\pm i\varphi}\frac{\partial}{\partial r} + \frac{1}{r} C\theta\, e^{\pm i\varphi}\frac{\partial}{\partial \theta} \pm \frac{i e^{\pm i\varphi}}{rS\theta}\frac{\partial}{\partial \varphi}\right)$$

$$\mp \hbar r S\theta\, e^{\pm i\varphi}\left(C\theta \frac{\partial}{\partial r} - \frac{S\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$L_\pm = \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\,\text{cotan}\,\theta\frac{\partial}{\partial\varphi}\right) \quad (8)$$

Now, since, $L_+L_- = (L_x+iL_y)(L_x-iL_y) = L_x^2+L_y^2 - i[L_x,L_y]$

$$L^2 = L_z^2 + L_+L_- + i[L_x,L_y] = L_+L_- + L_z^2 - \hbar L_z \quad (9)$$

second order diff. op. involving θ and φ.

There remains the task of solving the diff. equ. (1)

Alternative approach;

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\varphi}\frac{1}{rS\theta}\frac{\partial}{\partial\varphi} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} \quad (10)$$

$$\begin{cases} \hat{r} = S\theta C\varphi\,\hat{i} + S\theta S\varphi\,\hat{j} + C\theta\,\hat{k} \\ \hat{\varphi} = -S\varphi\,\hat{i} + C\varphi\,\hat{j} \\ \hat{\theta} = C\theta C\varphi\,\hat{i} + C\theta S\varphi\,\hat{j} - S\theta\,\hat{k} \end{cases} \quad (11)$$

$$L = r\times p = (r\hat{r})\times\left(\frac{\hbar}{i}\nabla\right) = \frac{\hbar}{i}\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{S\theta}\frac{\partial}{\partial\varphi}\right) \quad (12)$$

$$(5)\rightarrow\begin{cases} L_x = \frac{\hbar}{i}\left(-S\varphi\frac{\partial}{\partial\theta} - C\varphi\,\text{cotan}\,\theta\frac{\partial}{\partial\varphi}\right) \\ L_y = \frac{\hbar}{i}\left(C\varphi\frac{\partial}{\partial\theta} - S\varphi\,\text{cotan}\,\theta\frac{\partial}{\partial\varphi}\right) \\ L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \end{cases} \quad (13)$$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left[\frac{1}{S^2\theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{S\theta} \frac{\partial}{\partial \theta} \left(S\theta \frac{\partial}{\partial \theta} \right) \right] \quad (14)$$

$$(1) \rightarrow \left[\frac{1}{S^2\theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{S\theta} \frac{\partial}{\partial \theta} \left(S\theta \frac{\partial}{\partial \theta} \right) \right] Y(\theta,\varphi) = -\lambda Y(\theta,\varphi) \quad (15)$$

$$Y(\theta,\varphi) = \Phi(\varphi)\,\Theta(\theta) \quad (16)$$

$$(16) \text{ in } (15) \rightarrow \underbrace{-\frac{1}{\Phi} \frac{d^2\Phi}{d\varphi^2}}_{\text{const} = m^2} = \frac{S^2\theta}{\Theta} \underbrace{\left[\frac{1}{S\theta} \frac{d}{d\theta} \left(S\theta \frac{d}{d\theta} \right) + \lambda\Theta \right]}_{\text{Const} = m^2} \quad (17)$$

Const = real, as will soon be seen by virtue of boundary conds.

$$\frac{d^2\Phi}{d\varphi^2} + m^2 \Phi = 0 \quad (18)$$

$$\frac{1}{S\theta} \frac{d}{d\theta} \left(S\theta \frac{d}{d\theta} \right) - \frac{m^2}{S^2\theta} \Theta + \lambda\Theta = 0 \quad (19)$$

Fundamental Postulate of Q.M.:

The wave func. for a particle without spin must have definite value at every point in space.

$\longrightarrow$ The wave func. must be single-valued func. of the particles position.

In particular: $\Phi(\varphi) = \Phi(\varphi + 2\pi) \quad (20)$

$(18) \rightarrow \Phi(\varphi) = e^{im\varphi}$ (21)

The normalization: $\int_0^{2\pi} d\varphi\, |\Phi(\varphi)|^2 = 1 \rightarrow \Phi(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$

$(19) \rightarrow m = 0, \pm 1, \pm 2, \ldots$ (22)

Now, by a change of variables,

$$\xi = \cos\theta, \quad F(\xi) = \Theta(\theta) \quad (23)$$

Remark: $\frac{d}{d\xi} = \frac{-d}{\sin\theta\, d\theta}$ $\rightarrow \sin\theta \frac{d}{d\theta} = -(1-\xi^2)\frac{d}{d\xi}$

$$(19) \rightarrow \frac{d}{d\xi}\left[(1-\xi^2)\frac{dF}{d\xi}\right] - \frac{m^2}{1-\xi^2}F + \lambda F = 0 \quad (24)$$

$$\text{For } m = 0 \rightarrow \frac{d}{d\xi}\left[(1-\xi^2)\frac{dF}{d\xi}\right] + \lambda F = 0 \quad (25)$$

Legendre diff equ

Equ (25) does not change its form when $\xi \longrightarrow -\xi$

$\rightarrow$ The sols. are $\begin{cases} \text{either odd} \\ \text{or even} \end{cases}$ fncs. of ξ.

Since $\xi \longrightarrow -\xi \Rightarrow \theta \rightarrow \pi - \theta$

$\rightarrow$ These fncs are symmetric or antisymmetric w.r.t. xy-plane

$$F(\xi) = \sum_{k=0}^{\infty} a_k \xi^k \quad (26) \quad \text{regular sol.}$$

$$(26) \text{ in } (25) \rightarrow a_{k+2} = \frac{k(k+1) - \lambda}{(k+1)(k+2)} a_k \quad (27)$$

For even sol. $a_0 \neq 0$

" odd " $a_1 \neq 0$ (28)

$$(27) \longrightarrow \begin{cases} a_{-2} = 0 & \text{for even case} \\ a_{-1} = 0 & \text{" odd "} \end{cases} \quad (29) \quad \left(\text{due to } \frac{1}{(k+1)(k+2)} \text{ factor}\right)$$

This is in agreement with the assumption that F is regular at $\xi = 0$.

If $k \to \infty$ $\to$ $\dfrac{a_{k+2}}{a_k} \to \dfrac{k}{(k+2)}$ (30)

(26) behaves like $\sum \frac{1}{n}$ for $\begin{cases} \text{even } n \\ \text{odd } n \end{cases}$ and

$$F(\xi) = \sum_{k=0}^{\infty} a_k \xi^k \to \infty \quad \text{at } \xi = \pm 1, \text{ i.e. } \theta = 0, \pi$$

For the same reason we exclude the second linearly indep. sol. of (25). It has logarithmic singularities at $\xi = \pm 1$

Such singular fncs., although solutions of the diff. equ. for almost all values of ξ, are not acceptable eigenfncs. of L^2.

We conclude $\longrightarrow$ The power series must terminate at some finite value $k = l$, where l is nonnegative integer, and all higher powers vanish.

$$(27) \rightarrow \quad \lambda = \ell(\ell+1) \qquad (31)$$

Alternative approach:

The eigenfucs. of the Hermitian ops. are orthonormal, if the eigenvalues are different.

With a proper normalization we will write:

$$\langle Y_{\ell' m'} | Y_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{m m'} \qquad (32)$$

$$\langle Y_{\ell m} | L^2 | Y_{\ell m} \rangle = \langle Y_{\ell m} | (L_x^2 + L_y^2 + L_z^2) | Y_{\ell m} \rangle \qquad (33)$$

$$= \underbrace{\langle L_x Y_{\ell m} | L_x Y_{\ell m} \rangle}_{\geq 0} + \underbrace{\langle L_y Y_{\ell m} | L_y Y_{\ell m} \rangle}_{\geq 0} + \underbrace{m^2\hbar^2}_{\geq 0} \geq 0$$

$$(33) \rightarrow \ell(\ell+1)\hbar^2 \geq 0 \qquad (34)$$

We already saw that:

$$L^2 = L_+ L_- + L_z^2 - \hbar L_z \qquad (35)$$

In the same way

$$L^2 = L_- L_+ + L_z^2 + \hbar L_z \qquad (36)$$

$$\rightarrow \quad [L_+, L_-] = 2\hbar L_z \qquad (37)$$

$$[L_+, L_z] = [L_x + iL_y, L_z] = -i\hbar L_y - \hbar L_x = -\hbar L_+ \qquad (38)$$

and

$$[L_-, L_z] = \hbar L_- \qquad (39)$$

Since $[\bar{L}^2, \bar{L}] = 0 \rightarrow \begin{cases} [\bar{L}^2, L_\pm] = 0 \\ [\bar{L}^2, L_z] = \end{cases}$ (40)

$$\rightarrow L^2 \underbrace{L_\pm Y_{\ell m}} = L_\pm L^2 Y_{\ell m} = \underbrace{\ell(\ell+1)\hbar^2}_{\uparrow} \underbrace{L_\pm Y_{\ell m}} \quad (41)$$

On the other hand;

$$L_z \underbrace{L_+ Y_{\ell m}} = (L_+ L_z + \hbar L_+) Y_{\ell m} = m\hbar L_+ Y_{\ell m} + \hbar L_+ Y_{\ell m}$$

$$= \underbrace{\hbar(m+1)}_{\uparrow} \underbrace{L_+ Y_{\ell m}} \quad (42)$$

Similarly; $L_z \underbrace{L_- Y_{\ell m}} = \underbrace{\hbar(m-1)}_{\uparrow} \underbrace{L_- Y_{\ell m}}$ (43)

$$\rightarrow L_\pm Y_{\ell m} = C_\pm(\ell, m) \, Y_{\ell, m\pm 1} \quad (44)$$

Also, $L_\pm^+ = (L_x \pm i L_y)^+ = L_x \mp i L_y = L_\mp$ (45)

Since, $\langle L_\pm Y_{\ell m} | L_\pm Y_{\ell m} \rangle \geqslant 0$ (46)

(46),(45) $\rightarrow \langle Y_{\ell m} | L_\mp L_\pm Y_{\ell m} \rangle \geqslant 0$ (47)

(35)(36)(47) $\rightarrow \langle Y_{\ell m} | (L^2 - L_z^2 \pm \hbar L_z) | Y_{\ell m} \rangle \geqslant 0$ (48)

(48) $\rightarrow \begin{cases} \ell(\ell+1) \geqslant m^2 + m \\ \ell(\ell+1) \geqslant m^2 - m \end{cases}$ (49)

Since $\ell(\ell+1) \geqslant 0 \rightarrow \begin{cases} \ell \geqslant 0 \\ \text{or} \\ \ell \leqslant -1 \end{cases}$ (50)

We choose $l \geq 0$. If we were to find $l \leq -1$, we would merely define $L = -l-1$ and replace the old l, with the new positive L. Nothing would change, since $L(L+1) = l(l+1)$.

$$(49) \rightarrow \quad -l \leq m < l \qquad (51)$$

Recall:

$l(l+1) \geq m(m+1) \rightarrow l \geq m$

$l(l+1) \geq m(m-1)$ For $m = -l$

$\rightarrow l(l+1) \geq l(l+1)$

For $m = -l+1 \rightarrow l(l+1) \geq l(l-1)$

For $m = -l-1 \rightarrow l(l+1) \cancel{\geq} (l+1)(l+2)$

Let $m_- : m_{min}$ $(m_- = -l)$,

$$\rightarrow L_- Y_{lm_-} = 0 \qquad (52)$$

Now;
$$\langle Y_{lm_-} | L^2 | Y_{lm_-} \rangle = \langle Y_{lm_-} | (L_+ L_- + L_z^2 - \hbar L_z) | Y_{lm_-} \rangle$$

$$\rightarrow l(l+1)\hbar^2 = 0 + m_-^2 \hbar^2 - m_- \hbar^2 \qquad (53)$$

Similarly; $\quad L_+ Y_{lm_+} = 0 \quad (54)$

$$\langle Y_{lm_+} | L^2 | Y_{lm_+} \rangle = \langle Y_{lm_+} | (L_- L_+ + L_z^2 + \hbar L_z) | Y_{lm_+} \rangle$$

$$\rightarrow l(l+1)\hbar^2 = 0 + m_+^2 \hbar^2 + m_+ \hbar \qquad (55)$$

$$(53)(55) \rightarrow \begin{cases} l(l+1) = m_-(m_- - 1) \\ l(l+1) = m_+(m_+ + 1) \end{cases} \qquad (56)$$

$$m_-(m_- - 1) = m_+(m_+ + 1) \quad \rightarrow m_- = -m_+ \qquad (57)$$

$$\rightarrow \begin{cases} m_- = -l \\ m_+ = +l \end{cases} \qquad (58)$$

$$m = \underbrace{-l,\ -l+1, \ldots\ldots,\ l-1,\ l}_{2l+1 \text{ steps}} \qquad (59)$$

Now, $C_\pm(l,m) = ?$

$$(44) \Rightarrow \quad |C_\pm(l,m)|^2 \langle Y_{l,m\pm1} | Y_{l,m\pm1} \rangle = \langle L_\pm Y_{lm} | L_\pm Y_{lm} \rangle$$

$$= \langle Y_{lm} | L_\mp L_\pm Y_{lm} \rangle = \langle Y_{lm} | (L^2 - L_z^2 \mp \hbar L_z) | Y_{lm} \rangle$$

$$= \hbar^2 \left[l(l+1) - m(m\pm1) \right] \qquad (60)$$

$$C_\pm(l,m) = \hbar \sqrt{l(l+1) - m(m\pm1)} \qquad (61)$$

note that:
$|Y_{l,m\pm1}\rangle = L_\pm |Y_{lm}\rangle$
$\rightarrow \langle Y_{l,m\pm1}| = \langle Y_{lm}| L_\mp$

The eigenfuncs:

$$Y_{lm}(\theta,\varphi) = \Theta_{lm}(\theta)\, e^{im\varphi} \qquad (62)$$

$$(54) \Rightarrow \quad \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right) \Theta_{ll}(\theta)\, e^{il\varphi}$$

$$= \hbar\, e^{i(l+1)\varphi} \left(\frac{\partial}{\partial\theta} - l \cot\theta \right) \Theta_{ll}(\theta) = 0 \qquad (63)$$

$$\xrightarrow{\text{Sol.}} \quad \Theta_{ll}(\theta) = (S\theta)^l \qquad (64)$$

(up to normalization)

An arbitrary state is obtained by the lowering procedure;

$$Y_{lm}(\theta,\varphi) = C\,(L_-)^{l-m}\left[(S\theta)^l e^{il\varphi}\right] \quad (65)$$

Consider first;

$$L_- Y_{ll}(\theta,\varphi) = \hbar\, e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\,\mathrm{cotan}\,\theta\,\frac{\partial}{\partial\varphi}\right)\left[(S\theta)^l e^{il\varphi}\right]$$

$$= \hbar\, e^{i(l-1)\varphi}\left(-\frac{\partial}{\partial\theta} - l\,\mathrm{cotan}\,\theta\right)(S\theta)^l \quad (66)$$

Since $\left(\frac{d}{d\theta} + l\,\mathrm{cotan}\,\theta\right) f(\theta) = \frac{1}{(S\theta)^l}\frac{d}{d\theta}\left[(S\theta)^l f(\theta)\right]$ (67)

$$\rightarrow Y_{l,l-1}(\theta,\varphi) = C'\,\frac{e^{i(l-1)\varphi}}{(S\theta)^l}\left(-\frac{d}{d\theta}\right)\left[(S\theta)^l (S\theta)^l\right] \quad (68)$$

$$Y_{l,l-2}(\theta,\varphi) = C'' L_- Y_{l,l-1}(\theta,\varphi) \quad (69)$$

$$Y_{l,l-2}(\theta,\varphi) = C''\,\frac{e^{i(l-2)\varphi}}{(S\theta)^{l-1}}\left(-\frac{d}{d\theta}\right)\left[(S\theta)^{l-1}\frac{1}{(S\theta)^l}\left(-\frac{d}{d\theta}\right)(S\theta)^{2l}\right]$$

$$= C''(-1)^2\,\frac{e^{i(l-2)\varphi}}{(S\theta)^{l-1}}\,\frac{d}{d\theta}\left[\frac{1}{S\theta}\frac{d}{d\theta}(S\theta)^{2l}\right] \quad (70)$$

In terms of,

$$u = \cos\theta \quad , \quad \frac{-1}{S\theta}\frac{d}{d\theta} = \frac{d}{du}$$

$$(68) \rightarrow Y_{l,l-1} = C'\,\frac{e^{i(l-1)\varphi}}{(S\theta)^{l-1}}\,\frac{d}{du}\left[(1-u^2)^l\right] \quad (71)$$

$$(70) \rightarrow Y_{l,l-2} = C''\,\frac{e^{i(l-2)\varphi}}{(S\theta)^{l-2}}\,\frac{d^2}{du^2}\left[(1-u^2)^l\right] \quad (72)$$

The general form;

$$Y_{\ell m} = C\, \frac{e^{im\varphi}}{(\sin\theta)^m} \left(\frac{d}{du}\right)^{\ell-m} \left[(1-u^2)^\ell\right] \quad (73)$$

Normalization; $\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \quad (74)$

$$\langle Y_{\ell m} | Y_{\ell m}\rangle = 1 = \int_0^{2\pi} d\varphi \int_{-1}^{1} du\, |C|^2 \left[\frac{1}{(1-u^2)^{m/2}} \left(\frac{d}{du}\right)^{\ell-m} (1-u^2)^\ell\right]^2$$

(75)

The integration is tedious.

$$Y_{\ell m}(\theta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\, \frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi} \quad (76)$$

For $m \geq 0$ $\quad Y_{\ell,-m} = (-1)^m Y_{\ell m}^* \quad (77)$

The associated Legendre polynomials are given by $(m \geq 0)$

$$P_\ell^m(u) = (-1)^{\ell+m} \frac{(\ell+m)!}{(\ell-m)!} \frac{(1-u^2)^{-m/2}}{2^\ell\, \ell!} \left(\frac{d}{du}\right)^{\ell-m} (1-u^2)^\ell \quad (78)$$

$$P_\ell^{-m}(u) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(u) \quad (79)$$

Ex. – Consider a classical rotor, rotating in the x-y plane.

$$\rightarrow E = \frac{L_z^2}{2I} \qquad (83)$$ I: moment of inertia

$$\rightarrow H = \frac{L_z^2}{2I} \qquad (84)$$

$$\rightarrow E_m = \frac{\hbar^2 m^2}{2I} \qquad (85)$$ (with the eigenfuncs. $e^{\pm im\varphi}$)

Since $[L_z, H] = 0$ $\rightarrow$ there is degeneracy

$e^{+im\varphi}$ and $e^{-im\varphi}$ have the same energy.

Now consider N-particle rigidly fixed on a circle with equal angles $\alpha = \frac{2\pi}{N}$ between neighboring particles.

If the particles are identical, then the sol. of the energy eigenvalue eqn.

$$H\,\varphi_E(\varphi) = E\,\varphi_E(\varphi) \qquad (86)$$

will again be $e^{\pm i\lambda\varphi}$ (87)

The physical system is unaltered under a rot. of $\beta = k\frac{2\pi}{N}$ (k: integer) and the sols. should reflect this.

$$\rightarrow \lambda = NK \qquad (88)$$ K: an integer

$$E = \frac{\hbar^2 (Nm)^2}{2I}$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\; e^{i\varphi} \sin\theta$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\; \cos\theta$$

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}}\; e^{2i\varphi} \sin^2\theta \qquad (80)$$

$$Y_{2,1} = -\sqrt{\frac{15}{8\pi}}\; e^{i\varphi} \sin\theta \cos\varphi$$

$$Y_{2,0} = \sqrt{\frac{15}{16\pi}}\; \left(3\cos^2\theta - 1\right)$$

Now;

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}\right] u_E(\bar{r}) + V(r)\, u_E(\bar{r}) = E\, u_E(\bar{r}) \qquad (81)$$

$$\rightarrow -\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{1}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right] R_{Elm}(r) + V(r)\, R_{Elm}(r) = E R_{Elm}(r) \qquad (82)$$

We see that there is no dependence on m in the eqn..

Thus, for a given l there will always be (2l+1)-fold degeneracy (with $V(\bar{r}) = V(r)$).