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Quantum Mechanics-II

Useful Book : Quantum Mechanics
Stephen Gasiorowicz

Chapter 8

N-Particle System

$$\psi(x_1, x_2, \cdots x_N)$$

$$\int \cdots \int dx_1\, dx_2 \cdots dx_N \, |\psi(x_1, x_2 \cdots x_N)|^2 = 1 \qquad (1)$$

$|\psi(x_1, x_2 \cdots x_N)|^2$: The probability density for finding particle 1 at x_1, ..., particle N at x_N.

$$i\hbar \frac{\partial}{\partial t} \psi(x_1, \cdots x_N, t) = H \psi(x_1, \cdots x_N, t) \qquad (2)$$

where $$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x_1, x_2, \cdots x_N) \qquad (3)$$

$$H = -\hbar^2 \left(\frac{1}{2m_1} \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{1}{2m_N} \frac{\partial^2}{\partial x_N^2} \right) + V(x_1, \cdots x_N) \qquad (4)$$

$$[p_i, x_j] = \frac{\hbar}{i} \delta_{ij} \qquad (5)$$ i, j: particle indices

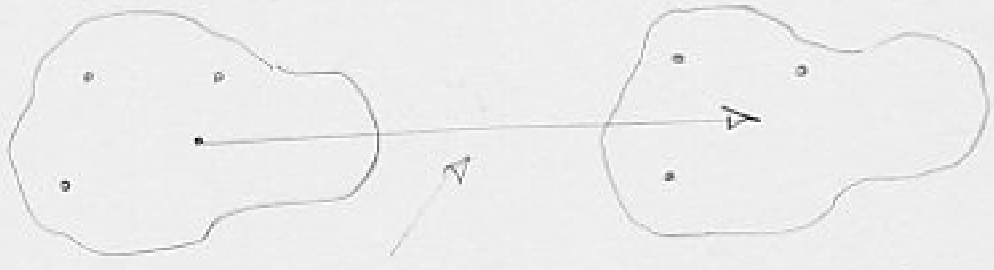
If External fields $= 0$ (like E, B, φ_{gr} ...)

$$\longrightarrow V = V(x_1 - x_2,\ x_1 - x_3,\ \cdots,\ x_{N-1} - x_N) \qquad (6)$$

relative coords.

(Relative distances)
Not absolute distances
measured from an origin.

External Field $= 0$ (No origin is determined)

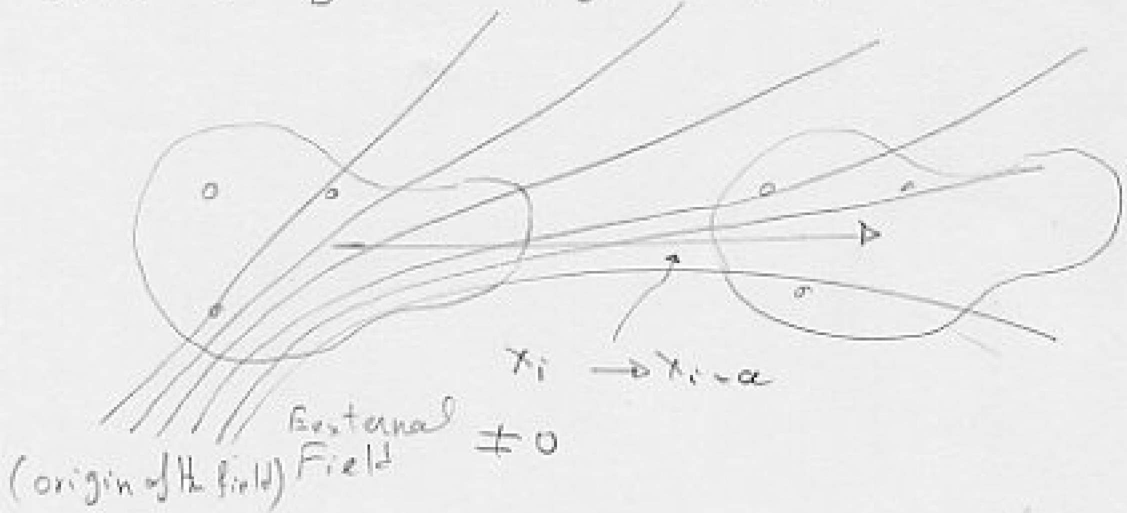


No change in the physical properties

$$x_i \xrightarrow{tr} x_i + a \qquad (7)$$

Because in the absence of any external agency, the displacement of the whole system should not change any physical properties of the system (Physical quantities invariance under $x_i \xrightarrow{tr} x_i + a$)

But if there exists an external field, there may occur some changes in the physical properties.

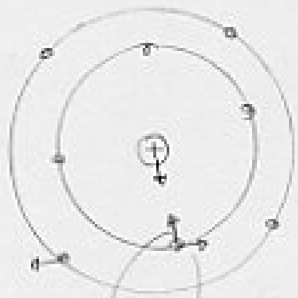


For two-body forces;

$$V = \sum_{i>j} V(x_i - x_j) \qquad (8)$$

In atom;

$$V = \sum_{i=1}^{N} W(x_i) + \sum_{i>j} V(x_i - x_j) \qquad (9)$$



In classical mechanics:

If $F_{external} = 0 \longrightarrow P = \text{const.}$ (10)

Because; $m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial}{\partial x_i} V(x_1 - x_2, x_1 - x_3 \dots x_{N-1} - x_N)$

(11)

$\longrightarrow \frac{d}{dt}\sum_i m_i \frac{dx_i}{dt} = -\sum_i \frac{\partial}{\partial x_i} V(x_1 - x_2, x_1 - x_3 \dots x_{N-1} - x_N)$

(12)

But $\sum_i \frac{\partial}{\partial x_i} V(x_1 - x_2, x_1 - x_3, \dots x_{N-1} - x_N) = 0$ (13)

Because for example, if $V = a(x_1 - x_2)^n$

then $\frac{\partial}{\partial x_1} V = -\frac{\partial}{\partial x_2} V$ and so, ..

$\longrightarrow P = \sum_i m_i \frac{dx_i}{dt} = \text{const.}$ (14)

In Q.M. the same conclusion holds. We shall demonstrate it by using the invariance of H under the tr. (7).

$H\, u_E(x_1, x_2 \dots x_N) = E_p\, u_E(x_1, x_2 \dots x_N)$ (15)

In the absence of any external field;

$H\, u_E(x_1 + a, x_2 + a \dots x_N + a) = E_p\, u_E(x_1 + a, x_2 + a, \dots x_N + a)$

(16)

Now let a = infinitesimal

$$U(x_1+a, x_2+a, \cdots x_N+a) \simeq U(x_1 \cdots x_N) + a\frac{\partial}{\partial x_1} U(x_1 \cdots x_N) +$$

$$+ a\frac{\partial}{\partial x_2} U(x_1 \cdots x_N) + \cdots \simeq U(x_1 \cdots x_N) + a\sum_i \frac{\partial}{\partial x_i} U(x_1, \cdots x_N) \qquad (17)$$

Subtracting (15) from (16);

$$a\, H\Big(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big) U_E(x_1 \cdots x_N) = a E \Big(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big) U_E(x_1 \cdots x_N)$$

$$= a\Big(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big) E\, U_E(x_1 \cdots x_N) = a\Big(\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big) H U_E(x_1 \cdots x_N) \qquad (18)$$

Define $\quad P = \frac{\hbar}{i} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \equiv \sum_{i=1}^{N} P_i \qquad (19)$

$$\rightarrow (HP - PH)\, U_E(x_1 \cdots x_N) = 0 \qquad (20)$$

Since $\{U_E(x_1 \cdots x_N)\}$: complete set

$$(20) \rightarrow (HP - PH)\, \Psi(x_1 \cdots x_N) = 0 \quad \forall \Psi \qquad (21)$$

$$\rightarrow (HP - PH) = 0 \rightarrow [H, P] = 0 \qquad (22)$$

Note that;

$$e^{+iHt/\hbar} [H, P]\, e^{-iHt/\hbar} = 0 \quad \rightarrow [H, P(t)] = 0 \qquad (23)$$

$$\frac{d P(t)}{dt} = \frac{i}{\hbar} [H, P(t)] = 0 \rightarrow \frac{dP(t)}{dt} = 0 \qquad (24)$$

P : const. of motion

Two-particle system:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \qquad \text{for two noninteracting particles} \qquad (25)$$

$$P(x_1, x_2) = \underbrace{P(x_1)\, P(x_2)}_{\text{independent probabilities}} : \quad \text{The probability of one at } x_1 \text{ and the other at } x_2 \qquad (26)$$

$$\left(-\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} \right) u(x_1, x_2) = E\, u(x_1, x_2) \qquad (27)$$

$$u(x_1, x_2) = \varphi_1(x_1)\, \varphi_2(x_2) \qquad \text{(we expect from (26))} \qquad (28)$$

$$(28) \xrightarrow{\text{in}} (27) \quad \rightarrow \quad \underbrace{\underbrace{\frac{-\left(\frac{\hbar^2}{2m_1}\right)\left(\frac{d^2 \varphi_1(x_1)}{dx_1^2}\right)}{\varphi_1(x_1)}}_{} + \underbrace{\frac{-\left(\frac{\hbar^2}{2m_2}\right)\left(\frac{d^2 \varphi_2(x_2)}{dx_2^2}\right)}{\varphi_2(x_2)}}_{}}_{\text{indep}} = E \qquad (29)$$

$$\rightarrow \quad E = E_1 + E_2 \qquad (30)$$

$$\rightarrow \quad \begin{cases} -\dfrac{\hbar^2}{2m_1} \dfrac{d^2 \varphi_1(x_1)}{dx_1^2} = E_1 \varphi_1(x) \\[2ex] -\dfrac{\hbar^2}{2m_2} \dfrac{d^2 \varphi_2(x_2)}{dx_2^2} = E_2 \varphi_2(x) \end{cases} \qquad (31)$$

$$\rightarrow \quad \varphi_1(x_1) \sim e^{ik_1 x_1} \qquad \varphi_2(x_2) \sim e^{ik_2 x_2} \qquad (32)$$

$$u(x_1, x_2) = C\, e^{ik_1 x_1 + ik_2 x_2} \qquad (33)$$

$$\text{where} \quad k_1^2 = \frac{2m_1 E_1}{\hbar^2} \qquad k_2^2 = \frac{2m_2 E_2}{\hbar^2} \qquad (34)$$

Let us now rewrite the sol. using the coords;

$$x = X_1 - X_2 \qquad X = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2} \qquad (35)$$

If we write $\quad k_1 X_1 + k_2 X_2 = \alpha (X_1 - X_2) + B \dfrac{m_1 X_1 + m_2 X_2}{m_1 + m_2} \qquad (36)$

$$\rightarrow \begin{cases} k_1 = \alpha + B \dfrac{m_1}{m_1+m_2} \\ k_2 = -\alpha + B \dfrac{m_2}{m_1+m_2} \end{cases} \qquad (37)$$

$$\rightarrow \begin{cases} B = k_1 + k_2 \equiv K \\ \alpha = \dfrac{m_2 k_1 - m_1 k_2}{m_1 + m_2} \equiv k \end{cases} \quad (38) \qquad \rightarrow u(x_1, x_2) = C e^{iKX} e^{ikx}$$

(39)

where K: wave number corresponding to the total momentum

k: " " " " " relative "

e^{iKX} : represents the motion of the center of mass

e^{ikx} : " " relative motion (internal wave func.)

$$E = \frac{\hbar^2 K^2}{2\underbrace{(m_1+m_2)}_{M}} + \frac{\hbar^2 k^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad (40)$$

$$E = \frac{\hbar^2 K^2}{2M} + \frac{\hbar^2 k^2}{2\mu} \qquad (41)$$

where $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2} \qquad (42)$

Remark:

$$E = \frac{\hbar^2 (k_1+k_2)^2}{2(m_1+m_2)} + \frac{\hbar^2 (k_1 m_2 - k_2 m_1)^2}{2(m_1+m_2)^2}\left(\frac{m_1+m_2}{m_1 m_2}\right) = \frac{\hbar^2 k_1^2}{2m_1} + \frac{\hbar^2 k_2^2}{2m_2} = E_1 + E_2$$

In the presence of $V = V(x_1 - x_2)$;

$$(25) \xrightarrow{\text{Changes}} \left(-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_2^2}\right) u(x_1,x_2) + V(x_1-x_2)\,u(x_1,x_2) = E\,u(x_1,x_2) \quad (43)$$

Using $$\begin{cases} x = x_1 - x_2 \\ X = \dfrac{m_1x_1+m_2x_2}{m_1+m_2} = \dfrac{\mu}{m_2}x_1 + \dfrac{\mu}{m_1}x_2 \end{cases} \quad (44)$$

$$\rightarrow \begin{cases} x_1 = X + \dfrac{\mu}{m_1}x \\ x_2 = X - \dfrac{\mu}{m_2}x \end{cases} \quad (45)$$

$$\rightarrow \left(-\frac{\hbar^2}{2(m_1+m_2)}\frac{\partial^2}{\partial X^2} \underbrace{- \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + V(x)}\right) u(x,X) = E\,u(x,X) \quad (46)$$

where we have used ;

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x_1} + \frac{\partial u}{\partial X}\frac{\partial X}{\partial x_1} = \frac{\partial u}{\partial x}(1) + \frac{\partial u}{\partial X}\frac{\mu}{m_2} \quad (47)$$

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial X}\frac{\mu}{m_2}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial X}\frac{\mu}{m_2}\right]\frac{\partial x}{\partial x_1} +$$
$$+ \frac{\partial}{\partial X}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial X}\frac{\mu}{m_2}\right]\frac{\partial X}{\partial x_1} = \frac{\partial^2 u}{\partial x^2} + \frac{\mu}{m_2}\frac{\partial^2 u}{\partial x\partial X} + \frac{\partial^2 u}{\partial x \partial X}\frac{\mu}{m_2}$$
$$+ \left(\frac{\mu}{m_2}\right)^2\frac{\partial^2 u}{\partial X^2} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{\mu}{m_2}\right)^2\frac{\partial^2 u}{\partial X^2} + \frac{2\mu}{m_2}\frac{\partial^2 u}{\partial x\partial X} \quad (48)$$

Similarly ; $\dfrac{\partial^2 u}{\partial x_2^2} = \dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\mu}{m_1}\right)^2\dfrac{\partial^2 u}{\partial X^2} - \dfrac{2\mu}{m_1}\dfrac{\partial^2 u}{\partial x\partial X}$

$$u(x,X) = e^{iKX}\,\varphi(x) \quad (49)$$

Remark: Note that $\varphi(x)$ is different from e^{ikx} because of $V(x)$.

$$\rightarrow \; -\frac{\hbar^2}{2\mu}\frac{d^2\varphi(x)}{dx^2} + V(x)\varphi(x) = \epsilon\varphi(x) \qquad (50)$$

(one-particle Schrödinger equ. with reduced mass μ)

$$\epsilon = E - \frac{\hbar^2 k^2}{2M} \qquad (51)$$

Identical Particles:

The electrons are indistinguishable. If this were not so, then the spectrum of an atom, say, helium, would vary from experiment to experiment, depending on what kind of electrons were contained in it.

This is purely quantum mechanical property.
In Cl. M. it is possible to follow the orbits of all particles (in principle) so that they are never really indistinguishable.

The spin has a further effect on the consequences of indistinguishability which we will discuss next.

$$H(1,2) = H(2,1) \quad \text{for identical particles} \qquad (52)$$

1, 2 : includes all deg. of freedom including spin

For example;

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1, x_2) \quad \text{for 2-particle system} \qquad (53)$$

with $V(x_1, x_2) = V(x_2, x_1)$ (54)

$$\psi(1,2,\ldots N) \equiv \psi(x_1,\alpha_1 \,; x_2,\alpha_2, \ldots x_N,\alpha_N) \quad (55)$$ the wave func. for N identical particles

For two-particle system;

$$H(1,2)\, u_E(1,2) = E\, u_E(1,2) \quad (56)$$

Since the labeling does not matter, we may write this as;

$$H(2,1)\, u_E(2,1) = E\, u_E(2,1) \quad (57)$$

On the other hand, using (52);

$$H(1,2)\, u_E(2,1) = E\, u_E(2,1) \quad (58)$$

Now we introduce; P_{12} : exchange op.

$$P_{12}\, \psi(1,2) = \psi(2,1)$$

$$(58) \rightarrow H\, P_{12}\, u_E(1,2) = E\, u_E(2,1)$$
$$= E\, P_{12}\, u_E(1,2) = P_{12}\, E\, u_E(1,2) = P_{12}\, H u_E(1,2) \quad (59)$$

$$\rightarrow [H, P_{12}] = 0 \quad (60)$$

$\rightarrow P_{12}$: const of motion

Also; $(P_{12})^2 \psi(1,2) = \psi(1,2)$ (61)

The eigenvalues of $P_{12} = \pm 1$ (62)

$$\rightarrow \begin{cases} \Psi^{(S)}_{(1,2)} = \frac{1}{\sqrt{2}}\left[\Psi(1,2) + \Psi(2,1)\right] \\ \Psi^{(A)}_{(1,2)} = \frac{1}{\sqrt{2}}\left[\Psi(1,2) - \Psi(2,1)\right] \end{cases} \qquad (63)$$

$[H, P_{12}] = 0 \rightarrow \begin{cases} \Psi^{(S)}_{(1,2)} \text{ remains symmetric at all times.} \\ \Psi^{(A)}_{(1,2)} \text{ " Anti- " " " "} \end{cases}$

The Law of Nature;

The symmetry or antisymmetry under the interchange of two particles is a characteristic of the particles, and not something that can be arranged in the preparation of the initial state.

The law, which was discovered by Pauli, states that;

1- Systems consisting of identical particles of half-odd-integral spin (i.e. spin $\frac{1}{2}, \frac{3}{2}, \ldots$) are described by antisymmetric wave fu-cs (Fermions). They are said to obey Fermi-Dirac Statistics.

2 – Systems consisting of identical particles of integral spin (spin 0, 1, 2, ...) are described by symmetric wave funcs. (Bosons). They are said to obey Bose-Einstein statistics.

Ex. –

$$\psi^{(A)}(1,2,3) = \frac{1}{\sqrt{6}}\Big[\psi(1,2,3) - \psi(2,1,3) + \psi(2,3,1) - \psi(3,2,1) + \psi(3,1,2) - \psi(1,3,2)\Big] \tag{64}$$

$$\psi^{(S)}(1,2,3) = \frac{1}{\sqrt{6}}\Big[\psi(1,2,3) + \psi(2,1,3) + \psi(2,3,1) + \psi(3,2,1) + \psi(3,1,2) + \psi(1,3,2)\Big] \tag{65}$$

Special Case:

N – noninteracting Fermions;

They don't interact with each other but, do interact with a common potential;

$$H = \sum_{i=1}^{N} H_i \tag{66}$$

$$H_i = \frac{P_i^2}{2m} + V(x_i) \tag{67}$$

$$H_k U_{E_k}(x_k) = E_k U_{E_k}(x_k) \quad (68)$$

$U_{E_k}(x_k)$: eigestates of the one-particle pot.

$$H U_E(1,2,\dots N) = E U_E(1,2,\dots N) \quad (69)$$

$$U_E(1,2,\dots N) = U_{E_1}(x_1) U_{E_2}(x_2) \dots U_{E_N}(x_N) \quad (70)$$

where $E_1 + E_2 + \dots + E_N = E$ (71)

In (70) x_i: includes space and spin degs. of freedom.

For two-particle system;

$$U^{(A)}(1,2) = \frac{1}{\sqrt{2}}\left[U_{E_1}(x_1) U_{E_2}(x_2) - U_{E_1}(x_2) U_{E_2}(x_1)\right] \quad (72)$$

With 3-particles;

$$\begin{aligned} U^{(A)}(1,2,3) = \frac{1}{\sqrt{6}} \Big[& U_{E_1}(x_1) U_{E_2}(x_2) U_{E_3}(x_3) - U_{E_1}(x_2) U_{E_2}(x_1) U_{E_3}(x_3) \\ & + U_{E_1}(x_2) U_{E_2}(x_3) U_{E_3}(x_1) - U_{E_1}(x_3) U_{E_2}(x_2) U_{E_3}(x_1) \\ & + U_{E_1}(x_3) U_{E_2}(x_1) U_{E_3}(x_2) - U_{E_1}(x_1) U_{E_2}(x_3) U_{E_3}(x_2) \Big] \end{aligned} \quad (73)$$

$$
u^{(A)}(1,2,\dots N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_{E_1}(x_1) & u_{E_1}(x_2) & \dots & u_{E_1}(x_N) \\ u_{E_2}(x_1) & u_{E_2}(x_2) & \dots & u_{E_2}(x_N) \\ \vdots & & & \\ u_{E_N}(x_1) & u_{E_N}(x_2) & \dots & u_{E_N}(x_N) \end{vmatrix} \qquad (74)
$$

$i \rightleftarrows j$ $\overset{\text{involves}}{\sim\sim}$ the interchange of two columns

(indices of the particles)

$\longrightarrow$ this changes the sign

For example $1 \rightleftarrows 2$

$$
u^{(A)}(2,1,\dots N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_{E_1}(x_2) & u_{E_1}(x_1) & \dots & u_{E_1}(x_N) \\ u_{E_2}(x_2) & u_{E_2}(x_1) & \dots & u_{E_2}(x_N) \\ \vdots & & & \\ u_{E_N}(x_2) & u_{E_N}(x_1) & \dots & u_{E_N}(x_N) \end{vmatrix} \qquad (75)
$$

$$
\rightarrow \quad u^{(A)}(1,2,\dots N) = - u^{(A)}(2,1,\dots N) \qquad (76)
$$

Now if $E_k = E_\ell$ $\longrightarrow$ two lines are the same

(we assume $E_k = E_\ell$ and $\sigma_k = \sigma_\ell$)

Since the interchange of two rows $\longrightarrow$ changes the sign of the det.

$\rightarrow$ det $= 0$ $\longrightarrow$ $u^{(A)}(1,2,\dots N) = 0$ (77)

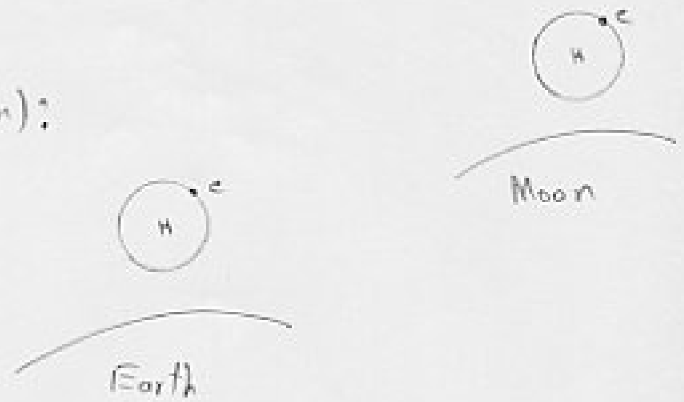
→ The requirement of antisymmetry introduces an <u>effective interaction</u> between two fermions;

Two particles in the <u>same</u> state tend to stay away from each other, since the joint wave func. vanishes when their separation goes to <u>zero</u>.

Thus even noninteracting particles behave as if there were a <u>repulsive</u> interaction between them.

Pauli: Exclusion principle (restricted version):

The state of given energy, ang. mom., parity, ... can be occupied by two electrons (spin $\frac{1}{2}$ particles) (of opposite spin variables) but by <u>no</u> more than two electrons



Remark: <u>No</u> two electrons can be in the <u>same</u> quantum state.

Question: Consider two H-Atoms in their ground states, one on the moon and the other one on the earth.

Must the two electrons be in opposite spin states?

To answer this question one must specify (in general);

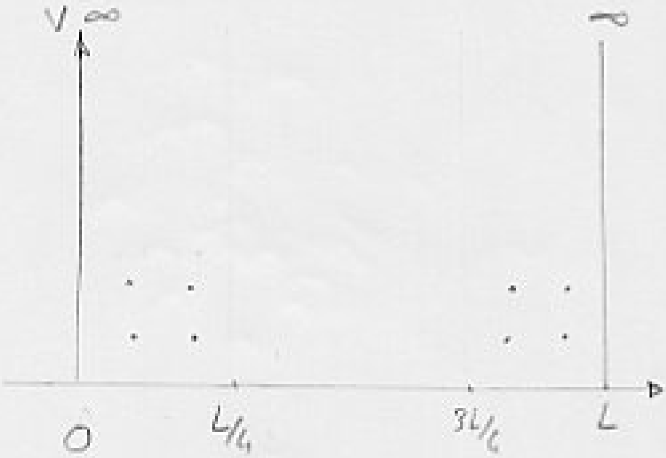
i) Spin orientation of the electrons

ii) Whether they are in the ground state of their respective atoms or not.

iii) The energy of the atoms.

How well do we know these?

Ex. – Consider the atoms are localized in $0 \le x \le L/4$ and $3L/4 \le x \le L$ respectively.



In chap. 4 for a particle in box we obtained;

$$E_n^{(-)} = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad E_n^{(+)} = \frac{(n-\frac{1}{2})^2\pi^2\hbar^2}{2ma^2} \qquad n = 1, 2, 3, \ldots$$

here $E_n^{(\pm)} = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$ $\quad n = 1, 2, 3, \ldots \quad (78)$ (Remember the wave func. must vanish at $x = 0, L$ $\xrightarrow{\text{sol.}}$ $\sin kx$ only)

Remark: Note that for classical Particles (m = large) we have localized wave func (λ), (i.e. $\psi = \sum a_n \sin k_n x$) So E and ΔE are different from eqns (78) and (82).

$$E = \frac{P^2}{2m} \qquad (79)$$

$$(78)(79) \rightarrow \quad P = \frac{\hbar\pi n}{L} \qquad (80)$$

$\Delta p \, \Delta x \gtrsim \hbar \quad \rightarrow \quad \Delta p \, \Delta x \sim n\hbar$ (can be proved using the wave-func)

$$\Delta p \sim \frac{n\hbar}{\Delta x} \sim \frac{n\hbar}{L} \qquad (81)$$

$$\Delta E \simeq \frac{p\Delta p}{m} = \left(\frac{\hbar\pi n}{L}\right)\left(\frac{n\hbar}{L}\right)\frac{1}{m} = \frac{\hbar^2 \pi n^2}{mL^2} \qquad (82)$$

This however is larger than;

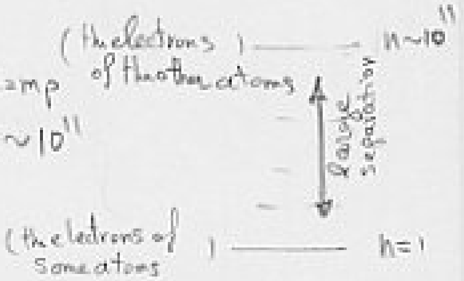
$$E_n - E_{n-1} \simeq \frac{\hbar^2 \pi^2 n}{mL^2} \qquad (83)$$

$$\Delta E > E_n - E_{n-1} \qquad \text{for } n > 1 \qquad (84)$$

In fact for atoms seperated by 1 meter, moving with 10^6 cm/sec

$\rightarrow n \sim 10^{11}$ (classical limit)

Note that (80) $\rightarrow n = \dfrac{pL}{\hbar\pi} = \dfrac{mvL}{\hbar\pi}$ (85) $\begin{cases} m \gg m_p \\ n \sim 10^{11} \end{cases}$



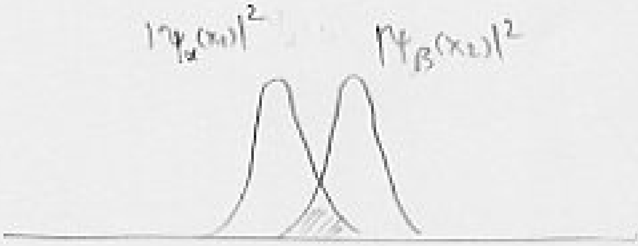
(Because of large seperation, the electrons are in different quantum states, so they may or may not be in the same spin state)

$\rightarrow$ So that there is no possibility that in a macroscopic situation there will be conflict with classical intuition.

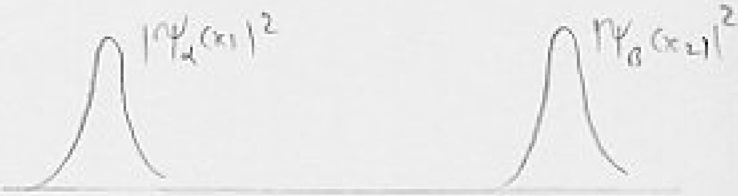
In fact, if the two atoms are labeled A and B, the question is whether there is a difference between using the wave func. $\psi_A(x_1)\,\psi_B(x_2)$

and $\frac{1}{\sqrt{2}}\left[\psi_A(x_1)\psi_B(x_2) - \psi_A(x_2)\psi_B(x_1)\right]$ (86)

to describe the two electron state ?



i) If the two electrons are close ($n \sim 1$), the results calculated with uncorrelated wave funcs and antisymmetrized wave funcs. are different



ii) But the results are the same if they are far enough ($n \to \infty$)

Remark: see P(170)

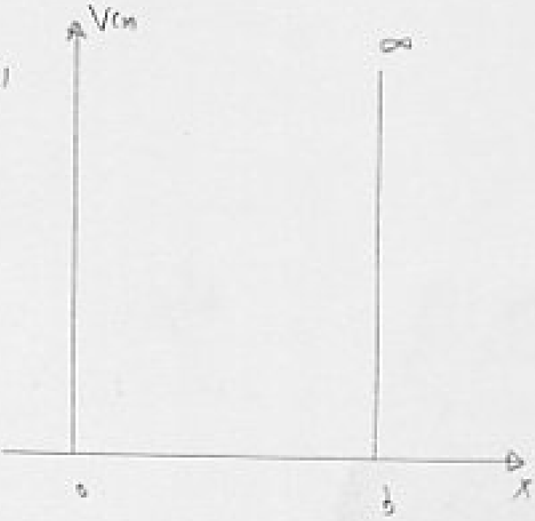
The ground state of N-Fermions and N-Bosons:
(both non-interacting)

Consider the infinite pot. box, $V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < b \\ \infty & b < x \end{cases}$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x) + V(x)\,u(x) = E\,u(x) \qquad (87)$$

$$\rightarrow u_n(x) = \sin\left(\frac{n\pi x}{b}\right) \qquad n = 1, 2, 3, \ldots \qquad (88)$$

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mb^2} \qquad (89)$$



For N non-interacting bosons, the ground state has all the particles in the $n=1$ state

$$E = N\frac{\hbar^2\pi^2}{2mb^2} \quad \rightarrow \quad \frac{E}{N} = \frac{\hbar^2\pi^2}{2mb^2} \qquad (90)$$

For N non-interacting fermions, only <u>2-electrons</u> can go into each of the states $n=1,2,3\ldots$

$\rightarrow$ $\frac{N}{2}$ of states are filled by N fermions;

$$E = 2\sum_{n=1}^{N/2}\frac{\hbar^2\pi^2 n^2}{2mb^2} = \frac{\pi^2\hbar^2}{mb^2}\frac{N^3}{24} \qquad (91)$$

where we have used;

$$\sum_{n=1}^{N/2} n^2 \approx \int_1^{N/2} n^2\,dn \approx \frac{1}{3}\left(\frac{N}{2}\right)^3 \qquad \text{for } n \rightarrow \text{large}$$

(92)

$$\rightarrow \frac{E}{N} = \frac{\hbar^2\pi^2}{24mb^2}N^2 \qquad (93)$$

grows with N^2

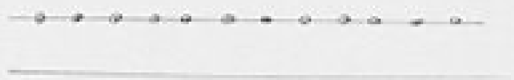
Therefore; for a given E;

$N \sim E$ number of bosons

$N \sim E^{1/3}$ " " fermions

The highest level to be filled in the fermion case has $n = \frac{N}{2}$

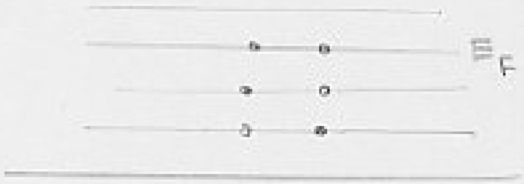
$$\rightarrow E_F = \frac{\hbar^2 \pi^2 N^2}{8 m b^2} \qquad (94)$$



Ground state of boson system

In terms of the density of fermions:

$$\rho = \frac{N}{b} \qquad \text{(in one-dim)}$$



Ground state of fermion system

$$\rightarrow E_F = \frac{\hbar^2 \pi^2}{8m} \rho^2$$

Remark! (From P 168);

The wave fnc. of two-electrons: Either $\psi = \psi_+ \chi_{Sing.}$ or $\psi_- \psi_- \chi_{trip.}$

where $\psi_\pm(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_\alpha(x_1)\psi_\beta(x_2) \pm \psi_\alpha(x_2)\psi_\beta(x_1)\right)$

The probability of observing electron $\underline{1}$ in d^3x_1 around x_1 and electron $\underline{2}$ in d^3x_2 around x_2 is given by:

$$P_\pm \sim \frac{1}{2}\left[\,|\psi_\alpha(x_1)|^2 |\psi_\beta(x_2)|^2 + |\psi_\alpha(x_2)|^2 |\psi_\beta(x_1)|^2 \pm \underbrace{2\,\mathrm{Re}\left(\psi_\alpha(x_1)\psi_\beta(x_2)\psi_\alpha^*(x_2)\psi_\beta^*(x_1)\right)}_{\text{exchange term}}\right] \cdot d^3x_1\, d^3x_2$$

$$P_-\big|_{x_1 = x_2} \simeq 0 \qquad P_+\big|_{x_1 = x_2} = \text{large}$$

If the separation of the two electrons is $\underline{\text{large}}$, then

exchange term $\approx 0 \quad \rightarrow \quad P_\pm \simeq |\psi_\alpha(x_1)|^2 |\psi_\beta(x_2)|^2$