Chapter 7

Scattering Theory

7.1 - The Lippmann Schwinger Equ.

Time-indep. formulation of scattering Processes;

Assume; $H = H_0 + V$ V: scatterer Potential

where $H_0 = \frac{p^2}{2m}$ $H_0 |p'\rangle = \frac{p'^2}{2m} |p'\rangle$

let $H_0 |\varphi\rangle = E |\varphi\rangle$

V $\xrightarrow{\text{causes}}$ energy eigenstate $|\psi\rangle$ to be different from a free particle state $|\varphi\rangle$

However if the scattering process is to be elastic (i.e. no change in energy) we are interested in obtaining a sol. to full-Hamiltonian Schrödinger equ. with the same energy eigenvalue -

Remark:

$|\varphi\rangle$: energy eigenstate either for free spherical wave or plane wave (free plane wave $|p\rangle$)

$|p'\rangle$: energy eigenstate with plane wave

We must solve;

$$(H_0+V)|\psi> = E|\psi>$$

Both H_0 and H_0+V exhibit continuous energy spectra (large quantum numbers) and $E>0$

$$\text{If } V \to 0 \implies |\psi> \to |\varphi>$$

(with the same energy eigen-value)

Now; $(E-H_0)|\psi> = +V|\psi>$

Remark: Since the scattering is elastic → $E = E_0$

Formal sol.:

$$|\psi> = \frac{1}{E-H_0} V|\psi> + |\varphi>$$

$|\varphi>$ is const. of integration, since when; $V \to 0$

we must have $|\psi> \to |\varphi>$

A check: - Apply $(E-H_0)$ operator on $|\psi>$

$$(E-H_0)|\psi> = (E-H_0)\left\{ \frac{1}{E-H_0} V|\psi> + |\varphi> \right\}$$

$$(E-H_0)|\psi> = V|\psi> + \cancel{(E-H_0)|\varphi>}$$

$$\to (H_0+V)|\psi> = E|\psi>$$

Remark: $\frac{1}{E-H_0}$ is singular, and not useful

The trick we used in time-indep perturbation (Projection op.) does not work well here, because both $|\varphi>$ and $|\psi>$ exhibit continuous eigenvalues.

Remedy- Let E slightly be complex;

$$E \longrightarrow E \pm i\epsilon_0$$

(the added small term is imaginary because of i- mathematical reason and ii- since E is continuous)

$$|\psi^{\pm}\rangle = |\varphi\rangle + \frac{1}{E - H_0 \pm i\epsilon_0} V |\psi^{\pm}\rangle$$

This is known as the Lippmann Schwinger equ.-

This is not a mathematical trick, but it has a physical root.

Consider the wave func. $\langle x|\psi^{\pm}\rangle$ (in position basis)

$$\langle x|\psi^{\pm}\rangle = \langle x|\varphi\rangle + \int d^3x' \langle x| \frac{1}{E - H_0 \pm i\epsilon} |x'\rangle \langle x'|V|\psi^{\pm}\rangle$$

(integral equ. of scattering)

If $|\varphi\rangle$ stands for a plane-wave state, with momentum $\vec{p}$

then;

$$\langle x|\varphi\rangle = \begin{cases} \dfrac{e^{\frac{i}{\hbar}p.x}}{(2\pi\hbar)^{3/2}} & \text{free space} \\ \dfrac{e^{\frac{i}{\hbar}p.x}}{L^{3/2}} & \text{box normalization} \end{cases}$$

On the other hand L-S equ. in momentum basis;

$$\langle p|\psi^{\pm}\rangle = \langle p|\varphi\rangle + \int d^3p' \langle p| \frac{1}{E - H_0 \pm i\epsilon} |p'\rangle \langle p'|V|\psi^{\pm}\rangle$$

But $\langle p|\frac{1}{E-H_0\pm i\epsilon}|p'\rangle = \frac{1}{E-(\frac{p^2}{2m})\pm i\epsilon}\langle p|p'\rangle = \frac{1}{E-(\frac{p^2}{2m})\pm i\epsilon}\delta^3(p-p'$

$$\rightarrow \langle p|\psi^\pm\rangle = \langle p|\varphi\rangle + \frac{1}{E-(\frac{p^2}{2m})\pm i\epsilon}\langle p|V|\psi^\pm\rangle$$

we have first evaluate the Kernel of the integral equ. defined by;

$$G_\pm(x,x') = \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0\pm i\epsilon}|x'\rangle$$

we claim $G_\pm(x,x')$ is given by

$$G_\pm(x,x') = -\frac{1}{4\pi}\frac{e^{\pm ik|x-x'|}}{|x-x'|}$$

where $E = \frac{\hbar^2 k^2}{2m}$

To show this, consider;

$$\frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0\pm i\epsilon}|x'\rangle = \frac{\hbar^2}{2m}\int d^3p'\int d^3p''\langle x|p'\rangle\langle p'|\frac{1}{E-(\frac{p'^2}{2m})\pm i\epsilon}|p''\rangle\langle p''|x'\rangle$$

where H_0 acts on $\langle p'|$

Now;

$$\langle p'|\frac{1}{E-(\frac{p'^2}{2m})\pm i\epsilon}|p''\rangle = \frac{\delta^3(p'-p'')}{E-(\frac{p'^2}{2m})\pm i\epsilon}$$

and $\langle x|p'\rangle = \frac{e^{\frac{i}{\hbar}p'\cdot x}}{(2\pi\hbar)^{3/2}}$ $\quad\langle p''|x'\rangle = \frac{e^{-\frac{i}{\hbar}p''\cdot x'}}{(2\pi\hbar)^{3/2}}$

$$\rightarrow \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0\pm i\epsilon}|x'\rangle = \frac{\hbar^2}{2m}\int\frac{d^3p'}{(2\pi\hbar)^3}\frac{e^{ip'\cdot(x-x')/\hbar}}{[E-(\frac{p'^2}{2m})\pm i\epsilon]}$$

let $E = \frac{\hbar^2 k^2}{m}$ and set $p' = \hbar q$

$$\rightarrow \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0\pm i\epsilon}|x'\rangle = \frac{\hbar^2}{2m}\int\frac{d^3q}{(2\pi)^3}\,\frac{e^{iq\cdot(x-x')}}{\left[\frac{\hbar^2k^2}{2m}-\frac{\hbar^2q^2}{2m}\pm i\epsilon\right]}$$

$$= \frac{1}{(2\pi)^3}\int d^3q\,\frac{e^{iq\cdot(x-x')}}{k^2-q^2\pm i\epsilon}$$

Note: $\frac{\pm i\epsilon}{\frac{\hbar^2}{2m}} \simeq \pm i\epsilon$

The volume element: $d^3q = q^2\,dq\,d\varphi\,d(\cos\theta)$

$$\rightarrow \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0\pm i\epsilon}|x'\rangle = \frac{1}{(2\pi)^3}\int_0^\infty q^2\,dq\int_0^{2\pi}d\varphi\int_{-1}^{+1}\frac{d(\cos\theta)\,e^{i|\vec{q}||x-x'|\cos\theta}}{k^2-q^2\pm i\epsilon}$$

$$= \frac{1}{(2\pi)^2}\int dq\,q^2\int_{-1}^{+1}d(\cos\theta)\,\frac{e^{iq|x-x'|\cos\theta}}{k^2-q^2\pm i\epsilon}$$

$$= \frac{1}{(2\pi)^2}\int dq\,q^2\left[\frac{1}{iq|x-x'|}\,\frac{e^{iq|x-x'|\cos\theta}}{k^2-q^2\pm i\epsilon}\right]_{-1}^{+1}$$

$$= \frac{-1}{(2\pi)^2}\int_0^\infty dq\,\frac{q}{i|x-x'|}\,\frac{e^{iq|x-x'|}-e^{-iq|x-x'|}}{-k^2+q^2\mp i\epsilon}$$

$$= \frac{-1}{8\pi^2}\,\frac{1}{i|x-x'|}\int_{-\infty}^{\infty}dq\,\frac{q\left(e^{iq|x-x'|}-e^{-iq|x-x'|}\right)}{q^2-k^2\mp i\epsilon}$$

Now; $$\frac{1}{q^2-k^2-i\epsilon} \equiv \frac{A}{q-k-i\epsilon}+\frac{B}{q+k+i\epsilon} = \frac{A(q+k+i\epsilon)+B(q-k-i\epsilon)}{q^2-(k+i\epsilon)^2}$$

$$= \frac{(A+B)q+k(A-B)+i\epsilon\overset{\emptyset}{(A-B)}}{q^2-k^2+\epsilon^{2\to 0}\underbrace{-2ki\epsilon}_{\approx -i\epsilon}} = \frac{q(A+B)+k(A-B)}{q^2-k^2-i\epsilon}$$

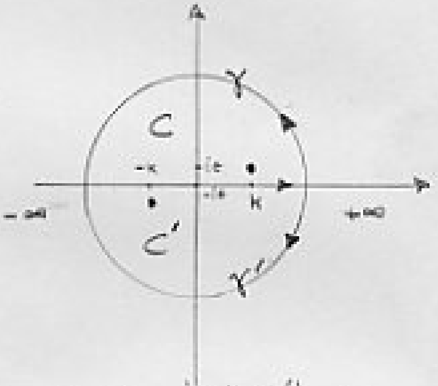
$$\Longrightarrow A+B=0\ ,\quad k(A-B)=1 \quad\Longrightarrow\quad A=-B\ ,\ A=\frac{1}{2k}$$

$$\rightarrow \frac{1}{q^2-k^2-i\epsilon} = \frac{1}{2k}\left(\frac{1}{q-k-i\epsilon} - \frac{1}{q+k+i\epsilon} \right)$$

Poles: $q = \begin{cases} k+i\epsilon \\ -(k+i\epsilon) \end{cases}$

$$\rightarrow \frac{\hbar^2}{2m} \langle x | \frac{1}{E-H_0+i\epsilon} | x' \rangle = -\frac{1}{8\pi^2} \frac{1}{i|x-x'|} \left\{ \int_{-\infty}^{\infty} \frac{q\,dq\, e^{iq|x-x'|}}{q^2-k^2-i\epsilon} - \int_{-\infty}^{-\infty} \frac{q\,dq\, e^{-iq|x-x'|}}{q^2-k^2-i\epsilon} \right.$$

$$\oint_C \frac{q\,dq\, e^{iq|x-x'|}}{q^2-k^2-i\epsilon} = \int_\gamma + \int_{-\infty}^{\infty}$$



$$\int_{\gamma\; R\to\infty} = 0 \quad \rightarrow \quad \oint_C = \int_{-\infty}^{\infty}$$

$$\oint_C = 2\pi i \sum \mathrm{Res}\left[\frac{q\, e^{iq|x-x'|}}{q^2-k^2-i\epsilon} \right] = 2\pi i\, \mathrm{Res}\left(\frac{q\, e^{iq|x-x'|}}{q^2-k^2-i\epsilon} \right)_{k+i\epsilon}$$

Since we have a simple pole;

$$\mathrm{Res}\left(\frac{q\, e^{iq|x-x'|}}{q^2-k^2-i\epsilon} \right)_{k+i\epsilon} = \left(\frac{q\, e^{iq|x-x'|}}{2q} \right)_{k+i\epsilon} \simeq \frac{1}{2} e^{ik|x-x'|}$$

Remark:

$$\oint_C f(z)\,dz = \sum \mathrm{Res}(f(z))$$

If $f(z) = \frac{P(z)}{q(z)}$ $\begin{cases} \text{where } P(z), q(z) \text{ are analytic at } z=a \\ z=a \quad \text{simple pole} \\ P(a) \neq 0 \end{cases}$

$$\rightarrow \underset{z=a}{\mathrm{Res}}(f(z)) = \lim_{z\to a} (z-a) \frac{P(z)}{q(z)}$$

$$\text{or}: \quad \underset{z=a}{\mathrm{Res}}(f(z)) = \underset{z=a}{\mathrm{Res}} \frac{P(z)}{q(z)} = \frac{P(a)}{q'(a)}$$

Similarly;

$$\oint_{c'} \frac{q\,dq\, e^{-iq|x-x'|}}{q^2-k^2-i\epsilon} = \int_{\gamma'} + \int_{-\infty}^{\infty}$$

$$\int_{\substack{\gamma' \\ R\to\infty}} = 0 \longrightarrow \oint_{c'} = \int_{-\infty}^{\infty}$$

$$\oint_{c'} = -2\pi i \sum \operatorname{Res}\left(\frac{q\, e^{-iq|x-x'|}}{q^2-k^2-i\epsilon} \right) = -2\pi i \operatorname{Res}\left(\frac{q\, e^{-iq|x-x'|}}{q^2-k^2-i\epsilon} \right)_{-(k+i\epsilon)}$$

$$\operatorname{Res}\left(\frac{q\, e^{-iq|x-x'|}}{q^2-k^2-i\epsilon} \right)_{-(k+i\epsilon)} = \left(\frac{q\, e^{-iq|x-x'|}}{2q} \right)_{-(k+i\epsilon)} \simeq \frac{1}{2} e^{+ik|x-x'|}$$

$$\rightarrow \frac{\hbar^2}{2m} \langle x | \frac{1}{E-H_0+i\epsilon} | x' \rangle = -\frac{1}{8\pi^2} \frac{1}{i|x-x'|} \left\{ 2\pi i \frac{1}{2} e^{ik|x-x'|} - (-2\pi i) \frac{1}{2} e^{ik|x-x'|} \right\}$$

$$= \frac{-1}{4\pi} \frac{e^{+ik|x-x'|}}{|x-x'|}$$

$$\rightarrow \frac{\hbar^2}{2m} \langle x | \frac{1}{E-H_0 \pm i\epsilon} | x' \rangle = -\frac{1}{4\pi} \frac{e^{\pm ik|x-x'|}}{|x-x'|}$$

$$\rightarrow \langle x | \psi^{\pm} \rangle = \underbrace{\langle x | \varphi \rangle}_{\text{initial wave}} + \underbrace{\frac{2m}{\hbar^2} \int d^3x'\, G_{\pm}(x,x') \langle x' | V | \psi^{\pm} \rangle}_{\text{scattered wave}}$$

$G_{\pm}(x,x')$ is nothing more than Green's func. which satisfies Helmholtz equ;

$$(\nabla^2+k^2)\, G_{\pm}(x,x') = \delta^3(x-x')$$

Remark: $(\nabla^2+k^2)\psi(x) = U(x)\psi(x)$, $U(x) = \frac{2\mu}{\hbar^2} V(x)$ $E = \frac{\hbar^2 k^2}{2\mu}$

$(\nabla^2+k^2)\, G_k(x,x') = \delta(x-x')$ $\psi(r) = \int G_k(x,x')\, U(x')\, \psi(x')\, d^3x'$ (satisfies the Schrödinger equ.)

We may add any sol. $\varphi(x)$ of the homogeneous equ. $(\nabla^2+k^2)\varphi(x) = 0$ to $\psi(x)$

$\Rightarrow \psi(x) = \varphi(x) + \int G_k(x,x')\, U(x')\, \psi(x)\, d^3x'$ $U(x')$: local pot.

we will see later at sufficiently large distances the spatial dependence of the

$$\langle x|\psi^{\pm}\rangle = \langle x|\varphi\rangle - \frac{2m}{\hbar^2}\int d^3x' \, \frac{e^{\pm i|k||x-x'|}}{4\pi|x-x'|}\langle x'|V|\psi^{\pm}\rangle$$

is $\dfrac{e^{\pm ikr}}{r}$

provided that the potential is of finite range.

This means that:

Positive sol. $\langle x|\psi^{+}\rangle$ = Plane wave + out going spherical wave

Negative sol. $\langle x|\psi^{-}\rangle$ = " + incoming " "

- In most physical Problems:

We are interested in the positive sol., because it is difficult to prepare a system satisfying the boundary cond. appropriate for the negative sol..

Local Potentials:

The potentials that are funcs only of the position operator x, belong to this category.

i.e. $\langle x'|V|x''\rangle = V(x')\,\delta^3(x'-x'')$

Remark: Example Nonlocal pot.: a) $V(x,x') = \frac{-\hbar^2}{2\mu}\lambda\, \mathcal{U}(|x|)\,\mathcal{U}(|x'|)$

b) $\langle x|V|x'\rangle = \delta(x-x')V_0(x) - \lambda\, \mathcal{V}(|x|)\,\mathcal{V}(|x'|)$ where $V_0(x) = \begin{cases} 0 & |x| > r_c \\ \infty & |x| \leq r_c \end{cases}$

and $\mathcal{V}(|x|) \equiv \mathcal{V}(r) = e^{-r/b}/r$

As a result;

$$\langle x'|V|\psi^{\pm}\rangle = \int d^3x'' \langle x'|V|x''\rangle\langle x''|\psi^{\pm}\rangle$$

$$= V(x')\langle x'|\psi^{\pm}\rangle$$

Hence;

$$\langle x|\psi^{\pm}\rangle = \langle x|\varphi\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|x-x'|}}{4\pi|x-x'|} V(x')\langle x'|\psi^{\pm}\rangle$$

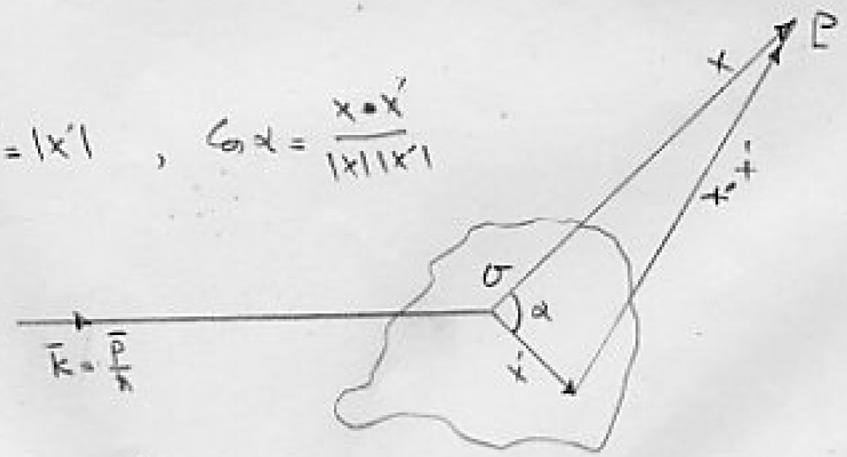
or;

$$\psi^{\pm}(x) = \varphi(x) - \frac{m}{2\pi\hbar^2}\int d^3x' \underbrace{\frac{e^{\pm ik|x-x'|}}{|x-x'|}}_{\text{amplitude that the scattered wave (at } x'\text{) reaches } x.} V(x')\psi^{\pm}(x')$$

$$\begin{cases} x: & \text{observation point} \\ x': & \text{scattering point} \end{cases}$$

Physically, we are interested in $|x| \gg |x'|$ (because we can not place a detector at short distance)

let $r = |x|$, $r' = |x'|$, $\cos\alpha = \frac{x \cdot x'}{|x||x'|}$



Hence

$$|x - x'| = (r^2 + r'^2 - 2rr'\cos\alpha)^{\frac{1}{2}}$$

$$= r\left(1 - \frac{2r'}{r}\cos\alpha + \frac{r'^2}{r^2}\right)^{\frac{1}{2}}$$

for $r >> r' \longrightarrow |x - x'| \simeq r\left(1 - \frac{r'}{r} G\alpha\right) = r - \hat{x}\cdot x'$

where $\hat{x} = \frac{\bar{x}}{|\bar{x}|}$

also we define $\bar{K}' \equiv k\hat{x}$

Then; $e^{\pm ik|x-x'|} \simeq e^{\pm ik(r - \hat{x}\cdot\bar{x}')} = e^{\pm ikr}\, e^{\pm ik'\cdot x'}$

and for $r >> r' \longrightarrow \frac{1}{|x-x'|} \simeq \frac{1}{r}$

also $\bar{k} \equiv \frac{\bar{p}}{\hbar} \Longrightarrow |\varphi\rangle = |p\rangle \longrightarrow |\varphi\rangle = |k\rangle$

$|k\rangle$ is normalized;

$$\langle k | k' \rangle = \delta^3(k - k')$$

$$\longrightarrow \langle x | k \rangle = \frac{e^{ik\cdot x}}{(2\pi)^{3/2}}$$

$$\langle x | \psi^+ \rangle \xrightarrow[\text{large } r]{} \langle x | k \rangle - \frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{e^{ikr}}{r}\int d^3x'\, e^{-ik'\cdot x'}\, V(x') \langle x' | \psi^+ \rangle$$

$$= \frac{1}{(2\pi)^{3/2}}\left[\underbrace{e^{ik\cdot x}}_{\text{original Plane wave}} + \underbrace{\frac{e^{ikr}}{r} f(k',k)}_{\text{scattered term (spherical wave) (out going)}}\right]$$

where

$$f(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x'\, \frac{e^{-ik'\cdot x'}}{(2\pi)^{3/2}}\, V(x')\, \underset{\substack{\uparrow \\ \text{K-dep.}}}{\langle x' | \psi^+ \rangle}$$

$$\theta = \frac{\bar{k}\cdot\bar{k}'}{|k||k'|}$$

$$f(k',k) = -\frac{4\pi^2 m}{\hbar^2}\int d^3x' \frac{e^{-ik'\cdot x'}}{(2\pi)^{3/2}} V(x') \langle x'|\psi^+\rangle$$

$$= -\frac{4\pi^2 m}{\hbar^2}\langle k'|V|\psi^+\rangle \quad \text{scattering amplitude}$$

Similarly:

$$\underset{\text{large } r}{\langle x|\psi^-\rangle} \longrightarrow \frac{1}{(2\pi)^{3/2}}\left[\underbrace{e^{ik\cdot x}}_{\text{original plane wave}} + \underbrace{\frac{e^{-ikr}}{r} f(-k',k)}_{\text{incomming wave}}\right]$$

$$f(-k',k) = -\frac{4\pi^2 m}{\hbar^2}\int d^3x' \frac{e^{ik'\cdot x}}{(2\pi)^{3/2}} V(x') \langle x'|\psi^-\rangle$$

$$= -\frac{4\pi^2 m}{\hbar^2}\langle -k'|V|\psi^-\rangle$$

Remark:

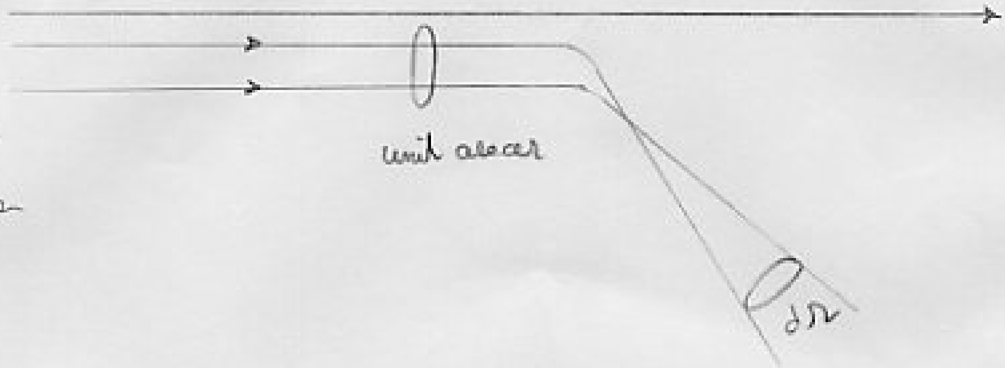
$$\langle k'|V|\psi^+\rangle = \int d^3x' \int d^3x'' \langle k'|x'\rangle \underbrace{\langle x'|V|x''\rangle}_{V\langle x'|x''\rangle}\langle x''|\psi^+\rangle$$

$$= \int d^3x' \langle k'|x'\rangle V(x') \langle x'|\psi^+\rangle$$

Cross section

Assumption: i) I_0 (incident intensity) is low enough → mutual int. among the particles can be neglected (But on the other hand this conflicts with the requirement for "good statistics")
ii) P of the particles is large enough → the deBroglie wave length $\frac{h}{p}$ is small enough So the scattering centers act independently (this may cause another problem like particle production).

⇒ Thus there are optimum values for I_0 and P for a given set up.



I_0: The number of incident particles per unit time per unit area perpendicular to incident

These identically prepared particles are characterized by the wave func.

$$\langle x|K\rangle = \frac{e^{iK\cdot x}}{(2\pi)^{3/2}}$$

$I(\theta,\varphi)\,d\Omega$: The number of scattered particles in the cone ($d\Omega$) per unit time.

$$\frac{d\sigma}{d\Omega} = \frac{I(\theta,\varphi)}{I_0}$$ differential scattering cross section.

$$d\sigma = \frac{d\sigma}{d\Omega}d\Omega = \frac{I\,d\Omega}{I_0} = \frac{(r^2 d\Omega)|J_{scat.}|}{|J_{incid}|} = \frac{\frac{\hbar k}{m}\frac{|f(k',k)|^2}{r^2}\overset{da}{(r^2 d\Omega)}}{\frac{\hbar k}{m}}$$

$$= |f(k',k)|^2 d\Omega \quad \longrightarrow \quad \frac{d\sigma}{d\Omega} = |f(k',k)|^2$$

Remark: It can be shown (Gasiorowicz) $\bar{J} = \underbrace{\frac{\hbar k}{m}}_{\text{incident}} + \frac{\hbar k}{m}\frac{|f(\theta)|^2}{r^2}\hat{r}$

$$J\cdot\hat{r} = \underbrace{\frac{\hbar k}{m}\cos\theta}_{\text{contributes only for } \theta \simeq 0} + \frac{\hbar k}{m}\frac{|f(\theta)|^2}{r^2}$$ (so we drop it)

Alternatively:

J_i : the flux of particles per unit area, per unit time

$dN = J_i\,\sigma(\theta,\varphi)\,d\Omega$: the number of particles per unit time scattered into solid angle $d\Omega$

$r^2 d\Omega$: the area of the detector

$dN = J_s\,(r^2 d\Omega)$

$J_s = J_i\,\sigma(\theta,\varphi)\,d\Omega / r^2 d\Omega$ the flux of scattered particles at the detector

$$\longrightarrow \sigma(\theta,\varphi) = \frac{r^2 J_s}{J_i} \quad \text{(dim. of area)}$$

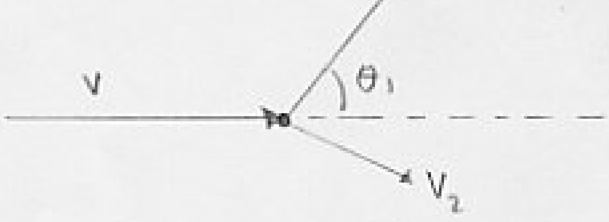
Since $J_i \sim \frac{1}{r^2}$ $\longrightarrow$ $\sigma(\theta,\varphi)$ r-indep.

$$\sigma = \int \sigma(\theta,\varphi)\,d\Omega$$

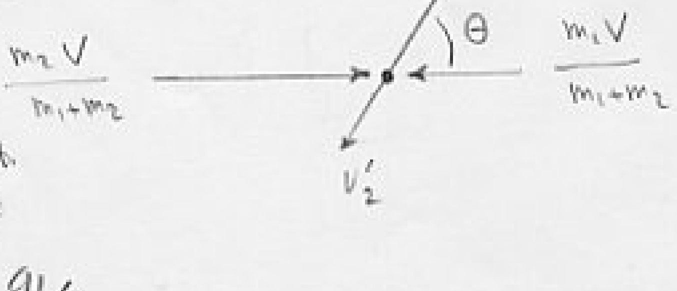
Laboratory and Center-of-Mass Frames

$\bar{P} = \sum m_i \frac{d\bar{r}_i}{dt} = M\frac{d\bar{R}}{dt} \rightarrow (m_1+m_2)\bar{V}_{cm} = m_1\bar{v}$ mom. cons.

$\rightarrow V_{cm} = \frac{m_1}{m_1+m_2}\bar{v} = \frac{\mu}{m_2}\bar{v}$ w.r.t. lab (1)



Transformation from lab. to CM. in which CM is at rest:



$$\begin{cases} \bar{r}_1 = \bar{R} + \bar{r}_1' \\ \bar{v}_1 = \bar{V} + \bar{v}_1' \end{cases}$$

(2)

The relation of coords. after scatt. in two different coords.

$$\begin{cases} \bar{V} - \bar{V}_{cm} = \dfrac{m_2\bar{V}}{m_1+m_2} & \text{(projectile)} \quad (3) \\[2ex] 0 - V_{cm} = -\dfrac{m_1\bar{V}}{m_1+m_2} & \text{before the scatt.} \quad (4) \end{cases}$$

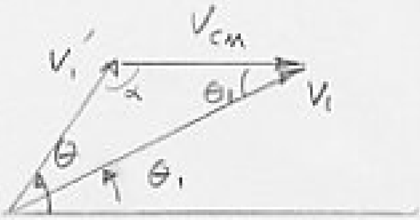


Fig. → $V_1 \sin\theta_1 = V_1' \sin\theta$ (5)

$V_1 \cos\theta_1 = V_1' \cos\theta + V_{cm}$ (6)

$$\to \tan\theta_1 = \frac{\sin\theta}{\cos\theta + B} \qquad B = \frac{V_{cm}}{V_1'} = \frac{\mu}{m_2}\frac{V}{V_1'} \qquad (7)$$

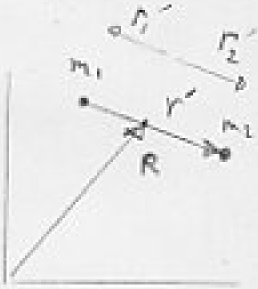
Now,

$$V_1^2 = V_1'^2 + V_{cm}^2 - 2V_1'V_{cm}\cos\alpha = V_1'^2 + V_{cm}^2 + 2V_1V_{cm}\cos\theta \qquad (8)$$

(1)(6)(8) $\to$ $$\cos\theta_1 = \frac{\cos\theta + B}{\sqrt{1 + 2B\cos\theta + B^2}} \qquad (9)$$

By <u>def.</u> of the center of mass;

$$\begin{cases} \bar{r}_1' = -\dfrac{m_2}{m_1+m_2}\bar{r}' = -\dfrac{\mu}{m_1}\bar{r}' \to V_1' = \dfrac{\mu}{m_1}V' \\[2ex] \bar{r}_2' = \dfrac{m_1}{m_1+m_2}\bar{r}' = \dfrac{\mu}{m_2}\bar{r}' \end{cases}$$



After scatt.

$$\to B = \frac{V_{cm}}{V_1'} = \frac{\mu}{m_2}V\left(\frac{m_1}{\mu}\frac{1}{V'}\right) = \frac{m_1}{m_2}\frac{V}{V'}$$

V: relative motion before scatt.

V': " " after "

For elastic scatt.;

$$K_1 + K_2 = K_1' + K_2' \quad \rightarrow v = v'$$

$$\rightarrow \beta = \frac{m_1}{m_2}$$

In inelastic scatt.; (Since the Kinetic energy of the CM motion must remain const., by conservation of linear mom.);

$$\frac{\mu v'^2}{2} = \frac{\mu v^2}{2} + Q \qquad Q<0$$

$$\rightarrow \frac{v'}{v} = \sqrt{1 + \frac{m_1+m_2}{m_2}\frac{Q}{E}}$$

E: the energy of incoming particle in <u>lab.</u>

$$\rightarrow \beta = \frac{m_1}{m_2\sqrt{1+\frac{m_1+m_2}{m_2}\frac{Q}{E}}}$$

(inelastic scatt.)

The connection between two functional forms of $\sigma(\theta)$ and $\sigma(\theta_1)$ is obtained from the observation that in a particular experiment the number of particles scattered into a given element of solid angle must be the same whether we measure the event in terms of θ or θ_1.

$$2\pi I\, \sigma(\theta) \sin\theta |d\theta| = 2\pi I\, \sigma'(\theta_1) \sin\theta_1 |d\theta_1|$$

$$\rightarrow \sigma'(\theta_1) = \sigma(\theta)\frac{\sin\theta}{\sin\theta_1}\left|\frac{d\theta}{d\theta_1}\right| = \sigma(\theta)\left|\frac{d\cos\theta}{d\cos\theta_1}\right|$$

using (9) $\rightarrow$ $$\sigma'(\theta_1) = \sigma(\theta)\,\frac{(1+2B\cos\theta+\rho^2)^{3/2}}{1+B\cos\theta} \qquad (10)$$

Both $\sigma'(\theta_1)$ and $\sigma(\theta)$ are measured in <u>lab</u>., they are merely expressed in terms of different coords..

i.e. $\sigma(\theta)$ is <u>not</u> the cross section an observer would measure in the <u>center of mass</u>.

An observer fixed in C.M. would see a different flux density of incident particles from that measured in the lab., and this transformation of flux density would have to be included if we wanted to relate the cross-section as measured in the <u>two different</u> system.

$$(9) \rightarrow \cos\theta_1 = \sqrt{\frac{1+\cos\theta}{2}} = \cos\frac{\theta}{2} \qquad \text{for } B = \frac{m_1}{m_2} = 1$$

(elastic)

$$\rightarrow \theta_1 = \frac{\theta}{2}$$

$\rightarrow$ with $m_1 = m_2$ $\qquad \theta_1 \leqslant \frac{\pi}{2}$

$$(10) \longrightarrow \sigma'(\theta_1) = 4\cos\theta_1\,\sigma(\theta) \qquad \theta_1 \leqslant \frac{\pi}{2} \quad B = 1$$

Even for $\sigma(\theta) = \text{Const.}$, $\sigma'(\theta_1)$ varries as the cosine of θ_1.

In inelastic scatt.:

$$(1)(7)(8) \quad \longrightarrow \quad \frac{v_1^2}{V^2} = \left(\frac{\mu}{m_2 B}\right)^2 \left[1 + 2B\cos\theta + B^2\right]$$

For elastic scatt. $B = \frac{m_1}{m_2}$

$$\frac{E_1}{E_0} = \frac{1 + 2B\cos\theta + B^2}{(1+B)^2}$$

For $B = \frac{m_1}{m_2} = 1 \quad \longrightarrow \quad \frac{E_1}{E_0} = \frac{1+\cos\theta}{2} = \cos\theta_1$

E_0: Initial kinetic energy of the incident particle in lab.

E_1: Corresponding energy after scatt.

At $\theta = \pi$, $\theta_1 = \frac{\pi}{2}$, the incident particle loses all its energy.

9/5

The Born Approximation,

Equ., $f(k',k) = -\frac{4\pi^2 m}{\hbar^2} \langle k'|V|\psi^+\rangle$

is still not directly useful in computing $\frac{d\sigma}{d\Omega}$, because of unknown ket $|\psi^+\rangle$

If the effect of scatterer is not very strong;

$$\langle x'|\psi^+\rangle \longrightarrow \langle x'|\varphi\rangle \quad \text{(under the integral sign)}$$

$$f^{(1)}(k',k) = -\frac{4\pi^2 m}{\hbar^2}\int d^3x' \frac{e^{-ik'\cdot x'}}{(2\pi)^{3/2}} V(x') \langle x'|\varphi\rangle$$

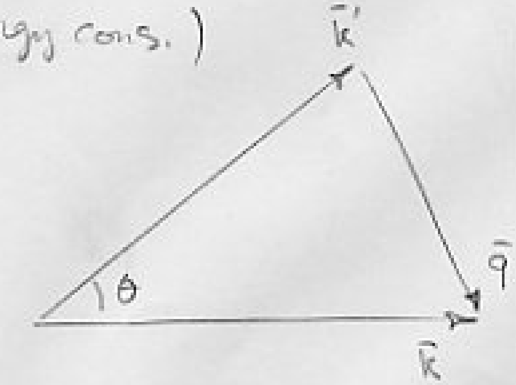
(we treat the potential to first order)

$$f^{(1)}(k',k) = -\frac{4\pi^2 m}{\hbar^2}\int d^3x' \frac{e^{-ik'\cdot x'}}{(2\pi)^{3/2}} V(x') \frac{e^{ik\cdot x'}}{(2\pi)^{3/2}}$$

$$= -\frac{m}{2\pi\hbar^2}\int d^3x' \, e^{i(k-k')\cdot x'} V(x') \quad \text{First order Born approx.}$$

Apart from the coeff. $\left(-\frac{m}{2\pi\hbar^2}\right)$, the first order amplitude is just three-dim Fourier transform of the potential V, with respect to $q = k - k'$

Since $|k'| = k$ (by energy cons.)

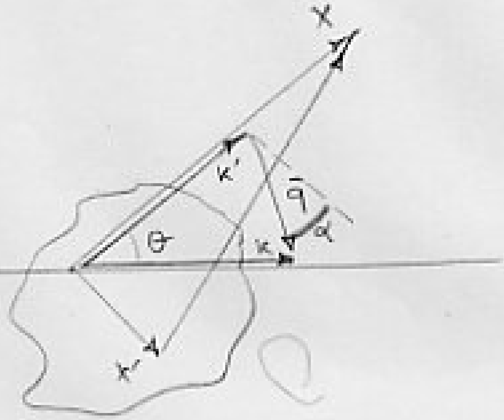


$$q^2 = k^2 + k^2 - 2k^2\cos\theta$$
$$= 2k^2(1-\cos\theta)$$
$$= 4k^2 \sin^2\frac{\theta}{2}$$
$$\rightarrow q = 2k\sin\frac{\theta}{2}$$

let $V = V(r)$ spherically symmetric potential.

$$f^{(1)}(q) = f^{(1)}(\theta) = -\frac{m}{2\pi\hbar^2}\int_0^\infty dr'\, r'^2 \int_0^{2\pi} d\varphi \int d(\cos\alpha)\; e^{iqr'\cos\alpha}\, V(r')$$

$$=\left(-\frac{m}{2\pi\hbar^2}\right)\int_0^\infty dr'\, r'^2\,(2\pi)\left[\frac{1}{iqr'}\, e^{iqr'\cos\alpha}\right]_{-1}^{+1} V(r')$$



$$f^{(1)}(q) = \frac{im}{\hbar^2 q}\int_0^\infty dr'\, r'\, V(r')\left(e^{iqr'} - e^{-iqr'}\right)$$

$$= -\frac{2m}{\hbar^2 q}\int_0^\infty dr'\, r'\, V(r')\, S qr'$$

Ex. – Consider scattering by Yukawa Potential;

$$V(r) = \frac{V_0 e^{-\mu r}}{\mu r} \qquad \frac{1}{\mu} : \text{range of Potential}$$

For $r \gg \frac{1}{\mu} \longrightarrow V \to 0$ rapidly.

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r\left(\frac{V_0 e^{-\mu r}}{\mu r}\right) S(qr)\, dr$$

$$= -\frac{2mV_0}{\mu\hbar^2}\,\frac{1}{q}\int_0^\infty e^{-\mu r}\, S(qr)\, dr$$

Since; $\displaystyle\int e^{ax} S(bx)\, dx = \frac{1}{a^2+b^2}\left(a\, S(bx) - b\cos(bx)\right)e^{ax} + C$

$$f^{(1)}(\theta) = -\frac{2mV_0}{\mu\hbar^2}\,\frac{1}{q}\left\{\frac{1}{\mu^2+q^2}\left(-\mu\, S qr - q\cos qr\right)e^{-\mu r}\right\}_0^\infty$$

$$f^{(1)}(\theta) = -\frac{2mV_0}{\mu\hbar^2}\frac{1}{q}\left\{\frac{1}{\mu^2+q^2}(+q)\right\} = -\left(\frac{2mV_0}{\mu\hbar^2}\right)\frac{1}{q^2+\mu^2}$$

Remark; Also, we could use;

$$\mathrm{Im}\left[\int_0^\infty e^{-\mu r} e^{iqr}\,dr\right] = -\mathrm{Im}\left(\frac{1}{-\mu+iq}\right) = \frac{q}{\mu^2+q^2}$$

$$\int e^{-\mu r}\sin(qr)\,dr = \mathrm{Im}\int e^{-\mu r} e^{iqr}\,dr$$

Notice; $q^2 = 4k^2\sin^2\frac{\theta}{2} = 2k^2(1-\cos\theta)$

$$\frac{d\sigma}{d\Omega} = \left|f^{(1)}(\theta)\right|^2 = \left(\frac{2mV_0}{\mu\hbar^2}\right)^2\frac{1}{[2k^2(1-\cos\theta)+\mu^2]^2}$$

If $\mu \to 0$, but $\frac{V_0}{\mu}$ = fixed

$\Longrightarrow$ Yukawa Potential $\to$ Coulomb Potential

for example; $\frac{V_0}{\mu} \to zz'e^2$

$$\lim_{\mu\to 0} V(r) = \lim \frac{V_0 e^{-\mu r}}{\mu r} = \frac{zz'e^2}{r}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mzz'e^2}{\hbar^2}\right)^2\frac{1}{4k^4(1-\cos\theta)^2} = \frac{(2m)^2(zz'e^2)^2}{\hbar^4}\frac{1}{16k^4\sin^4(\frac{\theta}{2})}$$

Since $\hbar k = |\vec{P}|$ $\quad E_k = \frac{\hbar^2 k^2}{2m}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\left(\frac{zz'e^2}{E_k}\right)^2\frac{1}{\sin^4(\frac{\theta}{2})}$$

This is precisely Rutherford scattering cross section.

Some Results:

Consider the Born amplitude with $V=V(r)$

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty r\,V(r)\,S(qr)\,dr$$

1- $\frac{d\sigma}{d\Omega}$ or $f(\theta)$ is a func. of q only (k and θ)

or energy $\frac{\hbar^2 k^2}{2m}$ and θ.

2- $f(\theta)$ is always <u>real</u>.

3- $\frac{d\sigma}{d\Omega}$ is indep. of the sign of $V(r)$

4- For small k ($q=2k\,S\frac{\theta}{2}$ small)

$$f^{(1)}(\theta) = \lim_{q\to 0} \frac{-m}{2\pi\hbar^2}\int d^3x'\, e^{iq.x'}\,V(x') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\,V(x')$$

(indep. of θ)

5- $f(\theta)$ is small for large q, due to rapid oscillation of the integrand.

Validity of the first order Born approx. –

It is applicable if:

$\langle x'|\psi^+\rangle$ is not too different from $\langle x|\varphi\rangle$ inside the range of potential.

In other words, the distortion of the incident wave must be small.

Going back to:

$$\langle x|\psi^+\rangle = \langle x|\varphi\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|x-x'|}}{4\pi|x-x'|} V(x')\langle x'|\psi^{\pm}\rangle$$

we see that $\langle x|\psi^+\rangle$ is not too different from $\langle x|\varphi\rangle$ at the center of scattering potential $x \simeq 0$, if

$$\overset{\simeq 1}{\frac{\langle x\simeq 0|\varphi\rangle - \langle x\simeq 0|\psi\rangle}{\langle x\simeq 0|\varphi\rangle}} \simeq 0 \to \left|\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3x' \frac{e^{ikr'}}{r'}V(x')\right| \ll 1$$

(1) a crude estimate for smooth central pot. ($ik|x-x'| \simeq ikr'$ at $x \simeq 0$)

$$\langle x|\psi^+\rangle = \langle x|\varphi\rangle - \text{small}$$

(the term $e^{ik\cdot x'}$ in the integrand has been neglected)

Now consider the special case of the yukawa pot. $V(r) = \frac{V_0 e^{-\mu r}}{\mu r}$

(1) → at low energies (small k, $k \ll \mu$)

$$\to e^{ikr'} \longrightarrow 1 \qquad \left|\frac{2m}{\hbar^2}\frac{1}{4\pi}\frac{4\pi V_0}{\mu}\int_0^\infty \frac{e^{-\mu r'}}{r'^2}r'^2 dr'\right| \ll 1 \to \frac{2m}{\hbar^2}\frac{|V_0|}{\mu^2} \ll 1$$

Remark: Scattered term $= -\frac{2m}{\hbar^2}\int d^3x' \frac{e^{ik|x-x'|}}{4\pi|x-x'|}V(x')\frac{e^{ik\cdot x'}}{(2\pi)^{3/2}} = -\frac{2m}{\hbar^2}\frac{1}{4\pi}\int \frac{e^{ikr'}}{r'}V(r')\frac{e^{ik\cdot x'}}{(2\pi)^{3/2}}d^3x'$ near $x \simeq 0$

$= -\frac{2m}{\hbar^2}\frac{1}{(2\pi)^{3/2}}\int_0^\infty r'^2 dr' \frac{e^{ikr'}}{r'}V(r')\frac{\sin kr'}{kr'} \to \frac{2m}{k\hbar^2}\left|\int_0^\infty e^{ikr'}\sin kr'\, V(r')dr'\right| \ll 1$

more exact form of (1) (Merzbacher)

Remember $|\langle x|\varphi\rangle| \sim |e^{ik\cdot x}| = 1$

The cond. for Yukowa Potential to develop a bound state is:

$$\frac{2m}{\hbar^2}\frac{|V_0|}{\mu^2} \geq 2.7 \qquad (V_0 \text{ negative})$$

In other words:
If the potential is strong enough, to develop a bound state

→ The Born appox. will probably give a <u>misleading</u> result.

In the <u>opposite</u> high k-limit. The cond. that the second term in. $<x|\psi^+>$ equ. is <u>small</u>, can be shown to imply:

$$\frac{2m}{\hbar^2}\frac{|V_0|}{\mu k}\ln\left(\frac{k}{\mu}\right) << 1$$

As k becomes larger, this inequality is more easily satisfied.

Born approx. tends to get better at <u>high</u> energies.

Born approx. gives good results, when;

$$\begin{cases} \text{I- The potential is not strong.} \\ \text{II- k is large.} \end{cases}$$

Note: The Born approx. encounters a rather peculer dificulty in the case of the <u>Coulomb</u> field.

Higher Order Born Approx.-

Define the transition op. T such that:

$$V|\psi^+> = T|\varphi>$$

multiply the Lippmann-Schwinger equ.,

$$|\psi^\pm> = |\varphi> + \frac{1}{E-H_0 \pm i\epsilon} V|\psi^\pm>$$

by V;

$$T|\varphi> = V|\varphi> + V \frac{1}{E-H_0+i\epsilon} T|\varphi>$$

This is supposed to hold for $|\varphi>$ taken to be any plane-wave state.

Furthermore these momentum eigenkets $|\varphi>$ are complete, therefore;

$$\longrightarrow \quad T = V + V\frac{1}{E-H_0+i\epsilon} T \qquad \forall |\varphi>$$

Since; $f(k',k) = -\frac{1}{4\pi}(2\pi)^3 \frac{2m}{\hbar^2} <k'|V|\psi^+>$

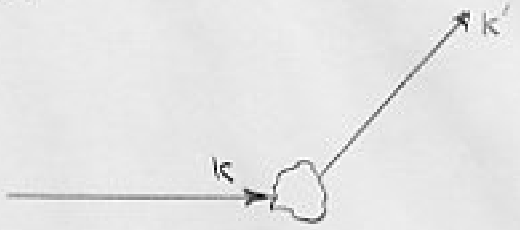
$$\longrightarrow f(k',k) = -\frac{1}{4\pi}(2\pi)^3 \frac{2m}{\hbar^2} <k'|T|k>$$

Now, we can obtain an iterative sol. for T;

$$T = V + V\frac{1}{E-H_0+i\epsilon}V + V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V + \dots$$

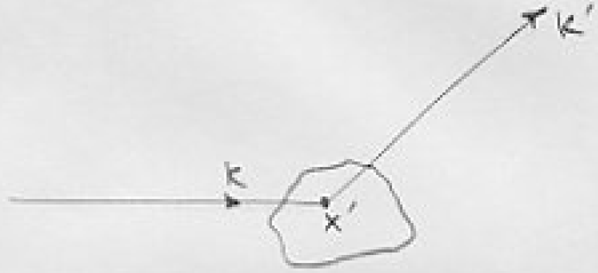
(Transition op.)

Diagramatic Representation of T:

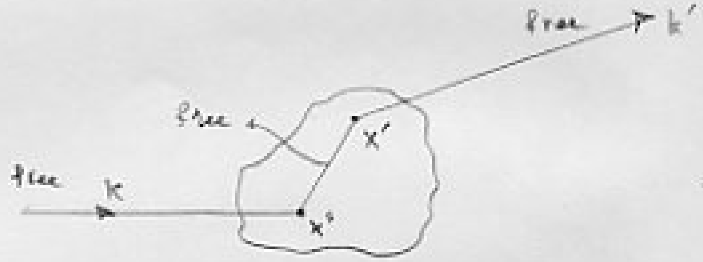


1) $f=0$

2) $T^{(1)}=V$



3) $T^{(2)} = V\frac{1}{E-H_0+i\epsilon}V$



Between the collisions the particle moves as a free particle. (because it is governed by H_0)

$$f(k',k) = \sum_{n=1}^{\infty} f^{(n)}(k',k)$$

$$f^{(1)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle k'|V|k\rangle$$

$$f^{(2)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle k'|V\frac{1}{E-H_0+i\epsilon}V|k\rangle$$

$\vdots$

For $f^{(2)}(k',k)$:

$$f^{(2)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x'\int d^3x''\langle k'|x'\rangle V(x')\langle x'|\frac{1}{E-H_0+i\epsilon}|x''\rangle$$
$$\cdot V(x'')\langle x''|k\rangle$$

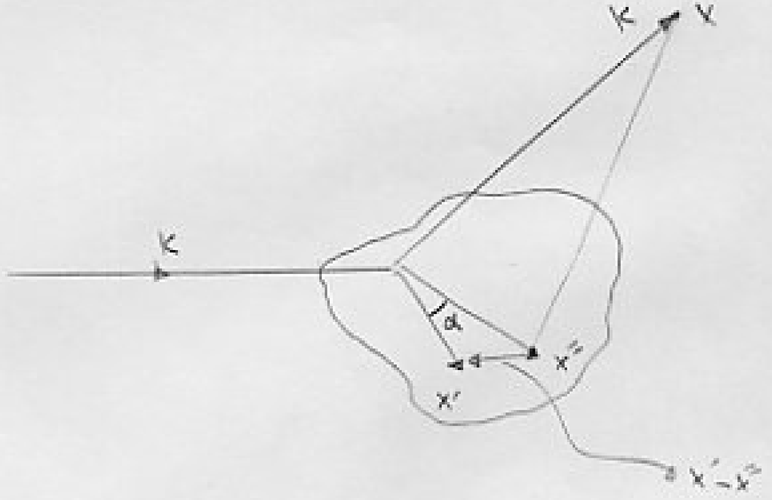
$$f^{(2)}(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' \int d^3x'' \, e^{-ik'\cdot x'} \, V(x')\left[\frac{2m}{\hbar^2}G_+(x',x'')\right]V(x'')\,e^{ik\cdot x''}$$

$$=\left(-\frac{1}{4\pi}\right)^2\left(\frac{2m}{\hbar^2}\right)^2\int d^3x' \, e^{-ik'\cdot x'} \, V(x')\int d^3x'' \, \frac{e^{ik|x'-x''|}}{|x'-x''|} \, V(x'')\,e^{ik\cdot x''}$$

At low energies $e^{ik|x'-x''|} \approx 1$

$$\frac{1}{|x'-x''|} = \sum_{\ell=0}^{\infty}\frac{r''^{\ell}}{r'^{\ell+1}}P_\ell(\cos\alpha) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r''^{\ell}}{r'^{\ell+1}}Y_\ell^{m*}(\theta'',\varphi'')\,Y_\ell^{m}(\theta',\varphi')$$

$$f^{(2)}(k',k) = \left(-\frac{1}{4\pi}\frac{2m}{\hbar^2}\right)^2\sum_{\ell}\sum_{m}\int d^3x' \, \frac{Y_\ell^m(\theta',\varphi')}{r'^{\ell+1}} \, e^{-ik'\cdot x'} \, V(x')\int d^3x'' \, r''^{\ell}\, Y_\ell^{m*}(\theta'',\varphi'')\, e^{ik\cdot x''} \, V(x'')$$



Optical Theorem;

Theorem: $\mathrm{Im}\, f(\theta=0) = \frac{k\sigma_{tot}}{4\pi}$

Consider forward scattering $\theta=0$, $k=k'$

$f(\theta=0) = f(k,k)$

$\sigma_{tot} \equiv \int \frac{d\sigma}{d\Omega}\, d\Omega$

Proof -

$$f(\theta=0) = f(k,k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 <k|T|k>$$

$\mathrm{Im}<k|T|k> = \mathrm{Im}<k|V|\psi^+>$, $|k>=|\varphi>$

Since, $|\psi^{\pm}> = |\varphi> + \frac{1}{E-H_0 \pm i\epsilon} V|\psi^{\pm}>$

$\rightarrow |\varphi> = |k> = |\psi^+> - \frac{1}{E-H_0+i\epsilon} V|\psi^+>$

Since $V = V^+$

$\rightarrow \mathrm{Im} <\underset{\underset{<\varphi|}{\uparrow}}{k}|V|\psi^+> = \mathrm{Im}\left[<\psi^+|V|\psi^+> - <\psi^+|V\frac{1}{E-H_0-i\epsilon}V|\psi^+>\right]$

Now, we use the well-known relation:

$$\frac{1}{E-H_0-i\epsilon} = P_r.\left(\frac{1}{E-H_0}\right) + i\pi\delta(E-H_0)$$

Not. $P_r.\left(\frac{1}{x}\right) = \frac{1}{2}\left(\frac{1}{x+i\epsilon} + \frac{1}{x-i\epsilon}\right)$

$$\mathrm{Im}\langle k|T|k\rangle = \mathrm{Im}\langle k|V|\psi^+\rangle = \mathrm{Im}\left[\langle\psi^+|V|\psi^+\rangle - \langle\psi^+|V\frac{1}{E-H_0-i\epsilon}V|\psi^+\rangle\right]$$

$$= \mathrm{Im}\left[\text{real terms} - i\pi\langle\psi^+|V\delta(E-H_0)V|\psi^+\rangle\right]$$

$$= -\pi\langle\psi^+|V\delta(E-H_0)V|\psi^+\rangle$$

$$= -\pi\langle k|T^+\delta(E-H_0)T|k\rangle$$

$$= -\pi\int d^3k'\langle k|T^+|k'\rangle\langle k'|T|k\rangle\,\delta\left(E-\frac{\hbar^2k'^2}{2m}\right)$$

Remark: $\begin{cases} V \text{ is hermitian} \\ V\left(\mathrm{Pr.}\frac{1}{E-H_0}\right)V \text{ is hermitan}\end{cases}$

$\longrightarrow$ their matrix elements are real.

The volume element: $d^3k' = k'^2dk'd\Omega' = k'^2d\Omega'\frac{dk'}{dE'}dE'$

δ-func. gives:

$E' = \frac{\hbar^2k'^2}{2m}$ $\qquad dE' = \frac{\hbar^2k'}{m}dk'$

$$d^3k' = k'^2d\Omega'dk' = k'^2d\Omega'\left(\frac{dk'}{dE'}\right)dE'$$

$$= k'^2d\Omega'\left(\frac{m}{\hbar^2k'}\right)dE' = \frac{m}{\hbar^2}k'd\Omega'dE'$$

$\underset{\sim}{E'}$

$$\mathrm{Im}\langle k|T|k\rangle = -\pi\int\left(\frac{m}{\hbar^2}k'd\Omega'dE'\right)|\langle\bar{k}'|T|\bar{k}\rangle|^2\,\delta\left(E-\frac{\hbar^2k'^2}{2m}\right)$$

$$= -\pi\int d\Omega'\,\frac{mk}{\hbar^2}\,|\langle\bar{k}'|T|\bar{k}\rangle|^2$$

Ω'-dependence

$\begin{cases} \text{Alternatively: } x \equiv \frac{\hbar^2k'^2}{2m} \to k'dk' = \frac{m}{\hbar^2}dx \\ \mathrm{Im}\langle k|T|k\rangle = -\pi\int\frac{m}{\hbar^2}dx\sqrt{\frac{2m}{\hbar^2}x}\,d\Omega'|\langle\bar{k}'|T|\bar{k}\rangle|^2\,\delta(E-x) \\ = -\pi\int d\Omega'\,\frac{mk}{\hbar^2}\,|\langle\bar{k}'|T|\bar{k}\rangle|^2 \\ \qquad \text{when } E = \frac{\hbar^2k^2}{2m}\end{cases}$

Now,

$$\text{Im}\, f(0) = -\frac{m}{2\pi\hbar^2}(2\pi)^3 \;\text{Im}\, \langle \bar{k} | T | \bar{k} \rangle$$

$$= \frac{m^2 k}{2\hbar^4}(2\pi)^3 \int d\Omega' \, |\langle \bar{k}' | T | \bar{k} \rangle|^2$$

Since;

$$\langle k' | T | k \rangle = \frac{\hbar^2}{m(2\pi)^2} \, f(k',k)$$

$$\text{Im}\, f(0) = \frac{m^2 k}{2\hbar^4}(2\pi)^3 \int d\Omega' \, \frac{\hbar^4}{m^2 (2\pi)^4} \, |f(k',k)|^2$$

$$\text{Im}\, f(0) = \frac{k}{4\pi} \int d\Omega' \, |f(k',k)|^2$$

$$= \frac{k}{4\pi} \int d\Omega' \, \frac{d\sigma}{d\Omega'} = \frac{k\,\sigma_{tot}}{4\pi}$$

Eikonal Approximation—

This approx. covers, situation in which;

$\left\{\begin{array}{l}\text{1- } V(x) \text{ Varies slowly over a distance of order of wave length } \lambda. \\ \text{2- As long as } E \gg |V|, \quad V \text{ itself need not be weak.}\end{array}\right.$

Thus the domain of validity is different from Born approx.

Under these conds., the <u>Semi classical</u> path becomes applicable.

$$\psi^{+} \sim e^{iS(x)/\hbar} \qquad \text{semiclassical wave-func.}$$

The schrödinger equ. gives;

$$\frac{1}{2m}\left(-i\hbar\nabla^2 S + (\nabla S)^2\right) = E - V$$

The classical limit is obtained if $\underline{\underline{\hbar}}|\nabla^2 S| \ll |\nabla S|^2$

$$\frac{(\nabla S)^2}{2m} + V = E \qquad \begin{array}{l}\text{Hamilton-Jacobi equ} \\ \text{Eikonal equ. (in optics)}\end{array}$$

where $E = \frac{\hbar^2 k^2}{2m}$

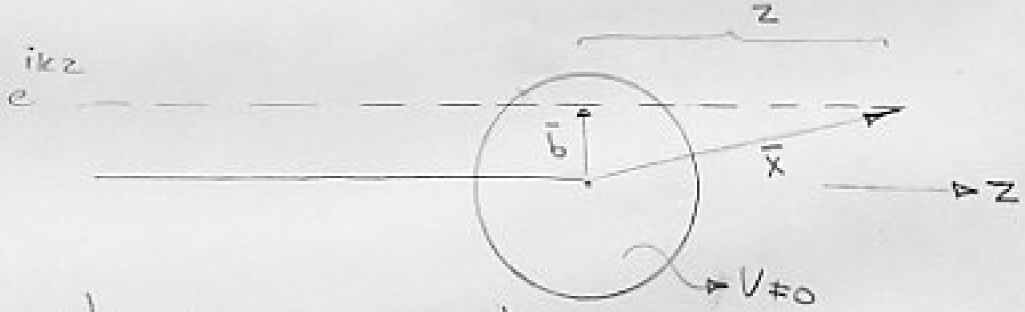
S: Hamilton Principal func.

Further approximation:

Classical trajectory is a straight line path (it is satisfactory for small deflections at high energy)

In other words scattering is confined to small angles and the trajectory will be nearly a straight line parralel to the direction of incidence.

→ Then semiclassical method is expected to be accurate.



$$\frac{\nabla S}{\hbar} = \left(k^2 - \frac{2m}{\hbar^2} V(x)\right)^{\frac{1}{2}} = \left(k^2 - \frac{2m}{\hbar^2} V(\sqrt{b^2+z^2}\right)^{\frac{1}{2}}$$

$$\rightarrow \frac{S}{\hbar} = \int_{-\infty}^{z} \left(k^2 - \frac{2m}{\hbar^2} V(\sqrt{b^2+z'^2})\right)^{\frac{1}{2}} dz' + C$$

C can be chosen in such a way:

$$\frac{S}{\hbar} \rightarrow kz \qquad \text{as } V \rightarrow 0$$

(i.e. $\varphi^+ \sim e^{iS(x)/\hbar} \longrightarrow e^{ikz}$ plane wave)

Then;

$$\frac{S}{\hbar} = kz + \int_{-\infty}^{z} \left\{ \sqrt{k^2 - \frac{2m}{k^2} V(\sqrt{b^2+z'^2})} - k \right\} dz'$$

If $E >> V(x)$:

$$\sqrt{k^2-\frac{2m}{\hbar^2}V} = k\left(1-\frac{2m}{\hbar^2}\frac{V}{k^2}\right)^{\frac{1}{2}} \simeq k - \frac{mV}{\hbar^2 k}$$

$$\longrightarrow \frac{S}{\hbar} \simeq kz - \frac{m}{\hbar^2 k}\int_{-\infty}^{z} V(\sqrt{b^2+z'^2})\,dz'$$

$$\psi^+(x) = \psi^+(\bar{b}+z\hat{z}) \simeq \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{ikz}\, e^{\frac{-im}{\hbar^2 k}\int_{-\infty}^{z} V(\sqrt{b^2+z'^2})\,dz'}$$

This wave func. does not have the correct asymptotic form appropriate for an incident plus spherically outgoing wave.

i.e. it is not of the form $e^{ikr} + f(\theta)\frac{e^{ikr}}{r}$ and indeed refers only to motion along the original direction.

But it can still be used to obtain $f(k',k)$.

$$f(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x' \frac{e^{-ik'\cdot x'}}{(2\pi)^{\frac{3}{2}}}\, V(x')\langle x'|\psi^+\rangle$$

$$= -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar^2}\langle k'|V|\psi^+\rangle$$

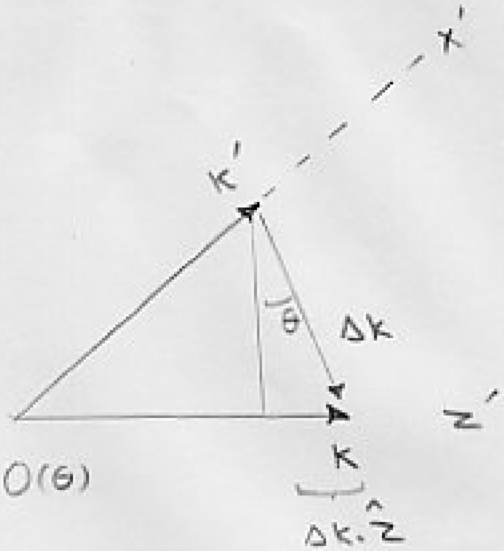
$$f(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\, e^{-ik'\cdot x'}\, V(\sqrt{b^2+z'^2})\, e^{ik\cdot x'}\left\{e^{\frac{-im}{\hbar^2 k}\int_{-\infty}^{z'} V(\sqrt{b^2+z''^2})\,dz''}\right\}$$

Note, that, without the last factor $\{--\}$ this gives the first order Born amplitude.

$(kz = k\cdot x')$

In cylindrical coord.:

$$d^3x' = b\,db\,d\varphi_b\,dz'$$



1- For small θ $|k-k'| = \Delta k = k\theta \sim O(\theta)$

2- $(k-k')\cdot\bar{x}' = (k-k')\cdot(b+z\hat{z}')$

$$= (k-k')\cdot z\hat{z}' - b\cdot k' + \underset{b\perp k}{\cancel{b\cdot k}}$$

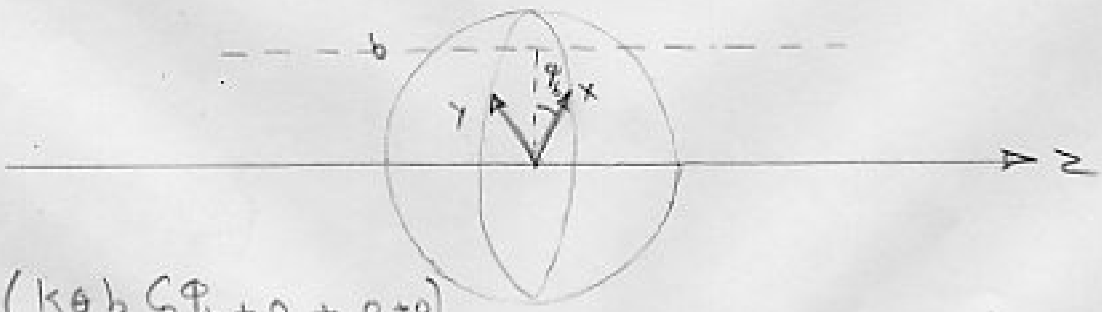
but $(k-k')\cdot\hat{z}' = \Delta\bar{k}\cdot\hat{z}' = k\theta\,\underbrace{S\theta}_{\simeq\theta} \simeq O(\theta^2) \simeq 0$

$$\longrightarrow (k-k')\cdot x' \simeq -b\cdot k'$$

Without loss of generality we choose scattering to be in the xz-plane.

$\bar{k}'\cdot\bar{b} = (kS\theta\hat{x} + kC\theta\hat{z})\cdot(bC\varphi_b\hat{x} + bS\varphi_b\hat{y})$ $\quad |k| = |k'|$

$C\theta \simeq 1 \quad S\theta \simeq \theta$



$$\bar{k}'\cdot\bar{b} = (k\theta bC\varphi_b + 0 + 0 + 0)$$

$$= kb\theta C\varphi_b$$

$$f(k',k) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int_0^\infty db\, b\int_0^{2\pi} d\varphi_b\, e^{-ikb\cos\varphi_b\theta}\int_{-\infty}^{\infty} dz\, V e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z} dz' V}$$

1- Using the identity;

$$\int_0^{2\pi} e^{-ikb\theta\cos\varphi_b}\, d\varphi_b = 2\pi J_0(kb\theta)$$

where $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ix\cos\theta}\, d\theta$

2- And also;

$$\int_{-\infty}^{\infty} dz\, V e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z} V dz'} = \frac{i\hbar^2 k}{m}\int_{-\infty}^{\infty} dz\, \frac{\partial}{\partial z} e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z} dz' V}$$

$$= \frac{i\hbar^2 k}{m}\left[e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z} V dz'}\right]_{z=-\infty}^{z=+\infty}$$

$$= \frac{i\hbar^2 k}{m}\left\{e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} V dz'} - e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{-\infty} V dz'}\right\}$$

$$= \frac{i\hbar^2 k}{m}\left[e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} V dz'} - 1\right]$$

$$= \frac{i\hbar^2 k}{m}\left[e^{2i\Delta(b)} - 1\right]$$

where $\Delta(b) = -\frac{m}{2\hbar^2 k}\int_{-\infty}^{\infty} dz'\, V(\sqrt{b^2+z'^2})$

$$\to f(k',k) = -ik\int_0^\infty db\, b\, J_0(kb\theta)\left[e^{2i\Delta(b)} - 1\right]$$

where $b <$ range of V (no contribution from $[e^{2i\Delta(b)} - 1]$ in the integral if $b >$ Range of V)

The Eikonal approximation satisfies the optical theorem;

Proof. –

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega \qquad \frac{d\sigma}{d\Omega} = |f(k',k)|^2$$

$$\sigma_{tot} = k^2 \int d\Omega \left\{ \int_0^\infty b\,db \int_0^\infty b'\,db' \, J_0(kb\theta)\, J_0(kb'\theta) \left[e^{2i\Delta(b)} - 1\right]\left[e^{-2i\Delta(b')} - 1\right] \right\}$$

$$d\Omega = d\varphi_b \, s\theta \, d\theta$$

$$\sigma_{tot} = k^2 \int_0^{2\pi} d\varphi_b \int_0^{\pi} s\theta \, d\theta \, \{\cdots\} = 2\pi k^2 \int_0^{\pi} s\theta \, d\theta \, \{\cdots\}$$

For small θ: $s\theta \to \theta \qquad \int_0^{\pi} \to \int_0^{\infty}$

The completeness relation for the Bessel fucs.;

$$\int_0^\infty J_0(x\theta)\, J_0(x'\theta)\, \theta \, d\theta = \frac{1}{x}\delta(x-x')$$

$$\sigma_{tot} = 2\pi k^2 \int_0^\infty b\,db \int_0^\infty b'\,db' \, \frac{1}{kb}\delta(kb-kb') \left[e^{2i\Delta(b)} - 1\right]\left[e^{-2i\Delta(b')} - 1\right]$$

but $\delta\, a(x-x') = \frac{1}{a}\delta(x-x')$

$$\sigma_{tot} = 2\pi k^2 \int_0^\infty b\,db \int_0^\infty b'\,db' \, \frac{1}{k^2 b}\delta(b-b') \left[e^{2i\Delta(b)} - 1\right]\left[e^{-2i\Delta(b')} - 1\right]$$

$$\sigma_{tot} = 2\pi \int_0^\infty b\,db \, \left|e^{2i\Delta(b)} - 1\right|^2 = 8\pi \int_0^\infty b\,db \, S^2(\Delta(b))$$

$$f(k,k') = -ik \int_0^\infty b\,db\; J_0(kb\theta)\left[e^{2i\delta(b)} - 1\right]$$

$$f(0) = -ik \int_0^\infty b\,db\; J_0(0)\left[e^{2i\delta(b)} - 1\right]$$

but $J_0(0) = 1$

$$f(0) = -ik \int_0^\infty b\,db\left(e^{2i\delta(b)} - 1\right)$$

$$\mathrm{Im}\, f(\theta=0) = -k \int_0^\infty b\,db\; \mathrm{Re}\left(e^{2i\delta(b)} - 1\right)$$

$$= -k \int_0^\infty b\,db\left(\cos 2\delta(b) - 1\right)$$

$$= -2k \int_0^\infty b\,db\; \sin^2\delta(b)$$

$$\longrightarrow \mathrm{Im}\, f(\theta=0) = \frac{k\sigma_{tot}}{4\pi}$$

7-5 Free Particle States

Plane Waves versus Spherical Waves

For free particle op. : $H_0 = \frac{P^2}{2m}$

$$[H_0, P] = 0$$

$$\rightarrow \begin{cases} H_0|K\rangle = E|K\rangle \\ P|K\rangle = \hbar K|K\rangle \end{cases}$$

Remark:
Since $[L_z, P_z] = i\hbar P_z \neq 0$
the wave func. can not
be diagonalized simultaneously
i.e. we have either
$P|P'\rangle = p'|P'\rangle$
or $L_z|E,l,m\rangle = m\hbar|E,l,m\rangle$

For free Particle we have also;

$$[H_0, L^2] = 0 \qquad [H_0, L_z] = 0$$

$\longrightarrow$ H_0, L^2, L_z have simultaneous eigen-ket.

Ignoring the spin, such state is denoted by

$|E, l, m\rangle$ spherical wave state.

More generally: (A free-particle state can be expanded in two different basis.)

$$|K\rangle = \sum\!\!\!\!\!\!\int \langle E,l,m|K\rangle |E,l,m\rangle$$ superposition (spherical wave) basis $\{|E,l,m\rangle\}$

$$|\varphi\rangle = \sum \langle K'|\varphi\rangle |K'\rangle$$ " (plane wave basis) $\{|K\rangle\}$

K' : different directions and different magnitudes.

Our Aim: We want to derive the tr. func. $\langle K|E,l,m\rangle$ which connects the plane-wave basis to the spherical basis

Remark: Here $|\varphi\rangle = |E,l,m\rangle$ is free spherical state,

$$\langle x|E,l,m\rangle = c_l j_l(kr) Y_l^m(\hat{r})$$

The quantity $\langle K|E,l,m\rangle$ can also be regarded as the momentum wave func. for the spherical wave $|E,l,m\rangle$

The normalization:

$$\langle E',l',m'|E,l,m\rangle = \delta_{ll'}\,\delta_{mm'}\,\delta(E-E')$$

A guess: $\langle K|E,l,m\rangle = g_{lE}(k)\,Y_l^m(\hat{k})\,\sqrt{\frac{4\pi}{2l+1}}$

Proof. — Consider:

$|k\hat{z}\rangle$: momentum eigenket, plane wave propagating along z-dir

$$L_z|k\hat{z}\rangle = (xP_y - yP_x)\,|k_x=0,\ k_y=0,\ k_z=k\rangle = 0$$

Classically: Ang. mom. component must vanish in the dir of propagation, because: $L\cdot P = (x\wedge P)\cdot P = 0$

$l_z = m\hbar \rightarrow m=0$

$$\longrightarrow \langle E',l',m'|k\hat{z}\rangle = 0 \qquad \text{for all } m\neq 0$$

$$|k\hat{z}\rangle = \sum_{l'}\int dE'\ |E',l',m'=0\rangle\langle E',l',m'=0|k\hat{z}\rangle$$

$\sum_{m'}$ is canceled

Now;

$$|\bar{k}\rangle = D(\alpha=\varphi,\ \beta=\theta,\ \gamma=0)\,|k\hat{z}\rangle$$

(dir. of $\bar{k}$ is specified by θ and φ)

$$\rightarrow |\bar{K}\rangle = \sum_{l'}\int dE'\ D(\alpha=\varphi,\beta=\theta,\gamma=0)\,|E',l',m'=0\rangle\langle E',l',m'=0|k\hat{z}\rangle$$

$$\langle E,l,m|K\rangle = \sum_{l'}\int dE'\langle E,l,m|D|E',l',m'=0\rangle\langle E',l',m'=0|k\hat{z}\rangle$$

$$ = \sum_{l'}\int dE'\ D^{l'}_{m0}(\alpha=\varphi,\beta=\theta,\gamma=0)\,\delta_{ll'}\,\delta(E-E')\,\langle E',l',m'=0|k\hat{z}\rangle$$

Note: $[J^2, J_k]=0 \rightarrow [J^2, f(J_k)]=0 \rightarrow [D(R), J^2]=0 \rightarrow D(R)$ can not change J-value.

But $[D(R), J_z]\neq 0$

$<E,l,m|k> = D^{l}_{m0}(\alpha=\varphi, \beta=\theta, \gamma=0) <E,l,m=0|k\hat{z}>$

Now $<E,l,m=0|k\hat{z}>$ is idep. of th orientation of $\bar{k}$ (i.e. θ,φ)

we call it $g^{*}_{lE}(k)$. $\quad <E,l,m=0|k\hat{z}> = g^{*}_{lE}(k) = \sqrt{\frac{2l+1}{4\pi}}\, g^{*}_{E}(k)$

using ; $D^{*l}_{m0}(\alpha,\beta,\gamma=0) = \sqrt{\frac{4\pi}{2l+1}}\, Y^{*m}_{l}(\theta,\varphi)\Big|_{\theta=\beta,\, \varphi=\alpha}$

$$<k|E,l,m> = \cancel{\sqrt{\frac{4\pi}{2l+1}}}\, g_{lE}(k)\, Y^{m}_{l}(\hat{k})$$

Remark:
$<l,m|\hat{z}> =$
$= Y^{*m}_{l}(\theta=0, \varphi = \text{undetermined}) \cdot \delta_{m0}$
$= \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta=1)\,\delta_{m0}$
$= \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0}$

Determination of $g_{lE}(k)$ –

$$(H_0 - E)|E,l,m> = 0$$

Also, $<k|(H_0-E) = \left(\frac{\hbar^2 k^2}{2m} - E\right) <k| = 0$

multiplying by $|E,l,m>$;

$$\underbrace{<k|(H_0-E)|E,l,m>}_{0} = \left(\frac{\hbar^2 k^2}{2m} - E\right) <k|E,l,m>$$

$$\rightarrow \left(\frac{\hbar^2 k^2}{2m} - E\right) <k|E,l,m> = 0$$

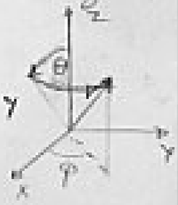
$\rightarrow <k|E,l,m>$ can be $\neq 0$ if $E = \frac{\hbar^2 k^2}{2m}$

$\rightarrow g_{lE}(k) = N\, \delta\left(\frac{\hbar^2 k^2}{2m} - E\right)$ (we will see in normalization that g_{lE} is indep. of l

$N = ?$

Remark: For any rot. 3-parameters are needed
α, β, γ
or
$\hat{n}$ (2-indep parameters) and φ (rotation angle)

Ex.



First rot. : about y by θ
Second rot:
about z by φ
In the notation of Euler angles
$D(R) = D(\alpha=\theta, \beta=\varphi, \gamma=0)$

Consider

$$\langle E', l', m' | E, l, m \rangle = \delta_{ll'} \delta_{mm'} \delta(E-E') \quad (1)$$

$$\langle E', l', m' | E, l, m \rangle = \int d^3k'' \langle E', l', m' | k'' \rangle \langle k'' | E, l, m \rangle$$

$$= \int dk'' \, k''^2 \int d\Omega_{k''} \left\{ |N|^2 \delta\left(\frac{\hbar^2 k''^2}{2m} - E'\right) \delta\left(\frac{\hbar^2 k''^2}{2m} - E\right) Y^{*m'}_{l'}(\hat{k}'') \, Y^{m}_{l}(\hat{k}'') \right\}$$

$$= \int \frac{k''^2 dE''}{\frac{dE''}{dk''}} \int d\Omega_{k''} \left\{ \quad \right\}$$

$$\frac{dE''}{dk''} = \frac{d}{dk''}\left(\frac{\hbar^2 k''^2}{2m}\right) = 2\frac{\hbar^2 k''}{2m} = 2\frac{\hbar k''}{\sqrt{2m}} \frac{\hbar}{\sqrt{2m}} = 2\sqrt{E''} \frac{\hbar}{\sqrt{2m}}$$

$$\langle E', l', m' | E, l, m \rangle = \int \frac{2m}{\hbar^2} E'' \frac{\sqrt{2m}}{2\sqrt{E''}\,\hbar} dE'' \int d\Omega_{k''} \left\{ \quad \right\}$$

where we have also used $E = \frac{\hbar^2 k^2}{2m}$ $\quad k = \sqrt{\frac{2m}{\hbar^2}} \sqrt{E}$

using this again;

$$\langle E', l', m' | E, l, m \rangle = \int \frac{m}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \sqrt{E''} \, dE'' \int d\Omega_{k''} \left\{ |N|^2 \right.$$

$$\left. \delta(E''-E') \, \delta(E''-E) \, Y^{*m'}_{l'}(\hat{k}'') \, Y^{m}_{l}(\hat{k}'') \right\}$$

$$= \frac{m}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \sqrt{E'} \, \delta_{ll'} \delta_{mm'} |N|^2 \delta(E-E')$$

$$= |N|^2 \frac{mk'}{\hbar^2} \delta(E-E') \, \delta_{ll'} \delta_{mm'} \quad (2)$$

$$(1)(2) \rightarrow N = \frac{\hbar}{\sqrt{mk}}$$

$$g_E(k) = \frac{\hbar}{\sqrt{mk}} \delta\left(\frac{\hbar^2 k^2}{2m} - E\right)$$

Hence:

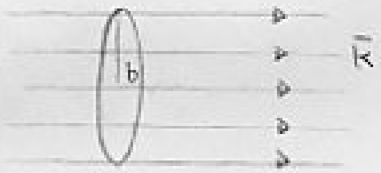
$$\langle K|E,l,m\rangle = \frac{\hbar}{\sqrt{mk}}\,\delta\left(\frac{\hbar^2k^2}{2m}-E\right)Y_l^m(\hat{k})$$

$$|K\rangle = \sum_l\sum_m\int dE\,|E,l,m\rangle\langle E,l,m|K\rangle$$

$$= \sum_l\sum_m |E,l,m\rangle\Big|_{E=\frac{\hbar^2k^2}{2m}}\left(\frac{\hbar}{\sqrt{mk}}\,Y_l^{m*}(\hat{k})\right) \quad \text{(The expansion in momentum space)}$$

The transverse dim. of the plane wave is infinite.

i.e.



$(b\to\infty)$

We expect the plane wave will contain all the b values.

(Semiclassical value $b=\frac{l\hbar}{p}$).

→ $|k\rangle$ will contain all possible values of l when written in spherical terms.

We have derived the wave func. for $|E,l,m\rangle$ in momentum space.

Now we consider the corresponding wave-func in position space.

From wave mechanics, we remember the wave-func for <u>free</u>-spherical wave is:

$$J_\ell(kr)\,Y_\ell^m(\hat{r})$$

The <u>second</u> sol! $\eta_\ell(kr)$, although it satisfies the appropriate differential equ. is inadmissiable, because it is <u>singular</u> at the origin.

$$\text{Thus;}\quad \langle x|E,\ell,m\rangle = C_\ell J_\ell(kr)\,Y_\ell^m(\hat{r})$$

$$C_\ell = ?$$

$$\langle x|k\rangle = \frac{e^{ik.x}}{(2\pi)^3} = \sum_\ell \sum_m \int dE\,\langle x|E,\ell,m\rangle\langle E,\ell,m|k\rangle$$

$$= \sum_\ell \sum_m \int dE\, C_\ell\, J_\ell(kr)\, Y_\ell^m(\hat{r})\,\frac{\hbar}{\sqrt{mk}}\,\delta\!\left(E-\frac{\hbar^2k^2}{2m}\right) Y_\ell^{m*}(\hat{k})$$

$$= \sum_\ell \sum_m C_\ell\, J_\ell(kr)\,\frac{\hbar}{\sqrt{mk}}\, Y_\ell^m(\hat{r})\, Y_\ell^{m*}(\hat{k})$$

Using the <u>addition</u> theorem;

$$\sum_m Y_\ell^m(\hat{r})\, Y_\ell^{m*}(\hat{k}) = \frac{2\ell+1}{4\pi}\, P_\ell(\hat{k}.\hat{r})$$

$$\langle x|k\rangle = \sum_\ell \frac{(2\ell+1)}{4\pi}\, P_\ell(\hat{k}.\hat{r})\,\frac{\hbar}{\sqrt{mk}}\, C_\ell\, J_\ell(kr) \qquad (1)$$

Now on the other hand:

$$\frac{e^{ik.x}}{(2\pi)^{3/2}} = \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1)(i)^\ell J_\ell(kr)\, P_\ell(\hat{k}.\hat{r}) \qquad (2)$$

which can be proved by the following integral representation for $J_\ell(kr)$

$$J_\ell(kr) = \frac{1}{2(i)^\ell}\int_{-1}^{+1} e^{ikr\cos\theta}\, P_\ell(\cos\theta)\, d(\cos\theta)$$

Defining z in the k-dir. and noting that $e^{ik\cdot r} = e^{ikr\cos\theta}$

$$(1)(2) \rightarrow C_\ell = \frac{i^\ell}{\hbar}\sqrt{\frac{2mk}{\pi}}$$

To summarize we have:

$$<k|E,\ell,m> = \frac{\hbar}{\sqrt{mk}}\,\delta\left(E - \frac{\hbar^2 k^2}{2m}\right) Y_\ell^m(\hat{k})$$

$$<x|E,\ell,m> = \frac{i^\ell}{\hbar}\sqrt{\frac{2mk}{\pi}}\, J_\ell(kr)\, Y_\ell^m(\hat{r})$$

7.6 Method of Partial waves expansion:

Let $V(r)$: spherically sym. pot.

$$[D(R), V(r)] = 0$$

It, then follows

$$[T, L^2] = 0 \qquad [T, L] = 0$$

where the transition operator is:

$$T = V + V \frac{1}{E - H_0 \pm i\epsilon} V + V \frac{1}{E - H_0 \pm i\epsilon} V \frac{1}{E - H_0 \pm i\epsilon} V + \cdots$$

In other words, T is a scalar operator;

Acc. to Wigner-Eckart Theorem;

$$\langle \alpha', j' m' | T_q^k | \alpha, j m \rangle = \langle \overset{j_1}{j} \overset{j_2}{k} ; \overset{m_1}{m} \overset{m_2}{q} | j k, \overset{j_3}{j'} \overset{m_3}{m'} \rangle \frac{\langle \alpha j' \| T^k \| \alpha j \rangle}{\sqrt{2j+1}}$$

For scalar op. $T_0^0 \equiv S$

$$\langle \alpha', j' m' | S | \alpha, j, m \rangle = \delta_{jj'} \delta_{mm'} \frac{\langle \alpha j' \| S \| \alpha j \rangle}{\sqrt{2j+1}}$$

Therefore; in spherical basis;

$$\langle E', l', m' | T | E, l, m \rangle = T_l(E)\, \delta_{ll'}\, \delta_{mm'}$$

In other words, T is diagonal both in l and m;

Furthermore, the (nonvanishing) diagonal element depends on $\underline{E \text{ and } l}$, but $\underline{\text{not } m}$.

We obtained ;

$$f(k,k') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle k'|T|k\rangle$$

$$\to f(k,k') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\sum_{\ell}\sum_{m}\sum_{\ell'}\sum_{m'}\int dE\int dE' \, \langle k'|E',\ell,m'\rangle$$

$$\langle E',\ell,m'|T|E,\ell,m\rangle\langle E,\ell,m|k\rangle$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\sum_{\ell}\sum_{m}\sum_{\ell'}\sum_{m'}\int dE\int dE' \, \frac{\hbar}{\sqrt{mk'}}\,\delta\left(E'-\frac{\hbar^2k'^2}{2m}\right)Y_{\ell'}^{m'}(\hat{k}')$$

$$\underbrace{\delta(E-E')\delta_{\ell\ell'}\delta_{mm'}\,T_\ell(E)}_{}\,\frac{\hbar}{\sqrt{mk}}\,\delta\left(E-\frac{\hbar^2k^2}{2m}\right)Y_\ell^{m*}(\hat{k})$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\frac{\hbar^2}{mk}\sum_{\ell}\sum_{m}T_\ell(E)\Big|_{E=\frac{\hbar^2k^2}{2m}}Y_\ell^m(\hat{k}')\,Y_\ell^{m*}(\hat{k})$$

$$= -\frac{4\pi^2}{k}\sum_{\ell}\sum_{m}T_\ell(E)\Big|_{E=\frac{\hbar^2k^2}{2m}}Y_\ell^m(\hat{k}')\,Y_\ell^{m*}(\hat{k})$$

Now choose; $\hat{k}\parallel\hat{z}$

$$\langle \ell,m|\hat{z}\rangle = Y_\ell^{m*}(\hat{k}\parallel\hat{z}) = Y_\ell^{m*}(\theta=0,\ \varphi\ \text{undetermined})\,\delta_{m0}$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)\Big|_{\cos\theta=1}\delta_{m0} = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}$$

$$Y_\ell^m(\hat{k}) = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0} \qquad (\hat{k}\parallel\hat{z})$$

If θ is the angle between k and k';

$$Y_\ell^0(\hat{k}') = \sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell(\cos\theta)$$

Remark. One may also use addition theorem;

$$\sum_m Y_{\ell m}(\hat{k})\, Y^*_{\ell m}(\hat{k}') = \frac{2\ell+1}{4\pi} P_\ell(\hat{k}.\hat{k}')$$

Define; the partial wave amplitude;

$$f_\ell(k) \equiv -\frac{\pi T_\ell(E)}{k}$$

Then since we have only m=0 contribution;

$$f(k';k) = f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$$

where $f(\theta)$ still depends on $\underline{k}$ (incident energy).

Let us consider the asymptotic form;

$$\langle x | \psi^+ \rangle \underset{r\to\infty}{\sim} \frac{1}{(2\pi)^{3/2}} \left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \right]$$

Also, we know;

$$e^{ik.x} = \sum_\ell (2\ell+1)(i)^\ell j_\ell(kr) P_\ell(\hat{k}.\hat{x})$$

$$\rightarrow e^{ikz} = \sum_\ell (2\ell+1)(i)^\ell j_\ell(kr) P_\ell(\cos\theta)$$ (z in k-dir.)

ad

$$j_l(kr) \xrightarrow[r\to\infty]{} \frac{1}{i2kr}\left[e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}\right] = \frac{1}{kr}\sin(kr-\frac{l\pi}{2})$$

$$\rightarrow \langle x|\psi^+\rangle \sim \frac{1}{(2\pi)^{3/2}}\left[\sum_l (2l+1)P_l(\cos\theta)\left(\frac{e^{ikr}-e^{-i(kr-l\pi)}}{2ikr}\right)\right.$$

$$\left. + \sum_l (2l+1) f_l(k) P_l(\cos\theta)\frac{e^{ikr}}{r}\right]$$

$\left\{\text{Remark: } i^l = e^{i\frac{\pi}{2}l}\right.$

$$= \frac{1}{(2\pi)^{3/2}}\sum_l (2l+1)\frac{P_l(\cos\theta)}{2ik}\left\{\left[1+2ikf_l(k)\right]\frac{e^{ikr}}{r} - \frac{e^{-i(kr-l\pi)}}{r}\right\}$$

In the absence of scattering potential ($V(r)=0$) we have; $f_l(k)=0 \quad \forall l$

The presence of scattering potential gives the following change on the outgoing wave;

$$1 \longrightarrow \left[1+2ikf_l(k)\right]$$

The incomming wave is completely unefected.

Alternative approach:

If 1 - The potential is spherically symmetric ($[V, D(R)]=0$)

2 - The incident wave has no azimuthal variation.

(1) → $[T, L^2] = [T, L] = 0$ T: transition op

→ T is scalar op.

(1)(2) → The wave func. $\psi(r)$ is axially symmetric about the direction of incidence $\hat{k}$.

In polar coord. (r,θ,φ)

1- with $\hat{k} \parallel \hat{z}$

2- origin at the center of force.

→ $\psi(\bar{r}) = \psi(r,\theta)$

Thus: $\psi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{1}{r} f_\ell(r) P_\ell(\cos\theta)$

Each term of this series is eigenfunc. of L^2 and L_z

$$L^2 P_\ell(C\theta) = \ell(\ell+1)\hbar^2 P_\ell(C\theta)$$

$$L_z P_\ell(C\theta) = 0 .$$

where $L^2 = -\hbar^2 \left\{ \frac{1}{S\theta}\frac{\partial}{\partial\theta}\left(S\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{S^2\theta}\frac{\partial^2}{\partial\varphi^2} \right\}$

$$L_z = -i\hbar\frac{\partial}{\partial\varphi}$$

The Schrödinger equ.:

$$(\nabla^2 - U(r) + k^2)\,\psi(r,\theta) = 0$$

where $E = \frac{\hbar^2 k^2}{2m}$ $\qquad U(r) = \frac{2m}{\hbar^2} V(r)$

but $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}$

Using the expansion of $\Psi(r,\theta)$:

$$\sum_{\ell=0}^{\infty} \frac{1}{r} P_\ell(\cos\theta)\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - U(r) + k^2\right] f_\ell(r) = 0$$

multiplying by $P_{\ell'}(\cos\theta)$ and integrate over $\cos\theta$,

$$\sum_{\ell=0}^{\infty} \int_{-1}^{1} d(\cos\theta)\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \left[\cdots \right] = 0$$

$$\sum_{\ell=0}^{\infty} \frac{2}{2\ell'+1}\, \delta_{\ell\ell'} \left[\qquad \right] = 0$$

$$\rightarrow \left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - U(r) + k^2\right] f_\ell(r) = 0$$

For potentials that are not too singular at the origin,

so that,

$$\lim_{r\to 0} r^{2+\epsilon} |U(r)| < c$$

$\left(\rightarrow |U(r)| << \frac{\ell(\ell+1)}{r^2} \text{ for } \ell \neq 0,\ r \to 0 \right)$

we can expand the radial func. as:

$$f_\ell(r) = \sum_n a_n r^n$$

An examination of the indicial equ., then shows that there are two sols.;

$f_\ell(r) \sim r^{\ell+1}$ regular at the origin

$f_\ell(r) \sim r^{-\ell}$ irregular " "

To describe a physical scattering situation; the wave-func. $\varphi(r)$ must be finite everywhere;

So, we choose, the regular sol.

At sufficiently large r;

$$U(r) << k^2$$

$\longrightarrow$ we can neglect $U(r)$ in comparison with k^2.

This is applicable to all potentials for which;

$$r^{1+\epsilon} V(r) \longrightarrow 0 \quad \text{as } r \longrightarrow \infty \qquad (\epsilon > 0)$$

(The coulomb potential is excluded)

Thus;

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 \right) f_l(r) = 0$$

Two indep. real sols. of this equ. are;

$$S_l(kr) = kr\, j_l(kr) \qquad j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^l \left(\frac{\sin\rho}{\rho} \right)$$

$$C_l(kr) = -kr\, n_l(kr) \qquad n_l(\rho) = -(-\rho)^l \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^l \left(\frac{\cos\rho}{\rho} \right)$$

where

$$\begin{cases} S_\ell(x) \underset{x\to 0}{\sim} \dfrac{x^{\ell+1}}{(2\ell+1)!!} \\[2ex] C_\ell(x) \underset{x\to 0}{\sim} \dfrac{(2\ell-1)!!}{x^\ell} \end{cases} \qquad \begin{array}{l} S_\ell(x) \underset{x\to\infty}{\sim} \sin(x-\frac{\ell\pi}{2}) \\[2ex] C_\ell(x) \underset{x\to\infty}{\sim} \cos(x-\frac{\ell\pi}{2}) \end{array}$$

The funcs. $S_\ell(x)$ and $C_\ell(x)$ can be expressed in terms of polynomials in $(\frac{1}{x})$ multiply $\cos x$ and $\sin x$;

For $\ell \leq 2$

$$\begin{cases} S_0(x) = \sin x \\ S_1(x) = \dfrac{\sin x}{x} - \cos x \\ S_2(x) = (\frac{3}{x^2}-1)\sin x - \frac{3}{x}\cos x \end{cases} \qquad \begin{cases} C_0(x) = \cos x \\ C_1(x) = \dfrac{\cos x}{x} + \sin x \\ C_2(x) = (\frac{3}{x^2}-1) + \frac{3}{x}\sin x \end{cases}$$

Sometimes it is convenient to use $e_\ell^{\pm}(x)$ as;

$$e_\ell^{\pm}(x) = C_\ell(x) \pm i\, S_\ell(x)$$

For large x; $e_\ell^{\pm} \sim e^{\pm i(x-\frac{\ell\pi}{2})}$

For large r;

$$f_\ell(r) \sim f_\ell^{A}(r) \quad \leftarrow \text{asymptotic}$$

where

1- $f_\ell^A(r) = A_\ell \left[S_\ell(kr) + K_\ell C_\ell(kr) \right]$

in the case that;

$$U(r) \ll K^2 - \frac{\ell(\ell+1)}{r^2}$$

2- $f_\ell^A(r) \underset{r \to \infty}{\sim} A_\ell \left[\sin(kr - \frac{\ell\pi}{2}) + K_\ell \cos(kr - \frac{\ell\pi}{2}) \right]$

in the case that; $\frac{\ell(\ell+1)}{r^2} \ll K^2$

where A_ℓ and K_ℓ are consts. -

A_ℓ: multiplicative normalization consts.

K_ℓ: determines the scattering amplitude.

Connection between the scattering amplitude and k_ℓ -

Consider the asymptotic part of $\psi(\bar{r})$, which is $\overset{A}{\psi}(\bar{r})$, as

$$\psi(r) \underset{r \to \infty}{\sim} \overset{A}{\psi}(r)$$

$$\overset{A}{\psi}(r) = \varphi_k(r) + f_k(\theta) \frac{e^{ikr}}{r}$$

The <u>first</u> term can be expanded as

$$\varphi_k(r) = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell \underbrace{\frac{1}{kr} S_\ell(kr)}_{j_\ell(kr)} P_\ell(\cos\theta)$$

<u>For large r:</u>

$$\varphi_k(r) \sim \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell \frac{1}{kr} \sin(kr - \frac{\ell\pi}{2}) P_\ell(\cos\theta)$$

The <u>second</u> term (outgoing scattered wave) can be expanded as;

$$f_k(\theta) \frac{e^{ikr}}{r} = \left[\sum_{\ell=0}^{\infty} (2\ell+1) \frac{1}{k} T_\ell \ P_\ell(\cos\theta) \right] \frac{e^{ikr}}{r}$$

T_ℓ coeffs. are called partial wave scattering amplitudes;

Then;

Since; $\psi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{1}{r} f_\ell(r) \, P_\ell(\cos\theta)$

$$\longrightarrow \begin{cases} \overset{A}{\psi}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{1}{r} \overset{A}{f_\ell} \ P_\ell(\cos\theta) \\ \overset{A}{\psi}(r,\theta) = \varphi_k(r) + f_k(\theta) \frac{e^{ikr}}{r} \end{cases}$$

$$\psi^A(r,\theta) \underset{r\to\infty}{\sim} \sum_{l=0}^{\infty} \left[\frac{(2l+1)\, i^l}{kr} \sin\left(kr - \frac{l\pi}{2}\right) + \frac{(2l+1)}{kr} T_l e^{ikr} \right] P_l(\cos\theta)$$

$$\rightarrow f_l^A(r) \underset{r\to\infty}{\sim} (2l+1)\left[\frac{i^l}{k} \sin\left(kr - \frac{l\pi}{2}\right) + \frac{T_l e^{ikr}}{k} \right]$$

writing:

$$e^{ikr} = e^{i\frac{l\pi}{2}} e^{i(kr - \frac{l\pi}{2})} = e^{i\frac{l\pi}{2}} \left\{ \cos\left(kr - \frac{l\pi}{2}\right) + i \sin\left(kr - \frac{l\pi}{2}\right) \right\}$$

$$f_l^A(r) \underset{r\to\infty}{\sim} (2l+1) \left\{ \left(\frac{i^l}{k} + T_l \frac{i e^{i\frac{l\pi}{2}}}{k} \right) \sin\left(kr - \frac{l\pi}{2}\right) + T_l \frac{e^{i\frac{l\pi}{2}}}{k} \cos\left(kr - \frac{l\pi}{2}\right) \right\}$$

Comparing with

$$f_l^A(r) \underset{r\to\infty}{=} A_l \left[\sin\left(kr - \frac{l\pi}{2}\right) + k_l \cos\left(kr - \frac{l\pi}{2}\right) \right]$$

$$\rightarrow \begin{cases} A_l = (2l+1)\left(\frac{i^l}{k} + T_l \frac{i e^{i\frac{l\pi}{2}}}{k} \right) \\ A_l k_l = (2l+1)\, T_l \frac{e^{i\frac{l\pi}{2}}}{k} \end{cases}$$

$$\longrightarrow K_l = \frac{T_l\, e^{i\frac{l\pi}{2}}}{i^l + T_l\, i\, e^{i\frac{l\pi}{2}}}$$

$$i = e^{i\frac{\pi}{2}} \qquad i^l = e^{i\frac{l\pi}{2}}$$

$$\longrightarrow K_l = \frac{T_l}{1 + i T_l}$$

$f_\ell^A(r)$ in terms of incoming and outgoing radial waves:

We obtained:

$$f_\ell^A(r) \underset{r\to\infty}{\sim} \frac{(2\ell+1)}{k}\left[i^\ell \sin\left(kr-\frac{\ell\pi}{2}\right) + T_\ell e^{ikr}\right]$$

$$\rightarrow f_\ell^A(r) \underset{r\to\infty}{\sim} \frac{(2\ell+1)}{k}\left[i^\ell \sin\left(kr-\frac{\ell\pi}{2}\right) + e^{i\frac{\ell\pi}{2}} T_\ell e^{i(kr-\frac{\ell\pi}{2})}\right]$$

$$\sim \frac{(2\ell+1)i^\ell}{k}\left[\sin\left(kr-\frac{\ell\pi}{2}\right) + T_\ell e^{i(kr-\frac{\ell\pi}{2})}\right]$$

$$\sim \frac{(2\ell+1)i^\ell}{k}\left[\frac{1}{2i}\left[e^{i(kr-\frac{\ell\pi}{2})} - e^{-i(kr-\frac{\ell\pi}{2})}\right] + T_\ell e^{i(kr-\frac{\ell\pi}{2})}\right]$$

$$\sim \frac{(2\ell+1)i^\ell}{2ik}\left[-e^{-i(kr-\frac{\ell\pi}{2})} + (1+2iT_\ell)e^{i(kr-\frac{\ell\pi}{2})}\right]$$

$$f_\ell^A(r) \underset{r\to\infty}{\sim} \frac{-(2\ell+1)i^\ell}{2ik}\left[e^{-i(kr-\frac{\ell\pi}{2})} - S_\ell(k)e^{i(kr-\frac{\ell\pi}{2})}\right]$$

where we took $S_\ell(k) = 2iT_\ell + 1$

Therefore:

$$\psi^A(r,\theta) \sim \sum_\ell \frac{1}{r} f_\ell^A(r) P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{-(2\ell+1)i^\ell}{2ikr}\left[e^{-i(kr-\frac{\ell\pi}{2})} - S_\ell(k)e^{i(kr-\frac{\ell\pi}{2})}\right] P_\ell(\cos\theta)$$

The conservation law requires that;

outgoing radial flux $\leqslant$ incoming radial flux

$$\longrightarrow \quad |S_\ell(k)| \leqslant 1$$

For real potential only elastic scattering is possible.

and;

outgoing radial flux = incomming radial flux

$$|S_\ell(k)| = 1$$

it follows: $\longrightarrow \quad S_\ell(k) = e^{2i\delta_\ell}$

where δ_ℓ: real const. (Phase shift of order ℓ)

Then, the partial wave scattering amplitude T_ℓ, is;

$$T_\ell = \frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right) = e^{i\delta_\ell}\, S\cdot\delta_\ell$$

$$\longrightarrow \quad K_\ell = \frac{T_\ell}{1+iT_\ell} = \frac{e^{i\delta_\ell}\, S\cdot\delta_\ell}{1+ie^{i\delta_\ell}\, S\delta_\ell} = \tan\delta_\ell$$

Remark:

$$K_\ell = \frac{S\delta_\ell(C\delta_\ell + iS\delta_\ell)}{1+iC\delta_\ell\, S\delta_\ell - S^2\delta_\ell} = \frac{S\delta_\ell(C\delta_\ell + iS\delta_\ell)}{C\delta_\ell(C\delta_\ell + iS\delta_\ell)} = \tan\delta_\ell$$

S_ℓ: is known as S-matrix, or scattering matrix element.

K_ℓ: reaction matrix element

$$f_k(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{1}{k} T_\ell P_\ell(\cos\theta)$$

$$f_k(\theta) = \sum_{\ell=0}^{\infty} \frac{1}{2ik} (e^{2i\delta_\ell} - 1) (2\ell+1) P_\ell(\cos\theta)$$

The total Cross-Section:

$$\sigma_{tot} = \sum_{\ell=0}^{\infty} \sigma_\ell \qquad \left(\sigma_{tot} = \int_{4\pi} \left(\frac{d\sigma}{d\Omega}\right) d\Omega \right)$$

where, $\sigma_\ell = \frac{4\pi}{k^2} (2\ell+1) \sin^2 \delta_\ell$

because; $$\sigma_{tot} = \sum_{\ell=0}^{\infty} \int d\Omega \left(\frac{1}{2ik}\right)\left(\frac{1}{-2ik}\right) (e^{2i\delta_{\ell'}} - 1)(e^{-2i\delta_\ell} - 1)(2\ell'+1)(2\ell+1) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$\sigma_{tot} = \sum_{\ell=0}^{\infty} \frac{1}{4k^2} (2\ell+1)^2 \frac{2(2\pi)}{2\ell+1} \left[2(1 - \cos(2\delta_\ell)) \right]$$

$$\sigma_{tot} = \sum_{\ell=0}^{\infty} \frac{4\pi}{k^2} (2\ell+1) \sin^2 \delta_\ell$$

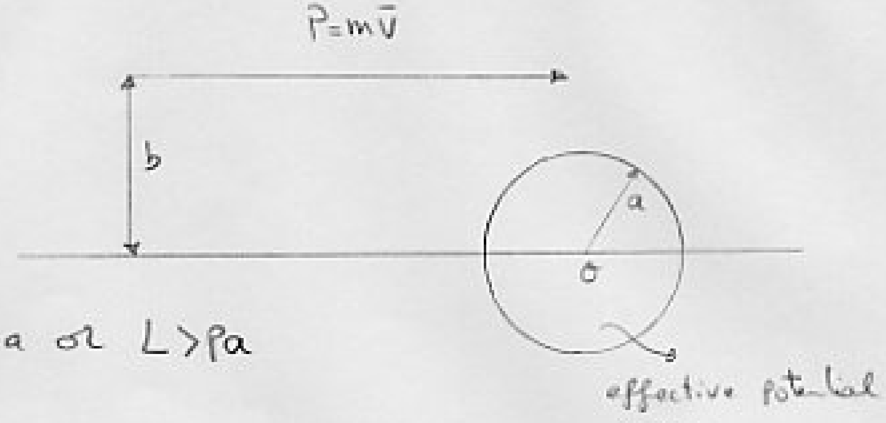
σ_ℓ: Partial Cross-Section

Since $\sin \delta_\ell \leqslant 1 \longrightarrow \sigma_\ell \leqslant \frac{4\pi(2\ell+1)}{k^2}$

This method is useful, if the expasion converges.

Nonrigorous discussion:

$L = bP$



The particles with $b > a$ or $L > Pa$ will not be scattered

writing $L = \sqrt{\ell(\ell+1)}\,\hbar \approx \ell\hbar$ and $P = \hbar k$

$$\ell\hbar > \hbar k a \longrightarrow \ell > ka \quad (\text{small scattering})$$

When the scattering in a given partial wave (ℓ) is small, the corresponding phase shift is small, and only finite number of terms in the partial wave expansion of the scattering amplitude will be important. The number increases with energy.

Unitary and Phase Shifts

1) We now examine the consequences of probability conservation or unitary.

The flux current (J) must satisfy

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \quad \text{or} \quad \nabla \cdot J = -\frac{\partial |\psi|^2}{\partial t} = 0 \quad \left(\text{in a time-indep formulation}\right)$$

$$\longrightarrow \int_V \nabla \cdot J \, d x^3 = \int_{S \atop \text{spherical surface}} J \cdot ds = 0 \qquad \text{Gausses Theorem}$$

$\longrightarrow$ outgoing flux = incoming flux

$$|J_{out}| = |J_{in}|$$

2) Angular momentum conservation ⟶ The above relation must hold for each partial wave separately. (each l).

⟶ The coeff. of incomming and outgoing waves (for each l) must be the same in magnitude.

Define; $S_\ell(k) = 1 + 2ik\, f_\ell(k)$

$$|S_\ell(k)| = 1 \qquad \text{unitary Relation}$$

Take: $S_\ell = e^{2i\delta_\ell}$

$\left\{\begin{array}{l} \delta_\ell\text{: phase of outgoing wave} \\ \text{coeff } 2 \text{ in } 2\delta_\ell\text{: convention} \\ \delta_\ell\text{: Real} \\ \delta_\ell = \delta_\ell(k) \end{array}\right.$

$$\to f_\ell(k) = \frac{S_\ell - 1}{2ik} = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k}$$

$$f_\ell(k) = \frac{1}{k\left(e^{-i\delta_\ell}/\sin\delta_\ell\right)} = \frac{1}{k\left(\cos\delta_\ell - i\sin\delta_\ell\right)/\sin\delta_\ell}$$

$$f_\ell(k) = \frac{1}{k\,\mathrm{cotg}\,\delta_\ell - ik}$$

Thus: $f(k',k) = f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(k)\, P_\ell(\cos\theta)$

$$= \sum_{\ell=0}^{\infty} (2\ell+1) \left(\frac{e^{2i\delta_\ell} - 1}{2ik} \right) P_\ell(\cos\theta)$$

$$f(k',k) = f(\theta) = \frac{1}{k} \sum_{k=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\cos\theta)$$

This expression for $f(\theta)$ rests on the twin principles of:

1- Rotational invariance. ($V(r)$)

2- Probability conservation.

Differential Cross-Section:

$$\frac{d\sigma}{d\Omega} = |f(k',k)|^2 \quad \rightarrow \quad \sigma_{tot} = \int d\Omega \, |f(k',k)|^2$$

$$\sigma_{tot} = \int |f(\theta)|^2 d\Omega$$

$$= \frac{1}{k^2}\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \sum_\ell \sum_{\ell'} (2\ell+1)(2\ell'+1) e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$= \frac{2\pi}{k^2} \frac{2\delta_{\ell\ell'}}{2\ell+1} \sum_\ell \sum_{\ell'} (2\ell+1)(2\ell'+1) e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'}$$

$$= \frac{4\pi}{K^2} \sum_\ell (2\ell+1) \sin^2\delta_\ell$$

Now, - A check!

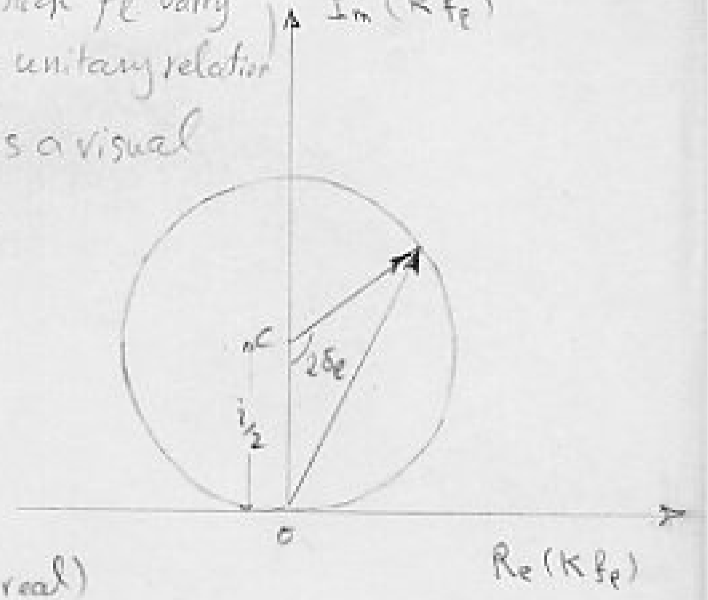
$$\text{Im}\, f(\theta=0) = \text{Im}\left(\frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell P_\ell(1)\right)$$

$$= \sum_\ell \frac{(2\ell+1)}{k} \sin^2\delta_\ell$$

$$\rightarrow \text{Im}\, f(\theta=0) = \frac{k\sigma_{tot}}{4\pi}$$

which satisfies the optical theorem.

Argand Diagram (The manner in which fℓ varry is given by the unitary relation. This diagram gives a visual immage of this.



$$f_\ell = \frac{e^{2i\delta_\ell}-1}{2ik}$$

$$K f_\ell = \frac{i}{2} + \frac{1}{2} e^{-(i\frac{\pi}{2})+2i\delta_\ell}$$

1- If δ_ℓ is small $\rightarrow \sin\delta_\ell \simeq \delta_\ell$

$$\longrightarrow \underset{\delta_\ell \to 0}{f_\ell} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{(1+i\delta_\ell)\delta_\ell}{k} \simeq \frac{\delta_\ell}{k}$$

(almost real)

It can be positive or negative.

2- If $\delta_\ell \simeq \frac{\pi}{2}$

$$\underset{\delta_\ell \to \frac{\pi}{2}}{f_\ell} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} \simeq \frac{i}{k}$$

(almost imaginary)

In this case the magnitude of $k f_\ell$ is <u>maximal</u>.

Under such a cond. the ℓ^{th} partial wave may be in <u>resonance</u>.

Note that the maximum partial cross-section:

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2\delta_\ell$$

$$\sigma_{tot} = \sum_\ell \frac{4\pi}{k^2} (2\ell+1) \sin^2\delta_\ell = \underset{\delta_\ell = \frac{\pi}{2}}{\sum_\ell} \frac{4\pi}{k^2} (2\ell+1)$$

$$\sigma_{tot_{max}} = \sum_\ell 4\pi \lambda^2 (2\ell+1)$$

$$\sigma_{\ell_{max}} = 4\pi\lambda^2(2\ell+1)$$

Connection with the Eikonal Approx.

Eikonal approx. is valid in high energies:

$\lambda << R$ range (high energy)

In classical arguments:

$$\begin{cases} L \simeq l\hbar \\ L = bP = b(\hbar k) \end{cases} \rightarrow l = bk$$

$$l_{max} = KR$$

Then we may make the following substitution in:

$$f(\theta) = \sum_{l=0} (2l+1)\left(\frac{e^{2i\delta_l}-1}{2ik}\right) P_l(\cos\theta)$$

$$\begin{cases} \sum\limits_{l=0}^{l_{max}} \rightarrow k\int db \quad \text{and} \\ P_l(\cos\theta) \underset{\substack{\downarrow \\ \text{large } l \\ \text{small } \theta}}{\approx} J_0(l\theta) = J_0(kb\theta) \quad (\text{high energies}) \end{cases}$$

where $l_{max} = KR$ implies

$$e^{2i\delta_l} - 1 = e^{2i\delta(b)} - 1 = 0 \quad \text{for } l > l_{max}$$

$$f(\theta) = \rightarrow k\int db \, \frac{2kb}{2ik}\left(e^{2i\delta(b)} - 1\right) J_0(kb\theta)$$

$$= -ik\int db \, b \, J_0(kb\theta)\left[e^{2i\delta(b)} - 1\right]$$

where $(2l+1) \simeq 2l = 2kb$ for large l

Determination of Phase Shift

Assume $\begin{cases} V \neq 0 & r \leq R \\ V = 0 & r > R \end{cases}$

Outside $r > R$ ⟶ The wave func.: Free spherical wave

Now, since $(r > R)$; there is no reason to exclude $\eta_\ell(kr)$ because <u>origin</u> is excluded.

$$\longrightarrow \psi^+ \text{ contains: } \begin{cases} J_\ell(kr) \\ \eta_\ell(kr) \end{cases}$$

$$\longrightarrow \psi^+ \text{ is a linear combination } \begin{cases} J_\ell(kr)\ P_\ell(\cos\theta) \\ \eta_\ell(kr)\ P_\ell(\cos\theta) \end{cases}$$

or equivalently;

$$\psi^+ \text{ is a linear combination } \begin{cases} h_\ell^{(1)}(kr)\ P_\ell(\cos\theta) \\ h_\ell^{(2)}(kr)\ P_\ell(\cos\theta) \end{cases}$$

where

$$\begin{cases} h_\ell^{(1)} = J_\ell + i\eta_\ell \\ h_\ell^{(2)} = J_\ell - i\eta_\ell \end{cases}$$

Asymplotic form:

$$\left\{\begin{array}{l} h_\ell^{(1)} \xrightarrow[r\to\infty]{} \dfrac{e^{i(kr-\frac{\ell\pi}{2})}}{ikr} \\ \\ h_\ell^{(2)} \xrightarrow[r\to\infty]{} \dfrac{-e^{-i(kr-\ell\frac{\pi}{2})}}{ikr} \end{array}\right.$$

The full wave func. at any r can then be written as:

$$\langle x|\psi^+\rangle = \frac{1}{(2\pi)^{3/2}} \sum (i)^\ell (2\ell+1) A_\ell(r)\; P_\ell(\cos\theta)$$

where $A_\ell = C_\ell^{(1)} h_\ell^{(1)}(kr) + C_\ell^{(2)} h_\ell^{(2)}(kr)$

The C_ℓ coeffs. are chosen in such a way;

when $V=0 \longrightarrow \quad A_\ell(kr) \longrightarrow j_\ell(kr)$ everywhere

Then for $r\to\infty$

$$\langle x|\psi^+\rangle \sim \frac{1}{(2\pi)^{3/2}} \sum_\ell (i)^\ell (2\ell+1) \left[C_\ell^{(1)} \frac{e^{i(kr-\frac{\ell\pi}{2})}}{ikr} - C_\ell^{(2)} \frac{e^{-i(kr-\frac{\ell\pi}{2})}}{ikr} \right] P_\ell(\cos\theta) \quad (1)$$

Compare this result with previous result:

$$\langle x|\psi^+\rangle \sim \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1) \frac{1}{2} \left[\underset{\substack{\downarrow \\ e^{2i\delta_\ell}}}{S_\ell} \frac{e^{ikr}}{ikr} - \frac{e^{-i(kr-\ell\pi)}}{ikr} \right] P_\ell(\cos\theta) \quad (2)$$

Noting that $(i)^\ell = e^{i\ell\frac{\pi}{2}}$

equ. (1) becomes:

$$\langle x|\psi^+\rangle = \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1) \left[C_\ell^{(1)} \frac{e^{ikr}}{ikr} - C_\ell^{(2)} \frac{e^{-i(kr-\ell\pi)}}{ikr} \right] P_\ell(\cos\theta)$$

$$\longrightarrow \quad C_\ell^{(1)} = \frac{1}{2} e^{2i\delta_\ell} \qquad C_\ell^{(2)} = \frac{1}{2}$$

The radial func. can be written as:

$$A_\ell(kr) = C_\ell^{(1)} h_\ell^{(1)}(kr) + C_\ell^{(2)} h_\ell^{(2)}(kr)$$

$$= \frac{1}{2} e^{2i\delta_\ell} \left(j_\ell(kr) + i\eta_\ell(kr) \right) + \frac{1}{2} \left(j_\ell(kr) - i\eta_\ell(kr) \right)$$

$$= \frac{1}{2} \left[j_\ell(kr) \left(e^{2i\delta_\ell} + 1 \right) + i\eta_\ell(kr) \left(e^{2i\delta_\ell} - 1 \right) \right]$$

but $e^{2i\delta_\ell} = \cos 2\delta_\ell + i \sin 2\delta_\ell$

$$\cos^2 \delta_\ell = \frac{1 + \cos 2\delta_\ell}{2} \longrightarrow \cos 2\delta_\ell = 2\cos^2 \delta_\ell - 1$$

$$\sin 2\delta_\ell = 2 \sin \delta_\ell \cos \delta_\ell$$

$$e^{2i\delta_\ell} = 2\cos^2 \delta_\ell - 1 + 2i \cos \delta_\ell \sin \delta_\ell$$

$$e^{2i\delta_\ell} - 1 = 2\left(\cos^2 \delta_\ell - 1\right) + 2i \cos \delta_\ell \sin \delta_\ell = -2 \sin^2 \delta_\ell + 2i \cos \delta_\ell \sin \delta_\ell$$

$$= 2i \sin \delta_\ell \left(\cos \delta_\ell + i \sin \delta_\ell \right) = 2i \sin \delta_\ell \, e^{i\delta_\ell}$$

Similarly:

$$e^{2i\delta_\ell} + 1 = 2\cos^2 \delta_\ell + 2i \cos \delta_\ell \sin \delta_\ell = 2 \cos \delta_\ell \left(\cos \delta_\ell + i \sin \delta_\ell \right)$$

$$= 2 \cos \delta_\ell \, e^{i\delta_\ell}$$

$$\longrightarrow A_\ell(kr) = e^{i\delta_\ell} \left[\cos \delta_\ell \, j_\ell(kr) - \sin \delta_\ell \, \eta_\ell(kr) \right] \qquad (r > R)$$

Using this relation we can evaluate logarithmic derivative at $r = R$ (with respect to kr)

$$B_\ell \equiv \left(\frac{r}{A_\ell}\frac{dA_\ell}{dr}\right)_{r=R} = k\left(\frac{r}{A_\ell}\,\underline{\frac{dA_\ell}{d(kr)}}\right)_{r=R}$$

$$= kR\left[\frac{j_\ell'(kR)\cos\delta_\ell - n_\ell'(kR)\sin\delta_\ell}{j_\ell(kR)\cos\delta_\ell - n_\ell(kR)\sin\delta_\ell}\right]$$

remark: $B_\ell'' \equiv \frac{\frac{d(rA_\ell)}{dr}}{rA_\ell} = \frac{\frac{dA_\ell}{dr}}{A_\ell} + \frac{1}{r}$, $B' = rB_\ell'' = \frac{r}{A_\ell}\frac{dA_\ell}{dr} + \underbrace{1}_{\text{const}}$ $\quad B_\ell = \frac{r}{A_\ell}\frac{dA_\ell}{dr}$

$$\rightarrow kR\left(j_\ell'(kR)\cos\delta_\ell - n_\ell'(kR)\sin\delta_\ell\right) = B_\ell\left(j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell\right)$$

$$\cos\delta_\ell\left(kRj_\ell'(kR) - B_\ell j_\ell(kR)\right) = \sin\delta_\ell\left(kRn_\ell'(kR) - B_\ell n_\ell(kR)\right)$$

$$\rightarrow \tan\delta_\ell = \frac{kRj_\ell'(kR) - B_\ell j_\ell(kR)}{kRn_\ell'(kR) - B_\ell n_\ell(kR)}$$

Now, B_ℓ should be determined.

Look the sols. of the Schrödinger equ. for $r<R$ (inside the range of the potential).

For <u>spherically</u> symmetric potential:

$$\frac{d^2u_\ell}{dr^2} + \left(k^2 - \frac{2m}{\hbar^2}V - \frac{\ell(\ell+1)}{r^2}\right)u_\ell = 0$$

Where $u_\ell(r) = rA_\ell(r)$ $\qquad$ $A_\ell(r)$: sol. for $r<R$

Remark: $\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$

if $U_{n\ell m}(r) = rR_{n\ell m}(r)$

$$\rightarrow \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\frac{U(r)}{r} = \frac{1}{r}\frac{d^2}{dr^2}U(r)$$

Subject to boundary cond.;

$$u_\ell(r)\Big|_{r=0} = 0$$

We integrate this equ. (if necessary numerically) up to $\underline{r=R}$ starting at $\underline{r=0}$

→ In this way we obtain logarithmic derivative at R.

By continuity;

$$B_{\ell\,\text{inside}} = B_{\ell\,\text{outside}}$$

$$\left(\begin{array}{l}\text{left hand side obtained by integrating}\\ \text{the schrödinger equ. up to } R\end{array}\right) = \left(\begin{array}{l}\text{right hand side, in terms of phase}\\ \text{shifts, that characterize the large}\\ \text{distance behavior of the wave-func.}\end{array}\right)$$

Alternatively, it is possible to derive an integral equ. for $A_\ell(r) \xrightarrow{\text{from which}}$ phase shifts can be obtained.

Remark;
$$\left\{\begin{array}{l}\psi_{in} = \psi_{out}\\ \dfrac{d\psi_{in}}{dx} = \dfrac{d\psi_{out}}{dx}\end{array}\right. \xrightarrow{\text{equivalent}} \left(\frac{1}{\psi}\frac{d\psi}{dx}\right)_{in} = \left(\frac{1}{\psi}\frac{d\psi}{dx}\right)_{out}$$

at r=R

Hard-sphere scattering:

$$V = \begin{cases} \infty & \text{for } \quad r<R \\ 0 & \text{"} \quad r>R \end{cases}$$

It is possible to find δ_ℓ without knowing B_ℓ.

The wave-func. must vanish at $r=R$ (the sphere is impenetrable)

$$A_\ell(kr) = C_\ell^{(1)} h_\ell^{(1)}(kr) + C_\ell^{(2)} h_\ell^{(2)}(kr) = 0 \Big|_{r=R}$$

We obtained before;

$$A_\ell(kr) = e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, \eta_\ell(kr)\right] \qquad \text{for } r>R$$

$$\longrightarrow \left[\cos\delta_\ell\, j_\ell(kR) - \sin\delta_\ell\, \eta_\ell(kR)\right] = 0 \qquad \text{at } r=R$$

$$\rightarrow \tan\delta_\ell = \frac{j_\ell(kR)}{\eta_\ell(kR)} \qquad (\text{without any approx.})$$

Now; if

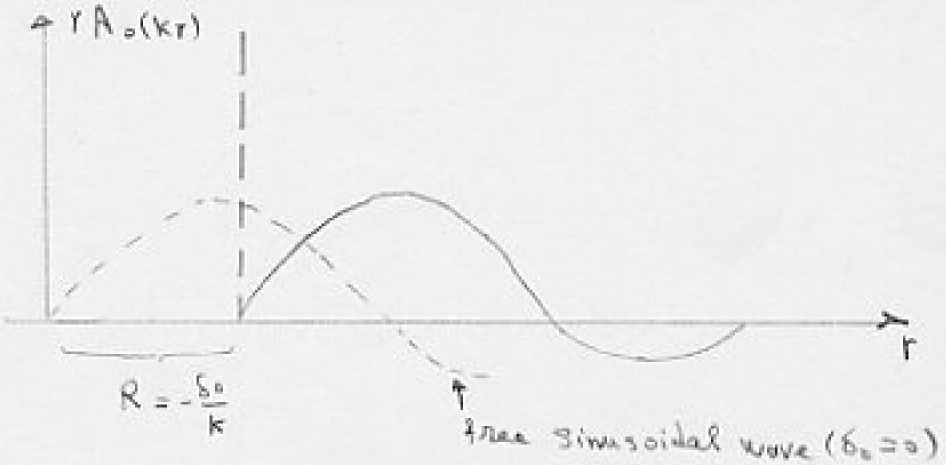
1- $\ell=0$

$$j_0(kr) = \frac{\sin(kr)}{kr} \qquad \eta_0(kr) = -\frac{\cos(kr)}{kr}$$

The radial func. with $e^{i\delta_0}$ omitted; varies as;

$$A_0(kr) \sim \frac{1}{kr}\left(\sin(kr)\cos\delta_0 + \cos(kr)\sin\delta_0\right) = \frac{1}{kr}\sin(kr+\delta_0)$$

Also $$\tan\delta_0 = \frac{\frac{\sin(kR)}{kR}}{-\frac{\cos(kR)}{kR}} = -\tan(kR) \qquad \rightarrow \delta_0 = -kR$$



2- Low energy limit: $KR \ll 1$

for $kr \ll 1$
$$\begin{cases} j_\ell(kr) \approx \dfrac{(kr)^\ell}{(2\ell+1)!!} \\ \eta_\ell(kr) \approx \dfrac{(2\ell-1)!!}{(kr)^{\ell+1}} \end{cases}$$

$$\rightarrow \tan\delta_\ell = \frac{-(kR)^{2\ell+1}}{(2\ell+1)[(2\ell-1)!!]^2}$$

So $\ell \neq 0$ terms are unimportant in low energy scattering -

→ We have S-wave scattering only for any finite-range potential at low energy.

Since $f(\theta) = \frac{1}{k}\sum_{\ell=0}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$

Considering $\ell = 0$ (only),

$$f(\theta) = \frac{\sin\delta_0 \, e^{i\delta_0}}{k}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \simeq \frac{\sin^2\delta_0}{k^2} \quad \text{for } KR \ll 1$$

$$\simeq \frac{\delta_0^2}{k^2} = \frac{k^2R^2}{k^2} = R^2$$

($\delta_0 = -KR$ regardless of whether K is small or large)

$$\sigma_{Tot} = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi R^2$$

This is 4-times the geometric cross-section πR^2

3 - High Energy Scattering:

$$\sigma_{Tot} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\ell=kR} (2\ell+1) \sin^2\delta_\ell \qquad L = r \times p \to \ell = kR$$

But we obtained: $\tan\delta_\ell = \frac{j_\ell(kR)}{\eta_\ell(kR)}$

$$\to \sin^2\delta_\ell = \frac{\tan^2\delta_\ell}{1+\tan^2\delta_\ell} = \frac{[j_\ell(kR)]^2}{[j_\ell(kR)]^2 + [\eta_\ell(kR)]^2}$$

$$\to \sin^2\delta_\ell \simeq \sin^2\left(kR - \frac{\pi\ell}{2}\right) \qquad (1)$$

Where we have used:

$$\begin{cases} j_\ell(kr) \sim \frac{1}{kr} \sin\left(kr - \frac{\ell\pi}{2}\right) \\ \eta_\ell(kr) \sim \frac{1}{kr} \cos\left(kr - \frac{\ell\pi}{2}\right) \end{cases} \qquad (\text{large } kr)$$

Now, consider:

$$\sin^2\delta_\ell + \sin^2\delta_{\ell+1} = \sin^2\delta_\ell + \sin^2\left(\underbrace{kR - \frac{\pi\ell}{2}} - \frac{\pi}{2}\right) \qquad (\text{using } (1))$$

$$= \sin^2\delta_\ell + \sin^2\left(\delta_\ell - \frac{\pi}{2}\right) = \sin^2\delta_\ell + \cos^2\delta_\ell = 1$$

$$\to \underset{\text{average}}{\overline{\left(\sin^2\delta_\ell + \sin^2\delta_{\ell+1}\right)}} = \frac{1}{2}$$

$$\overline{\left(\sum \sin^2\delta_\ell\right)} = \frac{1}{2}$$

The number of terms is roughly kR $\left(\sum\limits_{\ell=0}^{kR}\right)$,

Also;

$$\overline{(2\ell+1)} \simeq \overline{(2\ell)} = 2\left(\frac{kR}{2}\right) = kR$$

$$\rightarrow \sigma_{tot} = \frac{4\pi}{k^2}\sum_{\ell=0}^{kR}(2\ell+1)\sin^2\delta_\ell \simeq \frac{4\pi}{k^2}(kR)(kR)\left(\frac{1}{2}\right)$$

$$\sigma_{tot} = 2\pi R^2 \qquad \rightarrow \sigma_{tot} > \pi R^2 \text{ (Geometric Cross-Sec.)}$$

We expected in the $kR \gg 1$ limit (i.e. $\lambda = \frac{2\pi}{k} \ll$ size of the target) to get classical $\sigma = \pi R^2$, but this is not the case because the hard sphere has a sudden discontinuity at $r=R$ (so it can not be treated classically)

Classification;

1- Cross-Section in classical limit : πR^2

2- " " " long wave length : $4\pi R^2$

3- " " " short " " : $2\pi R^2$

To see the origin of factor 2, we may write;

$$f(\theta) = \sum_{\ell=0}^{kR}(2\ell+1)\left(\frac{e^{2i\delta_\ell}-1}{2ik}\right)P_\ell(\cos\theta)$$

$$= \frac{1}{2ik}\sum_{\ell=0}^{kR}(2\ell+1)e^{2i\delta_\ell}P_\ell(\cos\theta) + \frac{i}{2k}\sum_{\ell=0}^{kR}(2\ell+1)P_\ell(\cos\theta)$$

$$= f_{reflection} + f_{shadow}$$

1- Since $\{P_l(\cos\theta)\}$, form an orthogonal polynomials, in evaluating $\int |f_{ref}|^2 d\Omega$, there is no interference, amongst contribution from different l.

$$\int |f_{ref}|^2 d\Omega = \frac{2\pi}{4k^2} \sum_{l=0}^{l_{max}} \int_{-1}^{+1} (2l+1)^2 \left[P_l(\cos\theta)\right]^2 d(\cos\theta)$$

$$= \frac{2\pi}{4k^2} \sum_{l=0}^{l_{max}} (2l+1)^2 \frac{2}{(2l+1)} = \frac{\pi}{k^2} \sum_{l=0}^{l_{max}} (2l+1)$$

$$= \frac{\pi}{k^2} \underset{\substack{\downarrow \\ \text{number} \\ \text{of terms}}}{(l_{max})} \underset{\substack{\downarrow \\ \text{average}}}{(l_{max})} = \pi \frac{l_{max}^2}{k^2} = \frac{\pi k^2 R^2}{k^2} = \pi R^2$$

2- $f_{shad.}$: $\begin{cases} \text{Purely imaginary} \\ \text{Max at } \theta = 0 \ (P_l(\cos 0) = 1) \end{cases}$

The contribution from various l-values all add up coherently - (with the same phase). pure imaginary and positive

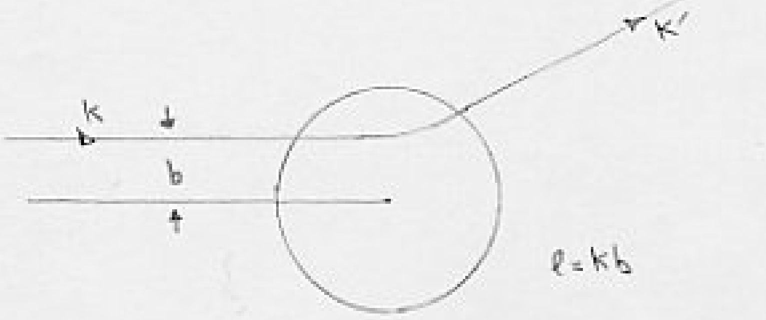
For small angle approx.:

$$P_l(\cos\theta) \sim \underset{\substack{\text{small } \theta \\ \text{large } l}}{J_0(l\theta)}$$

($\theta \simeq 0$ shaddow region behind the scatterer)
There is no contribution due to the first term near $\theta \simeq 0$ (see P150 up and down)

$$\sum_{l}^{l_{max}=kR} \longrightarrow k\int_0^R db \qquad 2l+1 \longrightarrow 2kb$$

$$f_{\text{shad.}} \simeq \frac{i}{2k} \sum (2l+1)\, j_0(l\theta) \simeq ik \int_0^R db\, j_0(kb\theta)\, b$$



$$f_{\text{shad}} = \frac{iR\, j_1(kR\theta)}{\theta} \qquad (1)$$

where $\begin{cases} j_0(\rho) = \frac{S\rho}{\rho} \\ j_1(\rho) = \frac{S\rho}{\rho^2} - \frac{C\rho}{\rho} \end{cases}$

(1) is Fraunhofer differaction in optics with a strong peaking near $\theta \approx 0$.

This has a strong peak at $\theta = 0$

let $\xi = kR\theta \rightarrow \frac{d\xi}{\xi} = \frac{d\theta}{\theta}$

$$\int d\Omega\, |f_{\text{shad}}|^2 = 2\pi \int_{-1}^{1} d(C\theta)\, \frac{R^2 |j_1(kR\theta)|^2}{\theta^2}$$

$$\int d\Omega\, |f_{\text{shad}}|^2 = 2\pi R^2 \int S\theta\, d\theta\, \frac{|j_1(\xi)|^2}{\theta^2}$$

$S\theta \approx \theta$

$$\int = 2\pi R^2 \int \frac{d\theta}{\theta} |j_1(\xi)|^2 = 2\pi R^2 \int_0^\infty \frac{|j_1(\xi)|^2}{\xi} d\xi$$

$j_1 \rightarrow 0$ as, $kR\theta \rightarrow \infty$

$$\int = \pi R^2$$

Remark:

$$f_{\text{shad}} \simeq ik \int_0^R db\, \frac{S(kb\theta)}{\cancel{kb\theta}}\, \cancel{b}$$

$$= -\frac{i}{\theta} \frac{1}{k\theta} C(kb\theta)\Big|_0^R$$

$$= \frac{i}{k\theta^2} \left[1 - C(kR\theta)\right]$$

Now $f_{\text{shad}} = \frac{iR\, j_1(kR\theta)}{\theta}$

$$= \frac{iR}{\theta} \left[\frac{S(kR\theta)}{(kR\theta)^2} - \frac{C(kR\theta)}{kR\theta} \right]$$

$$= \frac{iR}{\theta(kR\theta)} \left[1 - C(kR\theta)\right]$$

$\theta \rightarrow 0$.

Remark: The different procedure used in f_{shad} calculation is because we are trying to show the similarity between this and Franhofer diffraction in optics.

3- Interference term:

$$\mathrm{Re}\left(f^*_{\text{shad}}\, f_{\text{ref.}}\right) \simeq 0$$

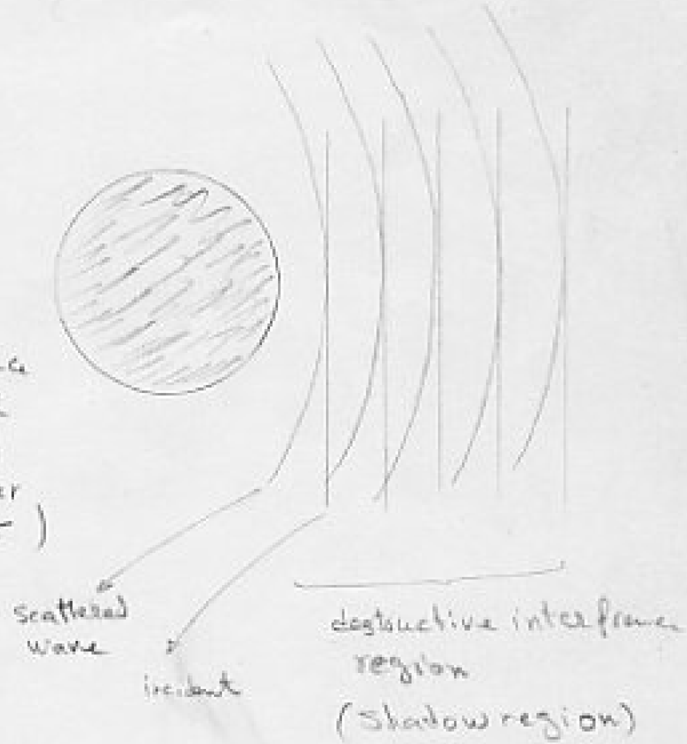
because the phase of f_{ref} oscillates $(2\delta_{\ell+1} = 2\delta_\ell - \pi)$ approximately averaging to <u>zero</u>, while f_{shad} is pure imaginary.

Then $\sigma_{\text{tot}} \simeq \pi R^2 + \pi R^2 = 2\pi R^2$

$\sigma_{\text{ref}} + \sigma_{\text{shad}}$

Behind the scatterer: Zero probability of finding the particle (shadow region)

Shadow is due to $\left\{ \right.$ destructive interference between the original wave and scattered wave. (i.e. e^{ikz} and $f(0)\frac{e^{ikr}}{r}$)



Since;

$$\psi^+(r) \underset{r\to\text{large}}{\sim} \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1) P_\ell(\cos\theta)\, \frac{1}{2ik}\left\{ (1+2ikf_\ell(k))\frac{e^{ikr}}{r} - \frac{e^{-i(kr-\ell\pi)}}{r} \right\}$$

$\rightarrow$ $f_\ell(k)$ must have positive imaginary part to get cancelation with (1) (destructive).

Acc. to optical theorem;

$$\frac{4\pi}{k}\,\mathrm{Im}\,f(\theta=0) = \frac{4\pi}{k}\left(\mathrm{Im}\,f_{shad}(\theta=0) + \mathrm{Im}\,f_{ref}(\theta=0)\right)$$

Since $\mathrm{Im}\,f_{ref}(\theta=0) \simeq 0$ due to oscillating phase

$$\left\{\begin{array}{l} \mathrm{Im}\,f_{ref}(\theta=0) = \\ \frac{1}{2k}\sum(2\ell+1)\underline{\cos 2\delta_\ell}\,P_\ell(1) \\ \mathrm{Im}\,f_{sh}(\theta=0) = \\ \frac{1}{2k}\sum(2\ell+1)\,P_\ell(1) \end{array}\right.$$

$$\longrightarrow \frac{4\pi}{k}\,\mathrm{Im}\,f(\theta=0) = \frac{4\pi}{k}\,\mathrm{Im}\,f_{shad}(\theta=0)$$

Using; $f(\theta) = f_{ref} + f_{shad} = \frac{1}{2ik}\sum_{\ell=0}^{kR}(2\ell+1)\,e^{2i\delta_\ell}\,P_\ell(\cos\theta) + \frac{i}{2k}\sum_{\ell=0}^{kR}(2\ell+1)\,P_\ell(\cos\theta)$

$$\frac{4\pi}{k}\,\mathrm{Im}\,f_{shad}(0) = \frac{4\pi}{k}\,\frac{1}{2k}\sum_{\ell=0}^{kR}(2\ell+1)\,P_\ell(1)$$

$$\simeq \frac{2\pi}{k^2}\,(kR)^2 = 2\pi R^2 \equiv \sigma_{tot}$$

Remark:

Since; $S\cdot\delta_\ell \simeq S\cdot(kR - \frac{\pi\ell}{2}) \rightarrow S\cdot 2\delta_\ell = S\cdot(2kR - \pi\ell)$

$\rightarrow S\cdot 2\delta_{\ell+1} = S\cdot(2kR - \pi(\ell+1)) = S\cdot(2\delta_\ell - \pi) = -S\cdot 2\delta_\ell \;\rightarrow (2\delta_{\ell+1} = 2\delta_\ell - \pi)$

$\longrightarrow S\,2\delta_{\ell+1} + S\,2\delta_\ell \simeq 0$

$$\left\{\begin{array}{l} \text{Also for this reason} \\ f_{ref}(\theta=0) \simeq 0 \quad (P_\ell(0)=1) \end{array}\right.$$

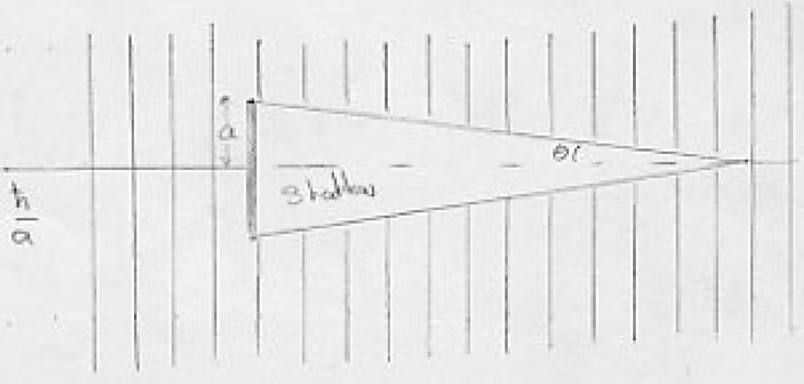
Also $C\,2\delta_{\ell+1} + C\,2\delta_\ell \simeq 0$

Remark:

Uncertainty in the lateral dir $\sim a$

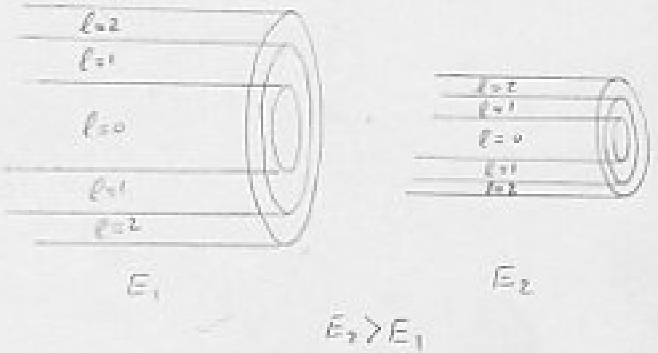
$\rightarrow$ Uncontrolled lateral mom. transfer $\sim P_\perp \sim \frac{\hbar}{a}$

$\theta \sim \frac{\hbar}{ap} \sim \frac{1}{ak}$



Low Energy Scattering

At low energies; $\begin{cases} \lambda\!\!\!^- = \frac{1}{k} \sim R \\ \text{or} \\ \lambda\!\!\!^- = \frac{1}{k} > R \end{cases}$



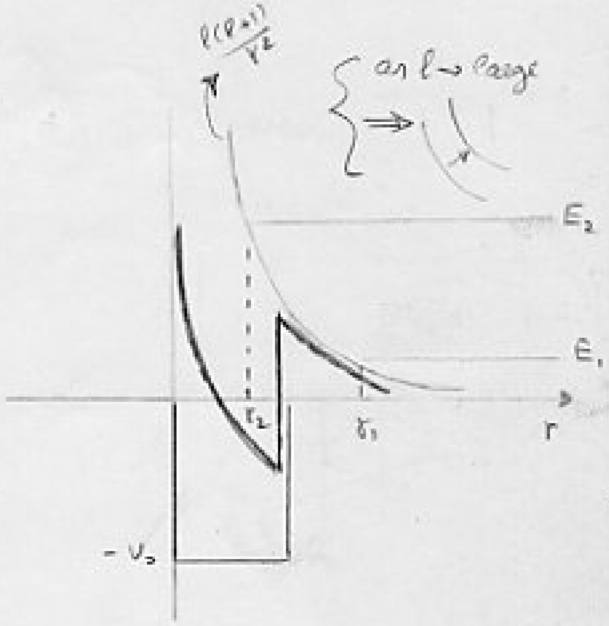
Partial waves for <u>higher ℓ</u> are in general <u>unimportant</u>.

Remark; $L = rP$,

Particles with $r > R$ or $L > RP$ will not be scattered;

$L = \sqrt{\ell(\ell+1)}\,\hbar \simeq \ell\hbar$ and $P = \hbar k$

$\ell\hbar > \hbar KR \longrightarrow \ell > KR$



$E_2 > E_1$
$r_2 < r_1$

→ Classically, because the particle can not penetrate the centrifugal barrier.

→ $V(r)$ has <u>no effect</u>.

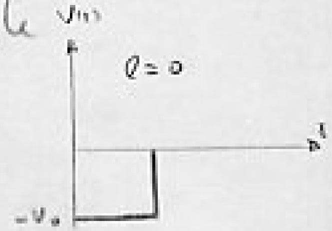
$V_{eff} = V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$

1- $V(r)$ is weak

The behavior of the radial wave-func. is largely determined by the <u>centrifugal</u> barrier term $(j_\ell(kr))$

2- V(r) is enough Strong:

Bound states for $\ell \neq 0$ near $E \simeq 0$ is available



It is possible to estimate the behavior of phase shift by:

(Prof. Gottfried)
Prob. 8

$$\frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = -\frac{2m}{\hbar^2}\int_0^\infty j_\ell(kr)\,V(r)\,A_\ell(r)\,r^2 dr$$

if $\begin{cases} 1- A_\ell(r) \text{ is not too different from } j_\ell(kr) \\ 2- \frac{1}{k} >> R \qquad (\text{small } k) \end{cases}$

$$\longrightarrow \int_0^\infty \cdots \sim k^{2\ell}$$

for small δ_ℓ : $\frac{e^{i\delta_\ell}\sin\delta_\ell}{k} \sim \frac{\delta_\ell}{k}$

$$\longrightarrow \delta_\ell \sim k^{2\ell+1}$$

For small k $\longrightarrow \delta_\ell \to 0$

This is known as threshold behavior.

Therefore:

At low energies with finite range potential,

S-wave scattering is important

i.e. $\delta_{\ell'} << \delta_\ell$ for $\ell' > \ell$, small k.

Rectangular Well or Barrier

Consider the S-wave scattering by;

$$V = \begin{cases} V_0 = \text{const.} & r<R \\ 0 & \text{otherwise} \end{cases} \qquad \begin{cases} V_0>0 & \text{repulsive} \\ V_0<0 & \text{attractive} \end{cases}$$

Many of the features we obtain here are common to more-complicated "finite-range" potentials.

We have already obtained the outside wave must behave like;

$$A_\ell(r) = e^{i\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right] \qquad r>R$$

$$A_0(r) = e^{i\delta_0}\left[\cos\delta_0\, j_0(kr) - \sin\delta_0\, n_0(kr)\right]$$

$$A_0(r) \sim e^{i\delta_0}\left[\cos\delta_0 \frac{\sin kr}{kr} - \sin\delta_0 \frac{\cos kr}{kr}\right] = \frac{e^{i\delta_0}}{kr}\sin(kr+\delta_0)$$

For hard-sphere;

1- $r>R$ $\qquad A_\ell(r)\Big|_{r=R} = 0 \longrightarrow \tan\delta_\ell = \dfrac{j_\ell(kR)}{n_\ell(kR)}$

$$\longrightarrow \delta_0 = -kR$$

2- Rectangular well; $\qquad$ <u>for</u> $E>0$

$r<R$;

$$u \equiv r\,A_{\ell=0}(r) \sim \sin k'r \qquad \left(k'^2 = \frac{2m(E-V_0)}{\hbar^2}\right)$$

Where we have used the boundary cond. $\qquad$ inside sol.
$u(0)=0$.

Because; $A_\ell(r) = C_\ell^{(1)} h_\ell^{(1)}(kr) - C_\ell^{(2)} h_\ell^{(2)}(kr)$

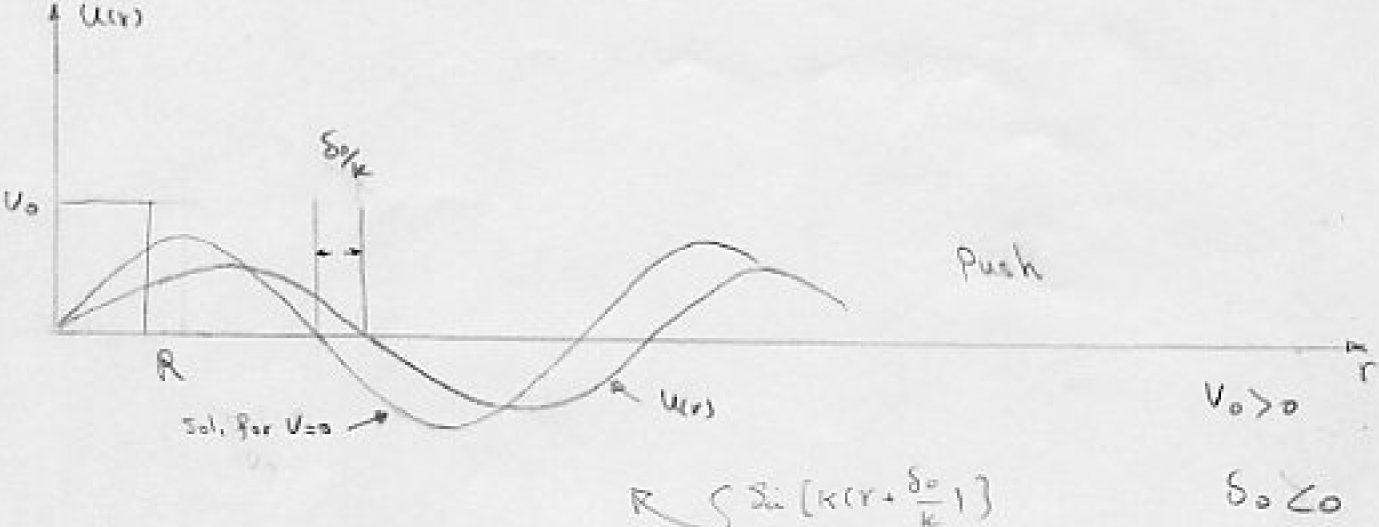
or

$$A_\ell(r) = C_\ell J_\ell(kr) + \underbrace{B_\ell \eta_\ell(kr)}_{\text{not regular at origin}}$$

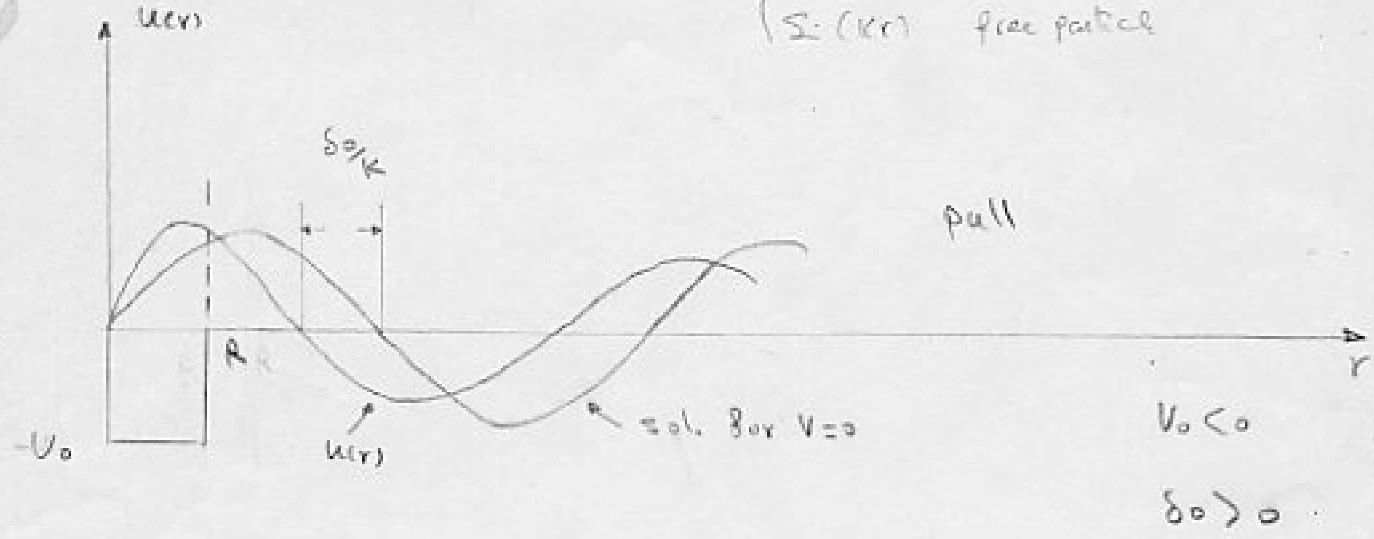
$\rightarrow$ for $r<R$ $\quad B_\ell = 0$

$\rightarrow A_\ell(r) = C_\ell J_\ell(kr)$

As long as $E > V_0 \rightarrow$ the inside wave-func. is sinesoidal.



$$\begin{cases} \sin[k(r+\frac{\delta_0}{k})] \\ \sin(kr) \quad \text{free particle} \end{cases}$$



If $V_0 > E$

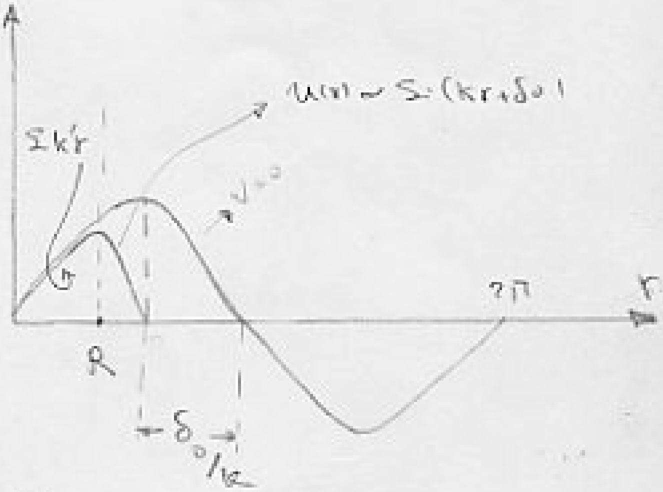
$u(r) \sim \sin(k'r)$

$$k'^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

Now, consider an attractive potential:

1- Suppose the attraction is <u>such</u> that:

The interval (0-R) just accommodates $\underline{\frac{1}{4}}$ cycle of the sinusoidal wave



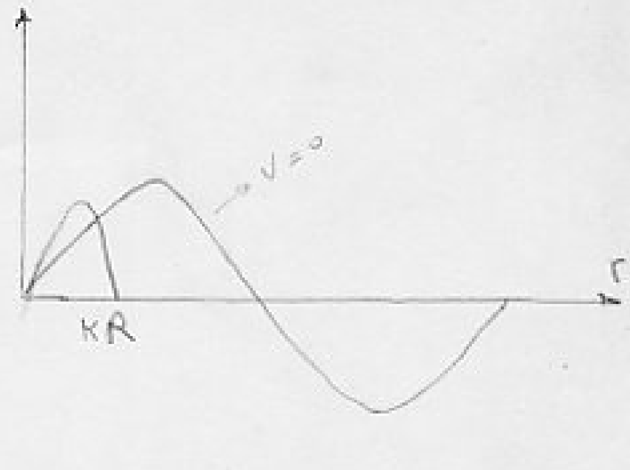
$\rightarrow \delta_0 = \frac{\pi}{2} \rightarrow \sin\delta_0 = 1$ (see Fig.)

$\rightarrow \sigma_0 = \text{Max}$ (for a given k)

Remark: $u(r) \sim \sin(kr + \delta_0)$ outside
$u(r) \sim \sin k'r$ inside
For $R = \frac{\lambda}{4}$ $\Rightarrow \sin k'R = \sin(kR + \delta_0) = 1$
$kR + \delta_0 = \frac{\pi}{2}$ for small k $\rightarrow \delta_0 \simeq \frac{\pi}{2}$

2- Now we increase the well depth V_0 even further.

Eventually the attraction is so strong that $\underline{\frac{1}{2}}$ cycle of the sinusoidal wave can be fitted within the range of the potential.



$\rightarrow \delta_0 = \pi \rightarrow \sin\delta_0 = 0$

$\rightarrow \sigma_0 = 0$

(despite the very strong attraction)

→ The wave-func outside R, is $\underline{180°}$ out of phase compared to the free-particle wave-func.

In addition, if the energy is low enough;

→ $l \neq 0$ waves unimportant.

→ therefore we have almost Perfect-transition of the incident wave

This is known as the Ramsauer Townsend effect.

Zero-Energy Scattering and Bound states:

Consider scattering of $K \simeq 0$ particles:

$$\text{For} \begin{cases} r > R \ (V(r) = 0) \\ l = 0 \\ K \simeq 0 \end{cases} \xrightarrow{\text{Schrödinger equ.}} \quad \frac{d^2u}{dr^2} = 0$$

$$\text{sol.} \longrightarrow \quad u(r) = c(r - a) \qquad \text{straight line}$$

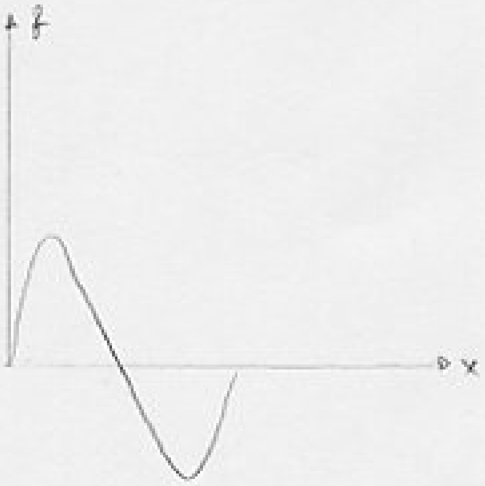
→ this can be interpreted as $\lambda \to \infty$ limit of usual outside solution.

$$\begin{cases} u(r) = r A_l(r) \\ A_{l=0}(r) = \frac{1}{kr} \sin(kr + \delta_0) \end{cases} \longrightarrow u_0(r) = \frac{1}{k} \sin(kr + \delta_0)$$

Now,

$$\lim_{k\to 0} \sin(kr+\delta_0) = \lim_{k\to 0} \sin\left[k\left(r+\frac{\delta_0}{k}\right)\right]$$

This looks like $u(r) = c(r-a)$



$f_1 = \sin k_1 x$

$f_2 = \sin k_2 x$

$k_2 << k_1$

The form of the wave is indep. of the phase.

Now; $\dfrac{u'(r)}{u(r)} = \dfrac{k\cos[k(r+\frac{\delta_0}{k})]}{\sin[k(r+\frac{\delta_0}{k})]} = k\,\mathrm{cotan}\left[k\left(r+\frac{\delta_0}{k}\right)\right]$

Since $\mathrm{cotan}\,x \simeq \dfrac{1}{x} - \dfrac{x}{3}$ as $x \to 0$

$$\frac{u'(x)}{u(x)} = k\,\mathrm{cotan}\left[k\left(r+\frac{\delta_0}{k}\right)\right] \xrightarrow[k\to 0]{} k\frac{1}{k(r+\frac{\delta_0}{k})} = \frac{1}{r+\frac{\delta_0}{k}}$$

$\rightarrow \dfrac{u'(r)}{u(r)} \xrightarrow{\text{looks}} \dfrac{1}{r-a}$ when $\dfrac{\delta_0}{k} = -a$

Remark:
$k(r+\frac{\delta_0}{k}) = kr+\delta_0$
$k\to 0 \to kr+\delta_0 \to 0$
because δ_0 is small for small k

With $u(r) = c(r-a)$ we obtain the same result.

Setting $r=0$ (even though at $r=0$, $u(r)=c(r-a)$ is not true wave-func.) we obtain

$$\frac{u'(0)}{u(0)} = k\,\mathrm{cotan}\left[k\left(0+\frac{\delta_0}{k}\right)\right] \xrightarrow{k\to0} -\frac{1}{a}$$

a: scattering length

Def.: $a = \lim_{k\to0} \frac{-\tan\delta_0(k)}{k}$

Since $f_\ell = \frac{e^{2i\delta_\ell}-1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\,\mathrm{cotan}\,\delta_\ell - ik}$

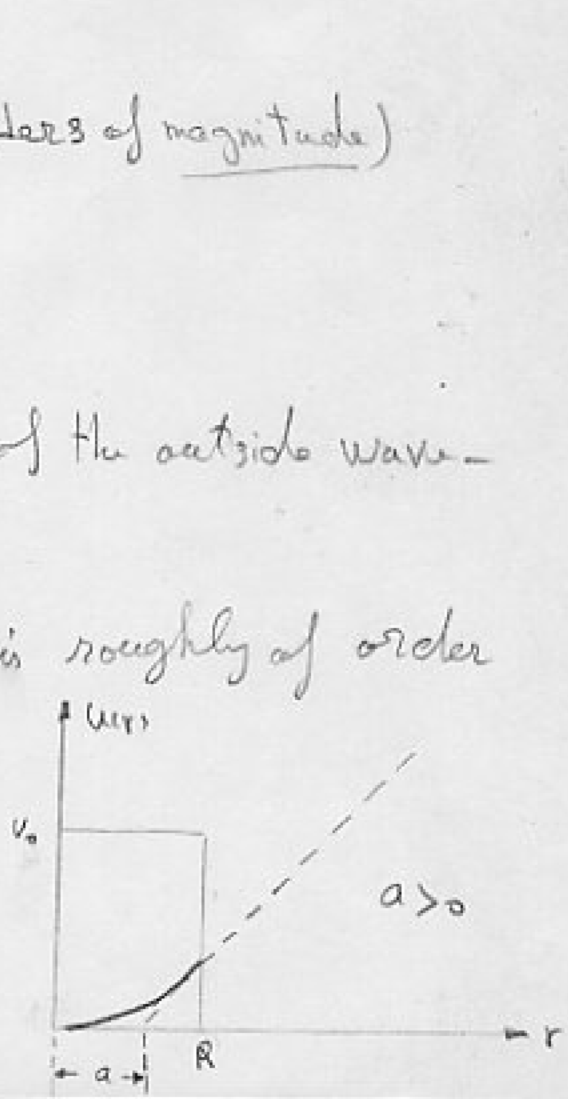
$$\sigma_{tot} \simeq \sigma_{\ell=0} = 4\pi \lim_{k\to0}\left[\frac{1}{k\,\mathrm{cotan}\,\delta_0 - ik}\right]^2 = 4\pi\left[\frac{1}{-\frac{1}{a}-0}\right]^2 = 4\pi a^2$$

a and R different, they differ by orders of magnitude)

What is the meaning of a?

a is nothing more than the intercept of the outside wave-func.

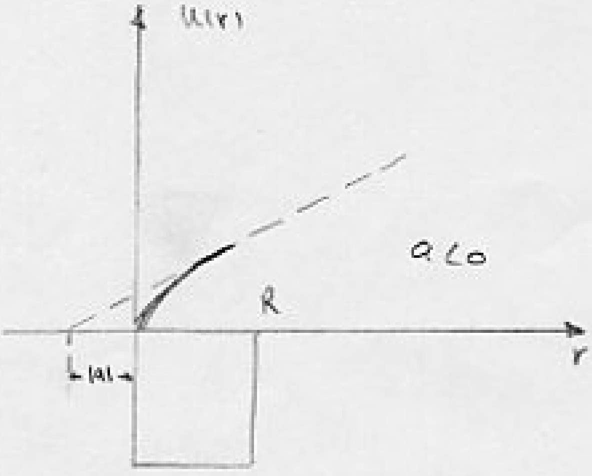
1- For repulsive potential; $a>0$ and is roughly of order of R.



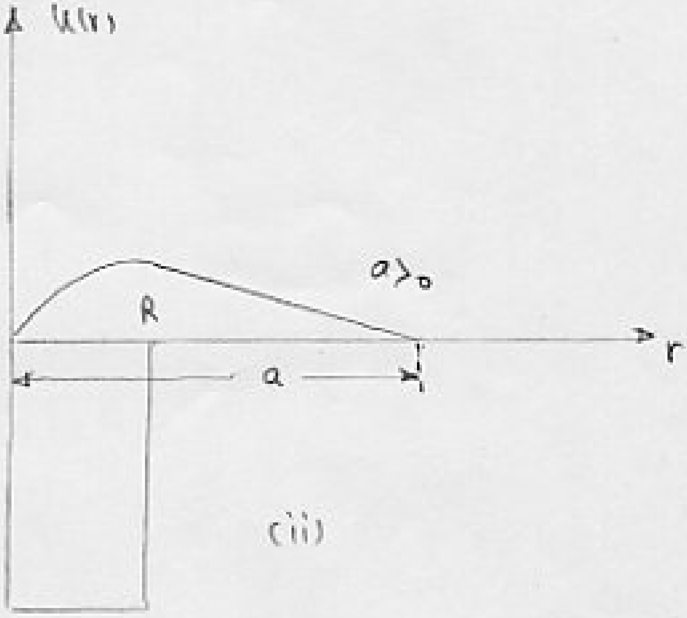
2- For attractive potential;

i) $a<0$

ii) If the attractive potential is increased, the out side wave-func. can again cross the r-axis on the positive side.



(i)



(ii)

The sign change resulting from increased attraction is related to the development of "bound state".

To see this quantiatively, note Fig (ii) that;

I) For $a>0$ and very large $\longrightarrow$ Wave-func. is flat for $r>R$

$$\begin{cases} u(r) \sim c(r-a) & (a \text{ large}) \\ e^{-kr} \simeq 1-kr & (K \approx 0) \end{cases} \longrightarrow \text{these two are not too different}$$

The later is a bound state wave-func. for $r>R$ with $E=0$

II) $r<R$, the inside wave-func. for;

$E = 0_+$ (scattering with Zero kinetic energy)

$E = 0_-$ (bound state with infinitesimally small binding energy)

are essentially the same.

In both cases; $u(r) \sim \sin k'r$

$$\text{where} \quad \frac{\hbar^2 k'^2}{2m} = \underset{\substack{\downarrow \\ E_{0-} \text{ or } E_{0+}}}{E} - V_0 \approx |V_0| \qquad V_0 < 0$$

For two physical situations (E_{0+} and E_{0-}) ⟶ the inside wave-funcs. are the same.

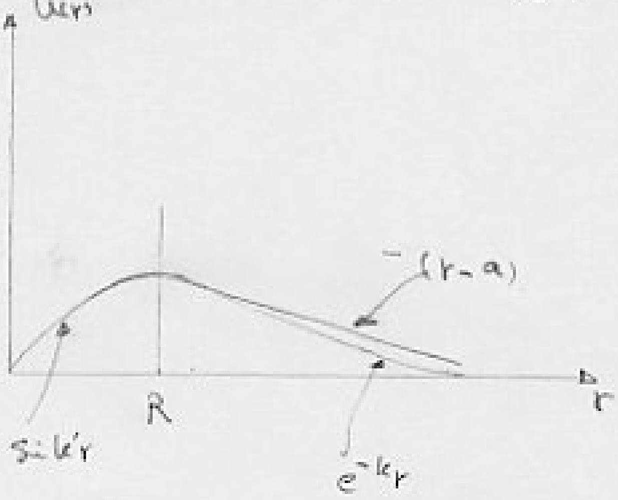
⟹ then inside the potential we can use E_{0+} sol. (scattering sol.) to represent E_{0-} sol. (bound state sol.)

$$\text{For } r<R \quad \begin{cases} \sin k'r & \text{bound state} \\ \sin k'r & \text{scattering} \end{cases}$$

$$\text{For } r>R \quad \begin{cases} e^{-kr} & \text{bound state} \\ r-a & \text{scattering} \end{cases}$$

Then we equate logarithmic derivative of bound state and zero-energy scattering sols. for $r \geq R$ at $r = R$. (Because the inside wave-funcs for E_{0-} and E_{0+} are the same).

$$\left.\frac{-k e^{-kr}}{e^{-kr}}\right|_{r=R} = \left.\frac{1}{r-a}\right|_{r=R}$$



$$\longrightarrow k \simeq \frac{-1}{R-a}$$

if $R << a \longrightarrow k \simeq \frac{1}{a}$

This is a relation between scattering and bound state energy:

$$E_{BE} = E_{\substack{\text{bound}\\\text{state}}} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2ma^2}$$

Therefore, if there exist a losely bound state; we can infer its binding energy by performing scattering experiment near zero kinetic energy, provided a, is measured to be large, compared with R.

Ex.- n-p system = deutron $^3S_1$ state

has a bound state $E_{BE} = 2.26$ Mev

the scattering length is measured to be

$$a_{\text{triplet}} = 5.4 \times 10^{-13} \text{ cm}$$

$$\longrightarrow \frac{\hbar^2}{2\mu a^2} = \frac{\hbar^2}{m_n a^2} = m_n c^2 \left(\frac{\hbar}{m_n c a} \right)^2$$

$$= (938 \text{ Mev}) \left(\frac{2.1 \times 10^{-14} \text{ cm}}{5.4 \times 10^{-13} \text{ cm}} \right)^2 = 1.4 \text{ Mev}$$

where $\mu = \frac{m_n m_p}{m_n + m_p} \simeq \frac{1}{2} m_n$

The agreement is not too satisfactory; $2.26 \neq 1.4$

The discrepancy is due to;

1- The inside wave-func. are not exactly the same.

2- $a \gg R$ is not really such a good approx. for deutron.

a better result can be obtained by keeping the next term in the expansion of k cotan δ as a func. of k.

$$k \cot \delta_0 \simeq -\frac{1}{a} + \frac{1}{2} r_0 k^2$$

r_0 : effective range (see P157)

$$\left(\cot x \simeq \frac{1}{x} - \frac{x}{3} \quad x \to 0 \right)$$

(we have to fit the inside-sol. with the outside-sol.)

Alternative approach:

In the zero-energy scatt. limit:

$\delta_\ell = 0$ for $\ell \neq 0$

$\rightarrow f(\theta) = f(\theta)_{\ell=0} = \frac{1}{k} e^{i\delta_0} \, S \, \delta_0$

Def:

$a = \lim_{k \to 0} [-f(\theta)] = -\frac{1}{k} e^{i\delta_0} \, S\delta_0$ (1)

$\rightarrow \sigma = 4\pi |f(\theta)|^2 = 4\pi a^2$

For $\begin{cases} r > r_0 & (V(r) = 0) \\ \ell = 0 \\ k \simeq 0 \end{cases} \rightarrow \frac{d^2 u_0}{dr^2} = 0$ Schrödinger equ.

$V_0 \equiv \{u_0(r)\}_{r \gg r_0} = br + c$

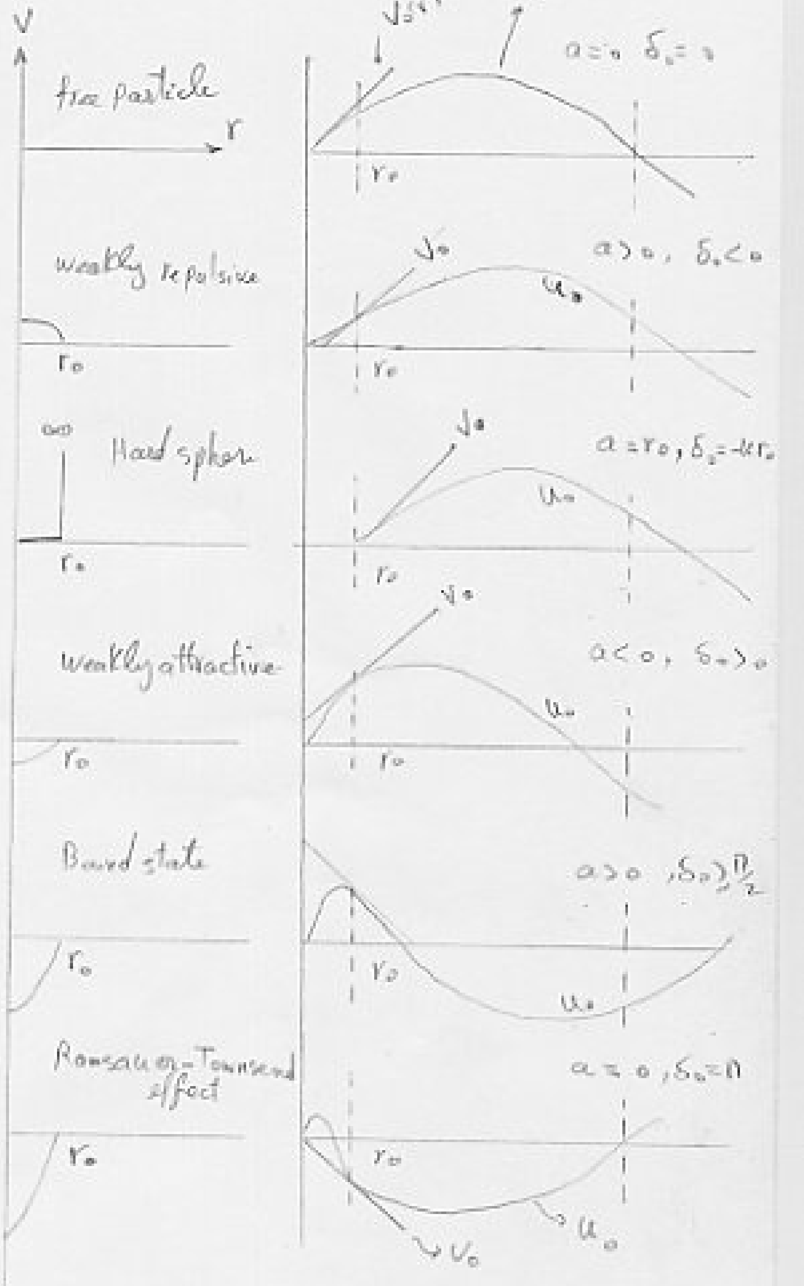
$u_0(r) = r R_0(r) \underset{r \gg r_0}{\sim} r e^{ik \cdot r} + f(\theta) e^{ikr}$

$\underset{k \to 0}{\approx} (r - a) = V_0(r)$ (using (1))

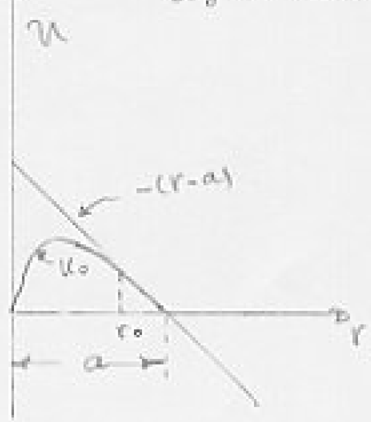
$V_0(r) = \alpha (r - a)$

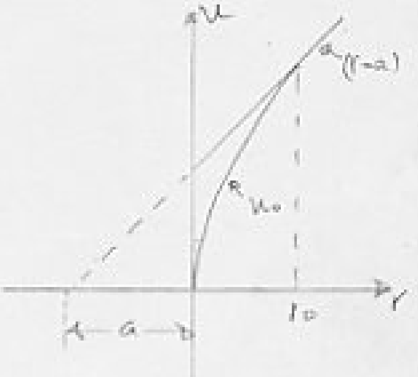
$V_0(r) = \pm (r - a)$ ($|\alpha| = 1$ normalization)

$r \gg r_0$



$u_0(r) \sim S(kr + \delta_0)$, $V_0 \sim \pm (r - a)$





Bound states as Poles of $S_\ell(k)$

We already obtained:

$$\langle x|\psi^+\rangle \sim \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1)\, \frac{P_\ell(\cos\theta)}{2ik} \left[\underbrace{(1+2ik f_\ell(k))}_{S_\ell(k)} \frac{e^{ikr}}{r} - \frac{e^{-i(kr-\ell\pi)}}{r} \right]$$

For $\ell=0$

$$\langle x|\psi^+\rangle \sim S_{\ell=0}(k)\, \frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r}$$

Compare this result with the wave-func. for a <u>bound-state</u> at large distance;

$$\frac{e^{-\kappa r}}{r} = \frac{e^{i(i\kappa)r}}{r} \qquad r \to \infty \quad \text{(outside)}$$

The existance of <u>bound states</u> implies that:

A nontrivial sol. to the schrödinger equ. with $E<0$ exists, only for a particular (discrete) values of k;

$$\frac{e^{-\kappa r}}{r} \longrightarrow \frac{e^{ikr}}{r} \qquad \text{(k purly imaginary)}$$

Apart from k being imaginary, the important difference between

$$S_{\ell=0}(k)\,\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r} \qquad \text{scatt. (outside)}$$

and $$\frac{e^{-\kappa r}}{r} \longrightarrow \frac{e^{ikr}}{r} \qquad (k \text{ purely imaginary})$$

bound state (outside)

is that, in the <u>bound state</u> case, $\frac{e^{-\kappa r}}{r}$ is present even without analouge of <u>incident</u> wave.

Only the ratio of the coeff. of $\frac{e^{ikr}}{r}$ to that of $\frac{e^{-ikr}}{r}$ is of physical interest, and this is given by: $S_\ell(k)$

1- In the bound state case:

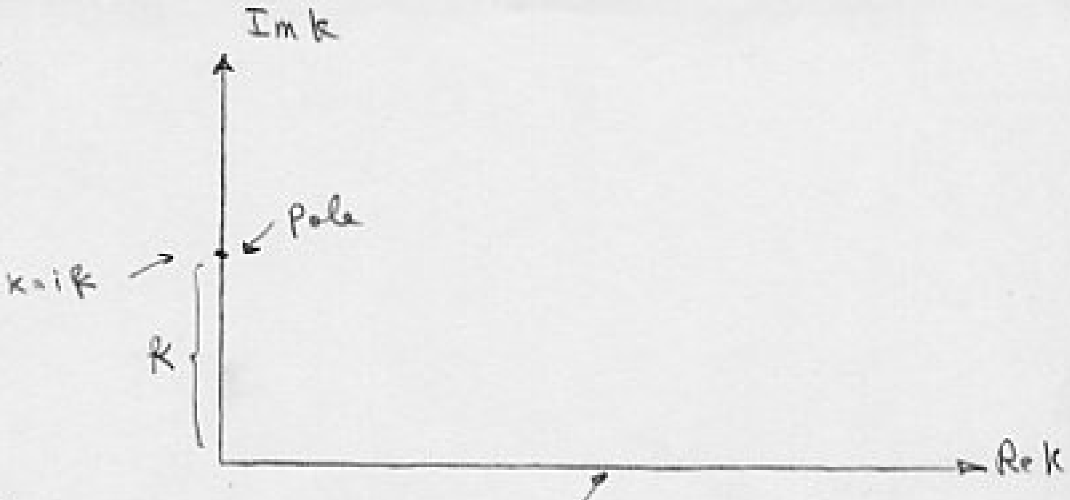
$$S_\ell(k) = \frac{\text{Coff. } U_{out}}{\text{Coff. } U_{in}} = \infty$$

Remark:
For bound state with the wave func. $\frac{e^{ikr}}{r}$ we have $k = i\kappa$, then the denominator of $S_0(k)$ must have a term like $k - i\kappa$. ($\ell=0$) case

$\longrightarrow$ $S_{\ell=0}(k)$ has a pole at $k = i\kappa$ (k = complex variable)

Thus a boundstates implies a <u>pole</u> (which is a simple pole) on the positive imaginary axis of the complex k-plane

2- $S_0(k) = \frac{\text{Coeff. } u_{out}}{\text{Coeff. } u_{in}}$



For Physical scattering we have:

$S_0 = e^{i2\delta_0}$ $(\delta_0 : \text{real})$

Furthermore we have already obtained:

as $k \to 0 \implies k \cot\delta_0 \to -\frac{1}{a}$ (finite)

$\xrightarrow{\text{So}} \cot\delta_0 \xrightarrow{\text{must}}$ infinite $\longrightarrow \delta_0 \to 0, \pm\pi, ..$

Hence; $S_0 = e^{i2\delta_0} \to 1$ as $k \to 0$ (1)

Now, let us attempt to construct a simple func. $S_\ell(k)$ satisfying:

1- Pole at $k = i\kappa$ (existance of bound state)

2- $|S_{\ell=0}| = 1$ for $k > 0$, real (unitary)

3 - $S_{\ell=0} = 1$ at $k = 0$ (threshold behavior) (using (1))
$k \to 0 \to \delta \to 0$

The simplest func. that satisfies all three conds; is:

$$S_{\ell=0}(k) = \frac{-k - i\kappa}{k - i\kappa}$$

Since;

$$f_{\ell=0} = \frac{S_{\ell=0}-1}{2ik} = \frac{1}{-\kappa-ik}$$

Also; $f_{\ell=0} = \dfrac{1}{k\cot\delta_0 - ik}$

$$\rightarrow \frac{1}{-\kappa-ik} = \frac{1}{k\cot\delta_0-ik} \longrightarrow k\cot\delta_0 = -\kappa$$

$$\rightarrow \lim_{k\to 0} k\cot\delta_0 = -\frac{1}{a} = -\kappa$$

This is precisely the relation between bound state and scattering length. (we obtained it before, i.e. $\kappa \approx \frac{1}{a}$)

→ By exploiting unitary and analicity of $S_\ell(k)$ in the k-plane we may obtain the kind of information which is available by solving the schrödinger equ. explicitly.

This technique is helpful in problems where the details of the potential are not known.

i.e. With the properties discussed for $S_\ell(k)$, one may construct it and get some kind of information without solving the schrödinger equ.

Using the experimental results we can constuct $S_\ell(k) = 1 + 2ik f_\ell(k)$ (when $f_\ell(k) = \frac{e^{2i\delta_\ell}-1}{2ik}$). Then the pole on $Im(k)$ axis is related to the bound state ($E \approx 0$) and $Re(k)$ gives information about the scatt.

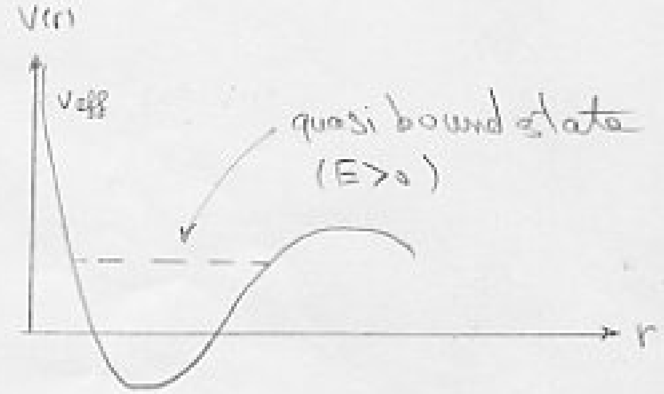
Resonance Scattering:

The scattering cross section for a partial wave exhibits a pronounced peak; This is called a resonance.

$$V_{eff}(r) = \underbrace{V(r)}_{\text{attractive}} + \underbrace{\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}}_{\text{repulsive centrifugal barrier}}$$

A valley from which the particle may tunnel:

Quasi bound state = Meta stable state
= Resonance state



1- Genuine bound states : (E>0)

Height of the barrier → ∞ ⇒ lifetime of the trapped particles → ∞ (even E>0)

In other words, they are stationary states with $\tau \to \infty$ (lifetime).

2- Quasi bound state : (E>0)

Height of the barrier = finite

In this case tunneling is probable.

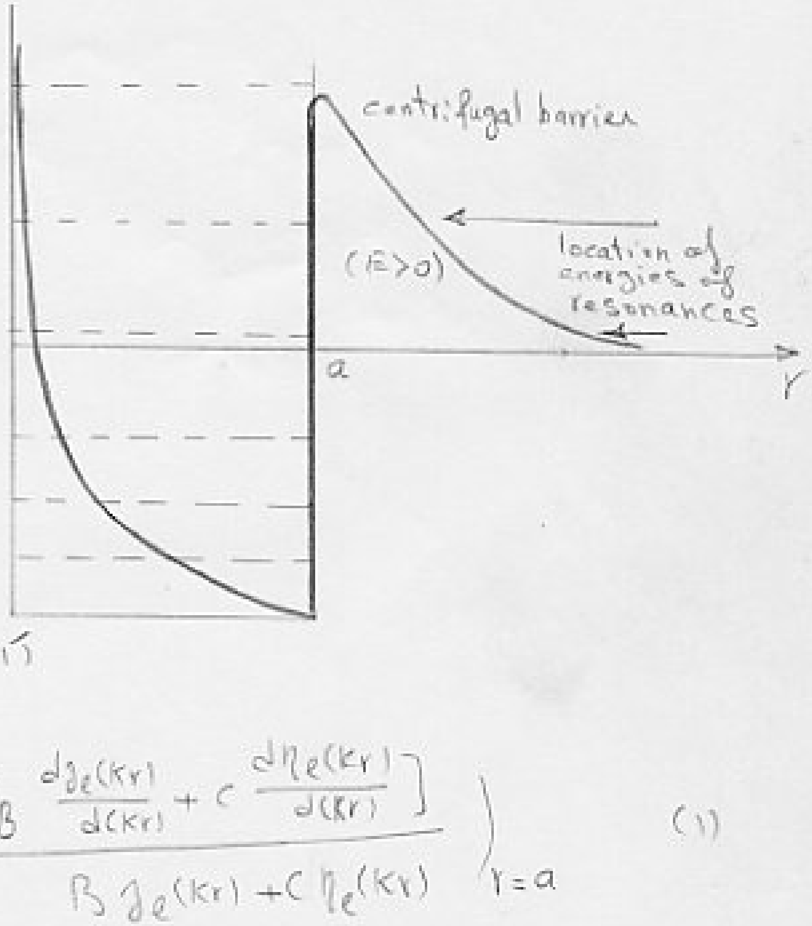
In such resonance state (quasi bound state);

$\delta_l \to \frac{\pi}{2}$ as $E \to E_{resonance}$ (energy of quasi bound state)

and $\sigma_l = \frac{4\pi}{k^2}(2l+1)$ (Max value)

Ex - We verify this point for the spherical well.

$$\begin{cases} \text{with } E>0 \\ K^2 = \frac{2ME}{\hbar^2} \quad (r>a) \text{ for square well} \\ k^2 = \frac{2M(E+V_0)}{\hbar^2} \quad (V_0>0) \quad (r<a) \end{cases}$$



$$\rightarrow \begin{cases} R_l(r) = A J_l(kr) & r<a \\ R_l(r) = B J_l(Kr) + C \eta_l(Kr) & (1') \end{cases}$$

$$k\left(\frac{r\frac{dJ_l(kr)}{d(kr)}}{J_l(kr)}\right)_{r=a} = K\left(\frac{r\left[B\frac{dJ_l(Kr)}{d(Kr)} + C\frac{d\eta_l(Kr)}{d(Kr)}\right]}{BJ_l(Kr) + C\eta_l(Kr)}\right)_{r=a} \qquad (1)$$

Now we compare the sol. of spherical square well at large r, and scattering sol.;

$$(1') \rightarrow \begin{cases} R_l(kr) \simeq \frac{B}{Kr}\left[\sin\left(Kr - \frac{l\pi}{2}\right) - \frac{C}{B}\cos\left(Kr - \frac{l\pi}{2}\right)\right] & (2) \text{ for spherical well} \\ R_l(kr) \simeq \frac{1}{Kr}\left[\sin\left(Kr - \frac{l\pi}{2}\right)\cos\delta_l + \cos\left(Kr - \frac{l\pi}{2}\right)\sin\delta_l\right] & (3) \end{cases}$$

for scatt.

$$(2)(3) \longrightarrow \tan\delta_\ell(k) = -\frac{C}{B} \qquad (4)$$

$$\frac{(1)(4)}{} \longrightarrow \tan\delta_\ell(k) = \frac{k\, j'_\ell(ka)\, j_\ell(Ka) - K\, j_\ell(ka)\, j'_\ell(Ka)}{k\, n'_\ell(ka)\, j_\ell(Ka) - K\, n_\ell(ka)\, j'_\ell(Ka)} \qquad (5)$$

$\left\{ \begin{array}{l} \text{or (P164) with } \beta_\ell = \left(\frac{r}{A_\ell}\frac{dA_\ell}{dr}\right)_{r=R} \longrightarrow \\ A_\ell = A\, j_\ell(Kr) \quad r<R \end{array} \right.$

$\left\{ \begin{array}{l} \text{Remark:} \\ L = \ell\hbar \\ L = bP = b\hbar k \\ \ell = bk \end{array} \right.$

i) For $ka \ll \ell$ (low energy or $a \ll b$)

using $j_\ell(x) \underset{x\to 0}{\simeq} \frac{x^\ell}{(2\ell+1)!!}$, $n_\ell(x) \underset{x\to 0}{\simeq} \frac{-(2\ell-1)!!}{x^{\ell+1}}$

$$\to \tan\delta_\ell(k) = \frac{2\ell+1}{[(2\ell+1)!!]^2}(ka)^{2\ell+1}\frac{\ell\, j_\ell(Ka) - Ka\, j'_\ell(Ka)}{(\ell+1)\, j_\ell(Ka) + Ka\, j'_\ell(Ka)} \qquad (6)$$

For large ℓ (even with $Ka \gg 1$) this drops faster than $e^{-\ell}$; (it can be shown).

$$\tan\delta_\ell(k) \sim k^{2\ell+1}$$

This result (with $ka \to 0$) is not restricted to square well potential, but is true for all reasonably smooth potentials. It is a consequence of the centrifugal barrier, which keeps waves of energy far below the barrier from feeling the effect of the potential.

Remark: Resonant scatt. occurs at low energies (see Fig. of P166/2 E>0 – levels)

ii) For certain values of the energy the denominator in (5) will vanish; so that;

$$\delta_l \longrightarrow \frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2} + n\pi$$

$$\Longrightarrow \sigma_l(k) = \frac{4\pi(2l+1)}{k^2}$$ largest possible value

Example;

Now, let us consider a very deep potential, and also l large, so that

$$Ka \gg l \gg ka$$

In such a case we may use equ.(6)

$$\tan\delta_l \rightarrow \infty \quad \text{if} \quad (l+1) j_l(Ka) + Ka\, j_l'(Ka) = 0$$

Since $Ka \gg l$, we may use the asymptotic form of the waves; (Ka = large)

$$\frac{(l+1)}{Ka} \cos\left(Ka - \frac{l+1}{2}\pi\right) - \sin\left(Ka - \frac{l+1}{2}\pi\right) = 0$$

$$\rightarrow \tan\left(Ka - \frac{l+1}{2}\pi\right) \simeq \frac{l+1}{Ka}$$

(Ka: large)

Since $\frac{l+1}{Ka} \ll 1 \rightarrow \tan x \simeq x$

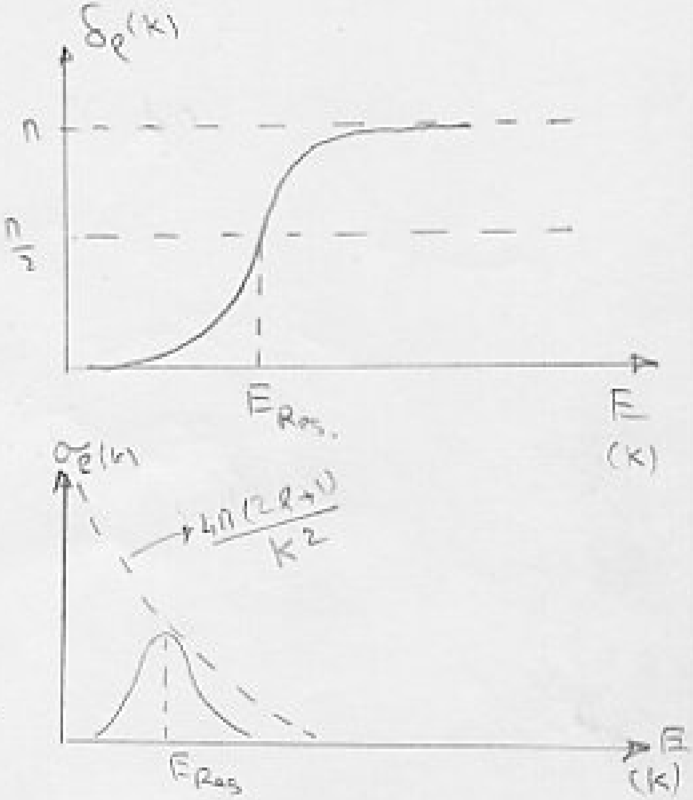
$$Ka - \frac{l+1}{2}\pi \simeq n\pi + \frac{l+1}{Ka}$$

Remark: $k^2 = \frac{2\mu E}{\hbar^2}$, $K^2 = \frac{2M(E+V_0)}{\hbar^2}$

E is the same in both cases

This is just the cond. for the existance of <u>discrete</u> levels in a three-dim box;

So the resonant scattering occure when the incident energy is just such to match an energy level ($E>0$).



We may represent this behavior by:

$$\tan\delta_\ell \simeq \frac{\gamma\,(ka)^{2\ell+1}}{E-E_{res}}$$

$$\sigma_\ell = \frac{4\pi(2\ell+1)}{k^2}\,\frac{\tan^2\delta_\ell}{1+\tan^2\delta_\ell}$$

$$= \frac{4\pi(2\ell+1)}{k^2}\,\frac{[\gamma\,(ka)^{2\ell+1}]^2}{(E-E_{res})^2+[\gamma\,(ka)^{2\ell+1}]^2}$$

(Breit-Wigner formula)

This is the characteristic of all potentials (not only square well) that have a shape such that metastable states can simulate bound states above $E=0$ in it.

Also;

$$f_\ell(k) = \frac{e^{2i\delta_\ell(k)}-1}{2ik} = \frac{\frac{1+i\tan\delta_\ell}{1-i\tan\delta_\ell}-1}{2ik} = \frac{\tan\delta_\ell}{k(1-i\tan\delta_\ell)}$$

$$f_\ell(k) = \frac{\gamma\,(ka)^{2\ell+1}/k}{E-E_{res}-i\gamma\,(ka)^{2\ell+1}}$$

(Q.M., V, ThanKappan, QC174.12.T48)

Resonance Satt.:

$$\gamma_\ell = \left[\frac{1}{R_\ell}\frac{dR_\ell}{dr}\right]_{r=r_0}$$

$$\to \tan\delta_\ell = \frac{k j_\ell'(kr_0) - \gamma_\ell j_\ell(kr_0)}{k n_\ell'(kr_0) - \gamma_\ell n_\ell(kr_0)} \quad (1)$$

r_0: range of the pot.

$$i\tan\delta_\ell = \frac{i\sin\delta_\ell}{\cos\delta_\ell} \quad (2)$$

Using $\frac{a}{b} = \frac{c}{d} \to \frac{a+b}{a-b} = \frac{c+d}{c-d}$ (3)

$$(1)(2)(3) \longrightarrow e^{2i\delta_\ell} = e^{2i(\xi_\ell+\varsigma_\ell)} \quad (4)$$

where; $$e^{2i\xi_\ell} = -\frac{j_\ell(kr_0) - i n_\ell(kr_0)}{j_\ell(kr_0) + i n_\ell(kr_0)} \quad (5)$$

$$e^{2i\varsigma_\ell} = +\frac{\beta_\ell - \Delta_\ell + i s_\ell}{\beta_\ell - \Delta_\ell - i s_\ell} \quad (6)$$

with $\beta_\ell = r_0\gamma_\ell$

$$\Delta_\ell(kr_0) = \frac{kr_0(j_\ell j_\ell' + n_\ell n_\ell')}{j_\ell^2 + n_\ell^2} \quad (7)$$

$$s_\ell(kr_0) = \frac{1}{kr_0(j_\ell^2 + n_\ell^2)} \quad (8)$$

(where we have used, $j_\ell(\rho)n_\ell'(\rho) - j_\ell'(\rho)n_\ell(\rho) = \frac{1}{\rho^2}$ in obtaining (8))

(5) $\xrightarrow{\text{equivalent}}$ $\tan \xi_\ell = \dfrac{j_\ell(kr_0)}{n_\ell(kr_0)}$ (9)

$\rightarrow \xi_\ell$: real

Similarly Δ_ℓ, s_ℓ and hence $\zeta_\ell = \tan^{-1}\left[\dfrac{s_\ell}{B_\ell - \Delta_\ell}\right]$ are also <u>real</u>. (10)

(4) $\rightarrow$ $\delta_\ell = \xi_\ell + \zeta_\ell$ (11)

ξ_ℓ : slowly varying func. of E.

In fact using $\begin{cases} j_\ell(\rho) \underset{\rho\to 0}{\sim} \dfrac{\rho^\ell}{(2\ell+1)!!} \\ n_\ell(\rho) \underset{\rho\to 0}{\sim} -(2\ell-1)!!\,\rho^{-(\ell+1)} \end{cases}$ ($\rho \to 0$, i.e. $E \approx 0$)

in (9) $\longrightarrow$ $\tan \xi_\ell \underset{kr_0\to 0}{\sim} - \dfrac{(kr_0)^{2\ell+1}}{(2\ell+1)\left[(2\ell-1)!!\right]^2}$ (12)

Since $E \sim k^2$ $\longrightarrow$ $\xi_\ell \sim E^{\ell+\frac{1}{2}}$ at low energies
(i.e. $\xi_\ell \to 0$ like $E^{\ell+1/2}$ at low energies)

Also, (6) $\rightarrow$ $\zeta \to 0$ as $B_\ell \to \infty$

$\rightarrow \delta_\ell = \xi_\ell$ when $B_\ell \to \infty$

But $B_\ell = \infty$ corresponds to a <u>hard sphere</u>.

For this reason ξ_l is called hard sphere phase shift.

Equ(10) shows that ξ_l is very sensetive to E,

because of $(\beta_l - \Delta_l)$ in the denominator.

Reason;

i- $$\beta_l(E_1) - \beta_l(E_2) = \frac{-(2\mu/\hbar^2)(E_1 - E_2)\int_0^{r_0} u_l(rk_1)\, u_l(rk_2)\, r_0\, dr}{u_l(r_0k_1)\, u_l(r_0k_2)} \tag{13}$$

$\rightarrow \beta_l$ monotonically decreasing func. of E.

ii- The energy dependence of Δ_l is clear from (7)

The sudden increase of ξ_l occurs at $\beta_l \simeq \Delta_l$

If $E \simeq$ small ; (12) $\longrightarrow$ ξ_l : negligibale compared with ξ_l at resonance E.

$\longrightarrow \delta_l \simeq \xi_l$

Also, the phaseshifts corresponding to the partial waves other than the one for which $\beta_l \simeq \Delta_l$ can be neglected (l: a certain l).

$$\sigma \simeq \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$$

$$\simeq \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell = \frac{4\pi(2\ell+1)}{k^2}\left[\frac{\tan^2\delta_\ell}{1+\tan^2\delta_\ell}\right] \qquad (14)$$

(contribution from other l's are neglected).

Let E_s^ℓ be the energy at which $B_\ell = \Delta_\ell$.

$$B_\ell(E) \simeq B_\ell(E_s^\ell) + (E-E_s^\ell)\left(\frac{\partial B_\ell}{\partial E}\right)_{E=E_s^\ell} + \cdots$$

$$\simeq \Delta_\ell - b_\ell(E-E_s^\ell) \qquad (15)$$

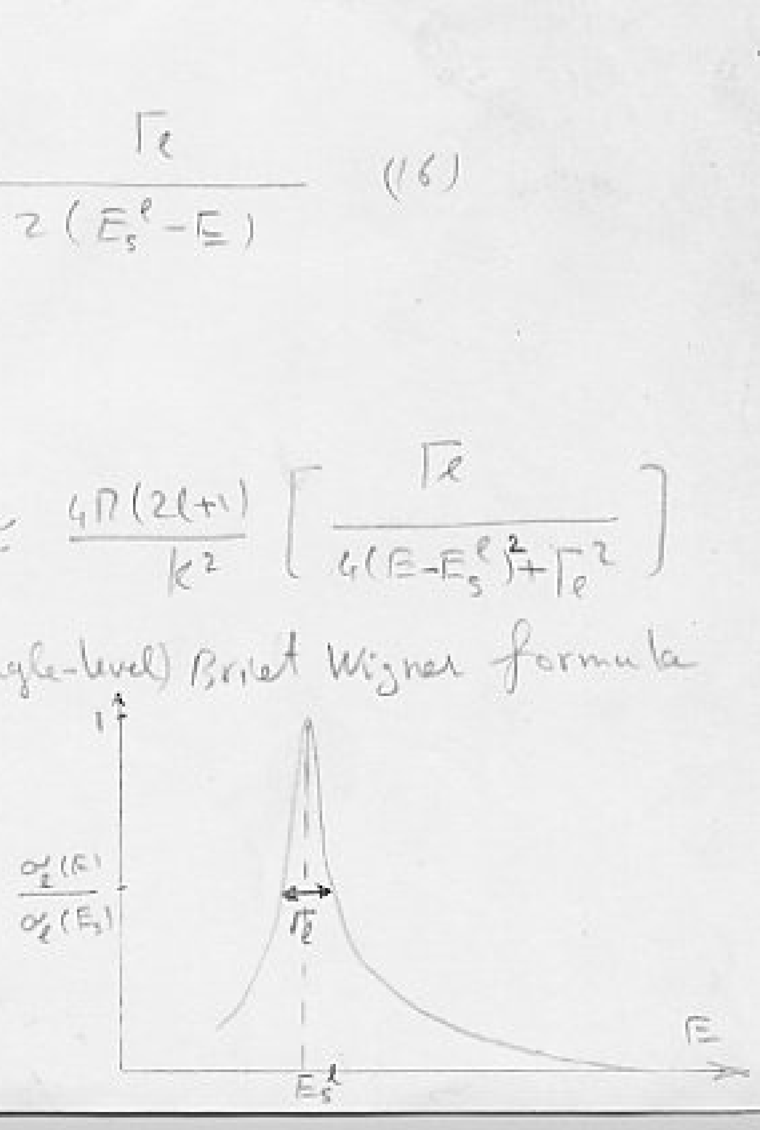
where $b_\ell > 0$ $\quad$ (B_ℓ : decreasing func. of E)

(15) in (10) $\rightarrow$ $\tan\delta_\ell = \dfrac{\Gamma_\ell}{2(E_s^\ell - E)}$ $\quad$ (16)

where $\Gamma_\ell = 2\dfrac{s_\ell}{b_\ell}$

(16) in (14) $\rightarrow$ $\sigma^\ell(E) \simeq \dfrac{4\pi(2\ell+1)}{k^2}\left[\dfrac{\Gamma_\ell}{4(E-E_s^\ell)^2+\Gamma_\ell^2}\right]$

(single-level) Briet Wigner formula



Determination of the Phase Shifts and Scattering Resonances.

The theoretical determination $\xrightarrow{\text{requires}}$ Solving the Schrödinger equ. for the given V and obtain Asymptotis form.

$$\Psi_{in} = \Psi_{out}$$

$$\underbrace{R_\ell(kr) = \frac{u_\ell(kr)}{r}}_{r<a} = \underbrace{A_\ell J_\ell(kr) + B_\ell \eta_\ell(kr)}_{r>a}$$

$$\beta_\ell = \left(\frac{a}{R_\ell}\frac{dR_\ell}{dr}\right)_{r=a}$$

$$\beta_{\ell\, in} = \beta_{\ell\, out} \longrightarrow \delta_\ell = ?$$

What about the energy dependence of β_ℓ ?

Using the continuity equ

$$\frac{\partial}{\partial t}(\psi_1^*\psi_2) + \frac{\hbar}{2\mu i}\nabla\cdot\left[\psi_1^*\nabla\psi_2 - (\nabla\psi_1^*)\psi_2\right] = 0$$

for two-separable sols. of the form;

$$R_\ell(k_1 r)\, Y_\ell^{m_1}(\theta,\varphi) \quad \text{and} \quad R_\ell(k_2 r)\, Y_\ell^{m_2}(\theta,\varphi)$$

with different energies E_1 and E_2 but both corresponding to the same angular momentum ℓ.

$$\frac{\partial}{\partial t}\left[(R_\ell(k_1r)\overset{*}{Y}{}_\ell^{m_1}(\theta,\varphi))\,(R_\ell(k_2r)Y_\ell^{m_2}(\theta,\varphi)\right] +$$

$$+\frac{\hbar}{2\mu i}\left(\hat{r}\frac{\partial}{\partial r}+\hat{\varphi}\frac{1}{rS\theta}\frac{\partial}{\partial\varphi}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\cdot$$

$$\cdot\left[(R_\ell(k_1r)Y_\ell^{*m_1}(\theta,\varphi))\left(\hat{r}\frac{\partial}{\partial r}+\hat{\varphi}\frac{1}{rS\theta}\frac{\partial}{\partial\varphi}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)(R_\ell(k_2r)Y_\ell^{m_2}(\theta,\varphi))-\right.$$

$$\left.-\left(\hat{r}\frac{\partial}{\partial r}+\hat{\varphi}\frac{1}{rS\theta}\frac{\partial}{\partial\varphi}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)(R_\ell(k_1r)Y_\ell^{*m_1}(\theta,\varphi))(R_\ell(k_2r)Y_\ell^{m_2}(\theta,\varphi)\right]=0$$

where
$$\begin{cases}\hat{r}=S\theta C\varphi\,\hat{i}+S\theta S\varphi\,\hat{j}+C\theta\,\hat{k}\\ \hat{\varphi}=-S\varphi\,\hat{i}+C\varphi\,\hat{j}\\ \hat{\theta}=C\theta C\varphi\,\hat{i}+C\theta S\varphi\,\hat{j}-S\theta\,\hat{k}\end{cases}$$

we obtain;

$$(E_2-E_1)R_\ell(k_1r)R_\ell(k_2r)\,r^2+\frac{\hbar^2}{2\mu}\frac{d}{dr}\left\{r^2\left[R_\ell(k_1r)\frac{dR_\ell(k_2r)}{dr}-\frac{dR_\ell(k_1r)}{dr}R_\ell(k_2r)\right]\right\}=0$$

Integrating from $r=0$ to $r=a$;

$$(E_2-E_1)\int_0^a R_\ell(k_1r)R_\ell(k_2r)\,r^2\,dr+\frac{\hbar^2}{2\mu}\left\{a^2\left[R_\ell(k_1a)\left.\frac{dR_\ell(k_2r)}{dr}\right|_{r=a}-\left.\frac{dR_\ell(k_1r)}{dr}\right|_{r=a}R_\ell(k_2a)\right]+0\right\}=0$$

$$\rightarrow \beta_\ell(E_2)-\beta_\ell(E_1)=-\frac{2\mu(E_2-E_1)\int_0^a R_\ell(k_1r)R_\ell(k_2r)r^2\,dr}{\hbar^2 a\,R_\ell(k_1a)\,R_\ell(k_2a)}$$

Hence, the logarithmic derivative $\beta_\ell(E)$ is a monotonically decreasing func. of E.

we have already obtained;

$$\beta_\ell(k) = ka\,\frac{j'_\ell(ka)\cos\delta_\ell - \eta'_\ell(ka)\sin\delta_\ell}{j_\ell(ka)\cos\delta_\ell - \eta_\ell(ka)\sin\delta_\ell}$$

We may solve this relation, and obtain;

$$S_\ell(k) = e^{2i\delta_\ell} = -\frac{j_\ell - in_\ell}{j_\ell + in_\ell}\left(1 + ka\,\frac{\dfrac{j'_\ell + in'_\ell}{j_\ell + in_\ell} - \dfrac{j'_\ell - in'_\ell}{j_\ell - in_\ell}}{\beta_\ell - ka\,\dfrac{j'_\ell + in'_\ell}{j_\ell + in_\ell}}\right)$$

(at $r = a$)

Now define a real phase angle ξ_ℓ by;

$$e^{2i\xi_\ell} = -\frac{j_\ell - in_\ell}{j_\ell + in_\ell}$$

and introduce the real parameters; Δ_ℓ and s_ℓ;

$$ka\,\frac{j'_\ell + in'_\ell}{j_\ell + in_\ell} = \Delta_\ell + is_\ell$$

Remark:

Using; $$j_\ell(z) = \frac{z^\ell}{2^{\ell+1}\ell!}\int_{-1}^{+1} e^{izs}(1-s^2)^\ell\,ds$$

and $$h^{(1)}_\ell(z) = \frac{z^\ell}{i2^\ell\,\ell!}\int_{\infty e^{i\alpha}}^{i} e^{zt}(1+t^2)^\ell\,dt$$

$$h_\ell^{(2)}(z) = \frac{z^\ell}{i\, 2^\ell\, \ell!} \int_{-i}^{\infty e^{i\alpha}} e^{zt} (1+t^2)^\ell \, dt$$

$$\begin{cases} J_\ell(z) = \frac{1}{2}\left[h_\ell^{(1)}(z) + h_\ell^{(2)}(z) \right] \\ n_\ell(z) = \frac{1}{2}\left[\quad '' \quad - \quad '' \quad \right] \end{cases}$$

it is easy to verify:

$$J_\ell(z)\, n'_\ell(z) - J'_\ell(z)\, n_\ell(z) = \frac{1}{z^2}$$

Now, $$ka \frac{J'_\ell + i n'_\ell}{J_\ell + i n_\ell} = ka \frac{J'_\ell J_\ell - i J'_\ell n_\ell + i n'_\ell J_\ell + n'_\ell n_\ell}{(J_\ell)^2 + (n_\ell)^2}$$

$$= \underbrace{ka \frac{J'_\ell J_\ell + n'_\ell n_\ell}{(J_\ell)^2 + (n_\ell)^2}}_{\Delta_\ell} + i \underbrace{ka \frac{J_\ell n'_\ell - J'_\ell n_\ell}{(J_\ell)^2 + (n_\ell)^2}}_{s_\ell}$$

Using the result of remark;

$$s_\ell = ka \frac{J_\ell n'_\ell - J'_\ell n_\ell > 0}{(J_\ell)^2 + (n_\ell)^2 > 0} \longrightarrow s_\ell > 0 \text{ definite}$$

$$s_\ell = ka \frac{\left(\frac{1}{ka}\right)^2}{(J_\ell)^2 + (n_\ell)^2} = \frac{1}{ka\left[(J_\ell)^2 + (n_\ell)^2\right]}$$

with these defs.

$$e^{2i(\delta_\ell - \xi_\ell)} = \frac{\beta_\ell - \Delta_\ell + i s_\ell}{\beta_\ell - \Delta_\ell - i s_\ell}$$

when we have also used:

$$ka\,\frac{j_\ell' - i n_\ell'}{j_\ell - i n_\ell} = \Delta_\ell - i s_\ell$$

Exersice - Show that:

$$e^{i\delta_\ell}\sin\delta_\ell = e^{2i\xi_\ell}\left(\frac{s_\ell}{\beta_\ell-\Delta_\ell-is_\ell} + e^{-i\xi_\ell}\sin\xi_\ell\right)$$

we know that: $\dfrac{e^{2i\delta_\ell}-1}{2ik} = \dfrac{e^{i\delta_\ell}\sin\delta_\ell}{k}$

$$\rightarrow e^{i\delta_\ell}\sin\delta_\ell = \frac{e^{2i\delta_\ell}-1}{2i}$$

$$e^{2i\delta_\ell} = 2ie^{i\delta_\ell}\sin\delta_\ell+1$$

Now $e^{2i(\delta_\ell-\xi_\ell)} = \dfrac{\beta_\ell-\Delta_\ell+is_\ell}{\beta_\ell-\Delta_\ell-is_\ell}$ can be written as:

$$e^{2i\delta_\ell} = e^{2i\xi_\ell}\,\frac{\beta_\ell-\Delta_\ell+is_\ell}{\beta_\ell-\Delta_\ell-is_\ell}$$

$$\rightarrow 2ie^{i\delta_\ell}\sin\delta_\ell+1 = e^{2i\xi_\ell}\,\frac{\beta_\ell-\Delta_\ell+is_\ell}{\beta_\ell-\Delta_\ell-is_\ell}$$

$$e^{i\delta_\ell}\sin\delta_\ell = \frac{1}{2i}\left(e^{2i\xi_\ell}\,\frac{\beta_\ell-\Delta_\ell+is_\ell}{\beta_\ell-\Delta_\ell-is_\ell} - 1\right)$$

From;

$$e^{i\delta_\ell} S\delta_\ell = e^{2i\xi_\ell}\left(\frac{s_\ell}{\beta_\ell-\Delta_\ell-is_\ell}+e^{-i\xi_\ell} S\xi_\ell\right)$$

If $\beta_\ell \longrightarrow \infty$, then;

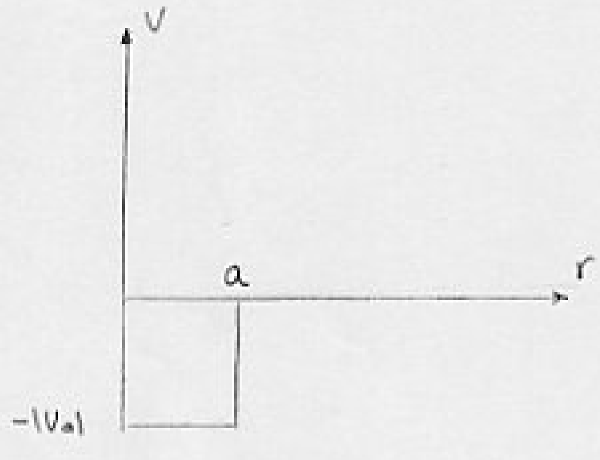
$$\delta_\ell \rightarrow \xi_\ell$$

This happens when we have a <u>hard-spherical</u> core of radius a, (the wave-func. <u>cannot penetrate</u> into the interior region at all).

$\longrightarrow$ ξ_ℓ : hard-sphere phase shifts.

$$\beta_\ell = \left(\frac{a\,R'_\ell}{R_\ell}\right)_{r=a} \longrightarrow \infty \quad \Longrightarrow R_\ell \rightarrow 0 \text{ at } r=a$$

Ex.- Square well, of range a, depth V_0;



$$R_\ell(r) = \frac{u_\ell(r)}{r} = j_\ell(k'r) \qquad r<a$$

where $\frac{\hbar^2 k'^2}{2m} = E + V_0$

$$\beta_\ell = k'a \frac{j'_\ell(k'a)}{j_\ell(k'a)}$$

For $l=0$, $j_0(z) = \frac{\Sigma z}{z}$, $\eta_0(z) = -\frac{\varsigma z}{z}$

$$\Delta_0 = ka \frac{j_0 j_0' + \eta_0 \eta_0'}{(j_0)^2 + (\eta_0)^2} = -1$$

$$s_0 = \frac{1}{ka[(j_0)^2 + (\eta_0)^2]} = ka$$

$$e^{2i\xi_0} = -\frac{j_0 - i\eta_0}{j_0 + i\eta_0} = -\frac{\frac{\Sigma ka}{ka} + i\frac{\varsigma ka}{ka}}{\frac{\Sigma ka}{ka} - i\frac{\varsigma ka}{ka}} = -\frac{\Sigma ka + i\varsigma ka}{\Sigma ka - i\varsigma ka}$$

$$e^{2i\xi_0} = -\frac{\Sigma^2 ka + 2i\Sigma ka\,\varsigma ka - \varsigma^2 ka}{\Sigma^2 ka + \varsigma^2 ka}$$

$$e^{2i\xi_0} = \varsigma 2ka - i\Sigma 2ka \rightarrow e^{2i\xi_0} = e^{-2ika}$$

$$\rightarrow \xi_0 = -ka$$

$$\beta_0 = k'a \frac{j_0'(k'a)}{j_0(k'a)} = k'a \frac{\frac{\varsigma k'a}{k'a} - \frac{\Sigma k'a}{(k'a)^2}}{\frac{\Sigma k'a}{k'a}}$$

$$\beta_0 = k'a \,\mathrm{ctg}\, k'a - 1$$

$$f_0 = \frac{1}{k} e^{i\delta_0} S\delta_0 = \frac{1}{k} e^{2i\xi_0}\left(\frac{S_0}{B_0 - \Delta_0 - iS_0} + e^{-i\xi_0} S\xi_0\right)$$

$$f_0 = \frac{1}{k} e^{-2ika}\left(\frac{k}{k'\cot k'a - ik} - e^{ika} Ska\right)$$

In the limit $E \to 0$, $k \to 0$, this gives a <u>nonvanishing</u> isotrope S-wave cross-section.

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1) S^2\delta_\ell$$

$$\sigma_0 = \frac{4\pi}{k^2} S^2\delta_0$$

$$\underset{k\to 0}{f_0} = \frac{1}{k'}\tan k'a - a \qquad (S ika \approx ka)$$

$$\frac{1}{k} e^{i\delta_0} S\delta_0 = \frac{1}{k'}\tan k'a - a$$

$$\to \frac{1}{k} S\delta_0 = \frac{1}{k'}(\tan k'a - a) e^{-i\delta_0}$$

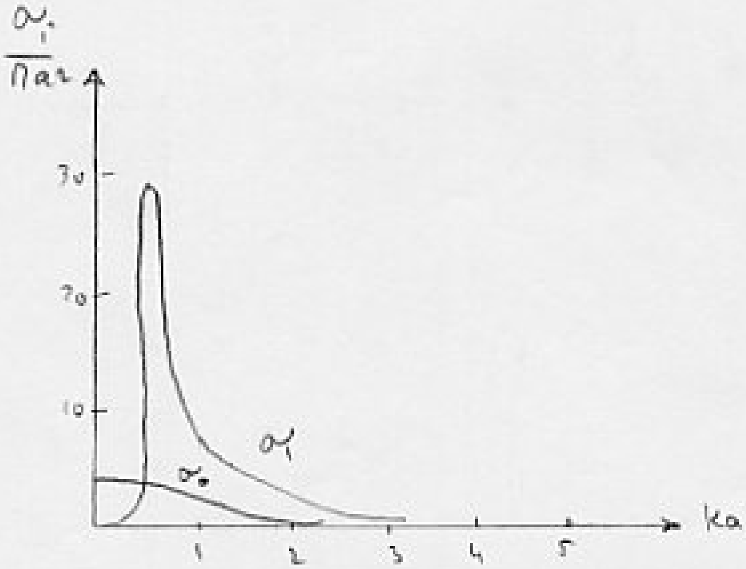
$$S\delta_0 = k\left(\frac{\tan k'a}{k'} - a\right) e^{-i\delta_0}$$

$$\sigma_0 = \frac{4\pi}{k^2} S^2\delta_0 = 4\pi\left(\frac{\tan k'a}{k'} - a\right)^2$$

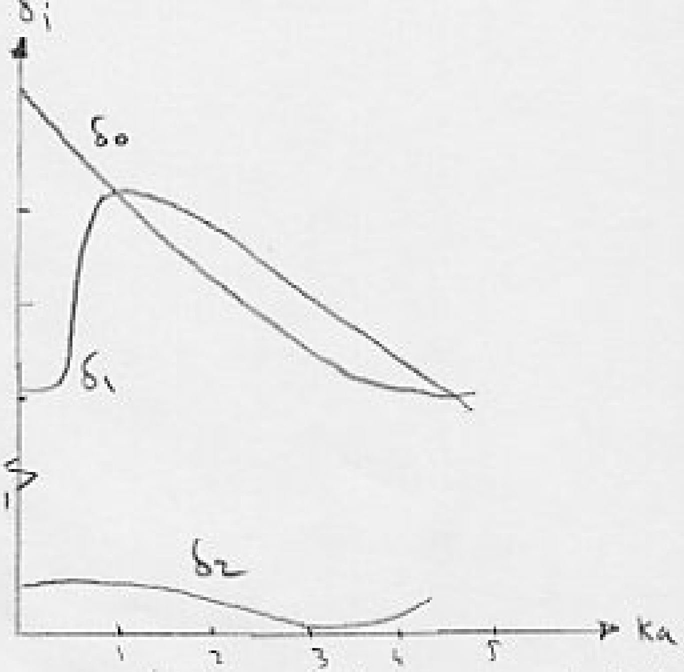
$$\sigma_0 = 4\pi a^2\left(\frac{\tan k'a}{k'a} - 1\right)^2$$

call $k' = k'_0 = \sqrt{\frac{2MV_0}{\hbar^2}}$ $\quad (E \to 0)$

$$\sigma_0 = 4\pi a^2 \left(\frac{\tan k' a}{k' a} - 1 \right)^2$$



Momentum-dep of the partial cross-section (σ_0 and σ_1) for S and P-wave, corresponding to the Phase shifts δ_0 and δ_1.



S, P and d-wave phase shifts ($\delta_0, \delta_1, \delta_2$) for scattering from a square-well of radius a with $k'a = \sqrt{\frac{2\mu V_0}{\hbar^2}}\, a = 6.2$

If, the quantities (ξ_ℓ, Δ_ℓ and s_ℓ) which characterize the external wave-func. vary slowly and smoothly with energy, a rapid change in δ_ℓ (hence σ_ℓ)

are usually attributed → to a strong energy-dep. of B_ℓ

$$e^{i\delta_\ell} \sin\delta_\ell = e^{2i\xi_\ell}\left(\frac{s_\ell}{\underset{\uparrow}{B_\ell - \Delta_\ell - i s_\ell}} + e^{-i\xi_\ell} \sin\xi_\ell \right)$$

If the rapid change of β_ℓ in small range of E can be represented by a linear approx.;

$$\beta_\ell(E) = c + bE$$

$$e^{2i(\delta_\ell-\xi_\ell)} = \frac{\beta_\ell - \Delta_\ell + i\Delta_\ell}{\beta_\ell - \Delta_\ell - i\Delta_\ell}$$

$$= \frac{c+bE-\Delta_\ell+is_\ell}{c+bE-\Delta_\ell-is_\ell} = \frac{E+\frac{c}{b}-\frac{\Delta_\ell}{b}+i\frac{s_\ell}{b}}{E+\frac{c}{b}-\frac{\Delta_\ell}{b}-i\frac{s_\ell}{b}} = \frac{E-E_0-i(\frac{\Gamma}{2})}{E-E_0+i(\frac{\Gamma}{2})}$$

where $E_0 = \frac{\Delta_\ell - c}{b}$ resonant energy $\quad \frac{1}{2}\Gamma = -\frac{s_\ell}{b}$ width of resonance

$$\begin{cases} \text{Since } \beta_\ell \text{ is decreasing-func. of } E \longrightarrow b<0 \\ \text{By def. } s_\ell > 0 \end{cases}$$

$$\longrightarrow \Gamma > 0$$

$$\frac{e^{2i(\delta_\ell-\xi_\ell)}-1}{2ik} = \frac{1}{k\,\text{cotg}(\delta_\ell-\xi_\ell)-ik}$$

$$\longrightarrow \frac{e^{2i(\delta_\ell-\xi_\ell)}-1}{2i} = \frac{1}{\text{cotg}(\delta_\ell-\xi_\ell)-i} \longrightarrow \text{cotg}(\delta_\ell-\xi_\ell) = \frac{2i}{e^{2i(\delta_\ell-\xi_\ell)}-1}+i$$

$$\tan(\delta_\ell-\xi_\ell) = -i\,\frac{e^{2i(\delta_\ell-\xi_\ell)}-1}{e^{2i(\delta_\ell-\xi_\ell)}+1}$$

$$\tan(\delta_\ell - \xi_\ell) = -i \frac{\frac{E-E_0-i(\Gamma/2)}{E-E_0+i(\Gamma/2)} - 1}{\frac{E-E_0-i(\Gamma/2)}{E-E_0+i(\Gamma/2)} + 1} = \frac{\Gamma}{2(E-E_0)}$$

These expressions are usefull if;

1- E_0 ad Γ are reasonably const.

2- Linear appr. $\beta_\ell(E) = c + bE$ is accurate over an energy range large compared with Γ.

Under this circumstance it can be seen;

$$e^{2i(\delta_\ell - \xi_\ell)} = \frac{E-E_0-i(\Gamma/2)}{E-E_0+i(\Gamma/2)}$$

that, the phase $2i(\delta_\ell - \xi_\ell)$, $\xrightarrow{\text{changes}}$ by 2Π over this energy range, (around resonance point).

$\xrightarrow{\text{Hence}}$ if $\xi_\ell \approx$ const (in this interval)

$\longrightarrow$ δ_ℓ changes by Π

$\longrightarrow$ $\sigma_\ell \sim \Sigma^2 \delta_\ell$ changes abruptly

Such sudden variations in the phase shifts are called resonances, with E_0 being the resonant energy, and Γ the width of resonance.

Now,

$$f_l = \frac{e^{i\delta_l} S\delta_l}{k}$$

$$f(\theta) = \sum_{l=0} (2l+1) \frac{e^{i\delta_l} S\delta_l}{k} P_l(\cos\theta)$$

$$f_l(\theta) = \frac{2l+1}{k} e^{2i\xi_l} \left(\frac{s_l}{\beta_l - \Delta_l - i s_l} + e^{-i\xi_l} Si\,\xi_l \right) P_l(\cos\theta)$$

Since $\frac{\Gamma}{2} = -\frac{s_l}{b}$, $E_0 = \frac{\Delta_l - c}{b}$, $\beta_l(E) = c + bE$

$$f_l(\theta) = \frac{2l+1}{k} e^{2i\xi_l} (\underbrace{\frac{\Gamma/2}{E_0 - E - i(\Gamma/2)}}_{\text{resonant part}} + \underbrace{e^{-i\xi_l} Si\,\xi_l}_{\text{non-resonant part}}) P_l(\cos\theta)$$

If the hard-sphere phase shifts ξ_l are negligible,

(as in low energies or high angular momenta)

$$\longrightarrow \sigma_l = \frac{4\pi}{k^2}(2l+1) S^2 \delta_l$$

$$Si\,\delta_l = e^{-i\delta_l} \underbrace{e^{2i\xi_l}}_{\simeq 1} \left(\frac{\Gamma/2}{E_0 - E - i(\Gamma/2)} + \underbrace{e^{-i\xi_l} Si\,\xi_l}_{\simeq 0} \right)$$

$$\sigma_l = \frac{4\pi(2l+1)}{k^2} \frac{\Gamma^2}{4(E-E_0)^2 + \Gamma^2}$$

if Γ: small ⟶ sharp max. centered at E_0.

7-9 Identical Particles and Scattering;

As an example:

Consider two identical spin integer (half-int.) $\underbrace{\text{charged particles}}_{\text{example}}$;

The wave-func. under the scattering by a $\underbrace{\text{Coulomb potential}}_{\text{example}}$, must be symmetric for the spinless particles.

The asymptotic form:

$\left\{ \begin{array}{l} \text{Recall:} \\ e^{ik\cdot x} + f(\theta)\frac{e^{ikr}}{r} \end{array} \right.$ unsymmetrized form

$$e^{ik\cdot x} + e^{-ik\cdot x} + [f(\theta,\varphi) + f(\pi-\theta,\varphi+\pi)]\frac{e^{ikr}}{r}$$ symmetric

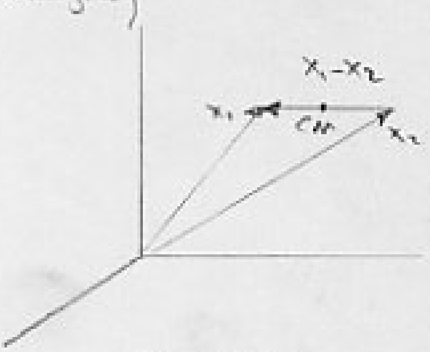
$$\rightarrow e^{ik\cdot x} + e^{-ik\cdot x} + [f(\theta + f(\pi-\theta)]\frac{e^{ikr}}{r}$$ for $V(\bar{r}) = V(r)$

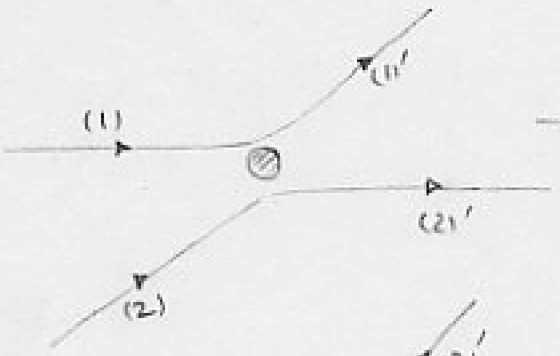
This is in the center of mass coord., where $\bar{X} = \bar{X}_1 - \bar{X}_2$

The position vector of the C.M. ($R = \frac{1}{2}(\bar{X}_1 + \bar{X}_2)$ for $m_1 = m_2$) is not affected by the interchange of $1 \rightleftarrows 2$.

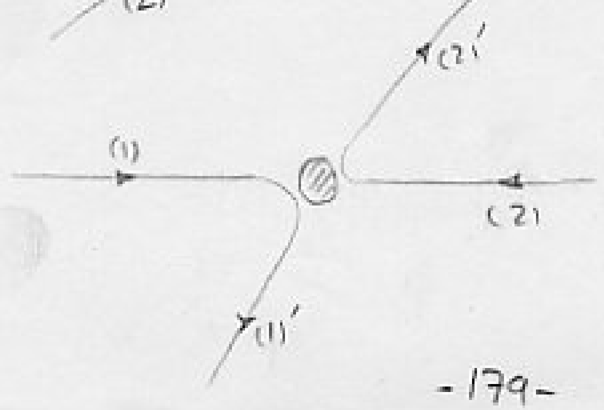
Therefore when the particles (1) and (2) are identical;

$$f \longrightarrow f = f(\theta) \pm f(\pi-\theta)$$ (sym. and antisym.)

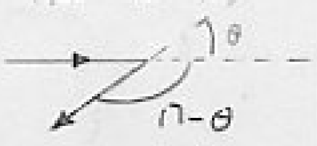




f(θ)
(for incident)



f(π-θ)
(for incident)



Remark:
By conservation of mom:
If one particle scatters through θ, the other goes in the dir π-θ -

i) For antisymmetric wave funcs.:
(spinless system)

$$\theta \longrightarrow \pi-\theta \xRightarrow{\text{Particle exchange}} \Psi \longrightarrow -\Psi$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)-f(\pi-\theta)|^2 = |f(\theta)|^2+|f(\pi-\theta|^2-2\,\mathrm{Re}\left[f(\theta)\,f^*(\pi-\theta)\right]$$

destructive interfrence at $\theta=\frac{\pi}{2}$, $\frac{d\sigma(\pi/2)}{d\Omega}=0$

ii) For symmetric wave funcs:
(spinless system)

$$\theta \longrightarrow \pi-\theta \Longrightarrow \Psi \longrightarrow \Psi$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)+f(\pi-\theta)|^2 = |f(\theta)|^2+|f(\pi-\theta)|^2+2\,\mathrm{Re}\left[f(\theta)\,f^*(\pi-\theta)\right]$$

Constructive interfrence at $\theta=\frac{\pi}{2}$ $\frac{d\sigma(\pi/2)}{d\Omega}=4\,|f(\frac{\pi}{2})|^2$

iii) Classical case;

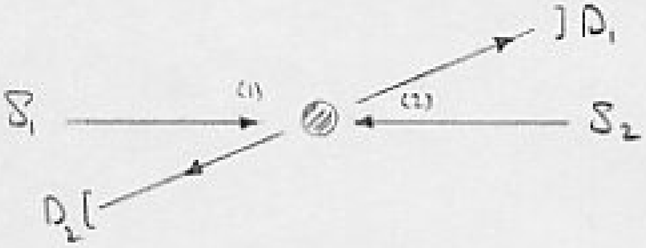
$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2+|f(\pi-\theta)|^2$$ (superposition principle)

at $\theta=\frac{\pi}{2}$; $\frac{d\sigma(\frac{\pi}{2})}{d\Omega}=2\,|f(\frac{\pi}{2})|^2$

Ex:- The unpolarized electron beam are colliding. If el.-el. scattering amplitude (indep of spin) is $f(\theta)$, find the total probability of scattering. (V indep. of spin)

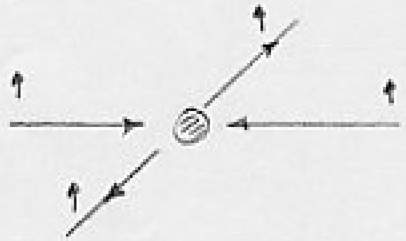
Occurance Probability	S_1	S_2	D_1	D_2	Probability
1/4	+	+	+	+	$\lvert f(\theta)-f(\pi-\theta)\rvert^2$
1/4	−	−	−	−	//
1/4	+	−	+	−	$\lvert f(\theta)\rvert^2$
			−	+	$\lvert f(\pi-\theta)\rvert^2$
1/4	−	+	−	+	$\lvert f(\theta)\rvert^2$
			+	−	$\lvert f(\pi-\theta)^2$

Remark:
Unpolarized beam can be analized to 4-different polarized beam.



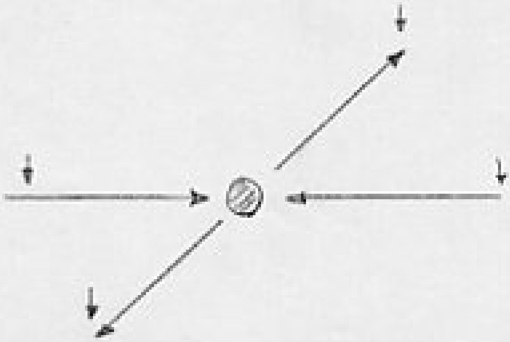
S: Source
D: Detector

I)

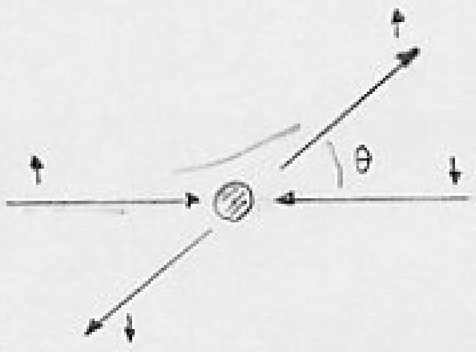


Spin up : ↑
" down : ↓

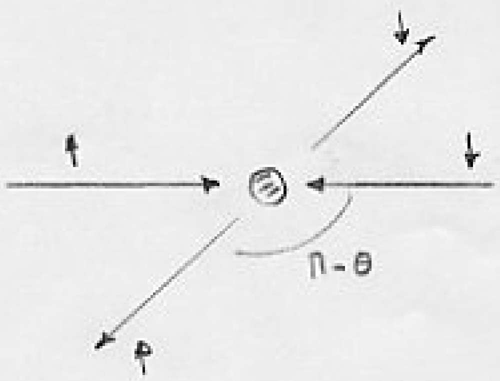
II)



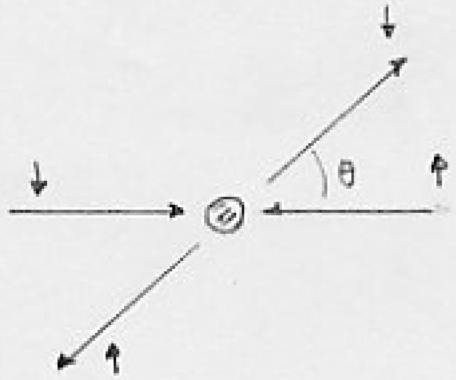
III)



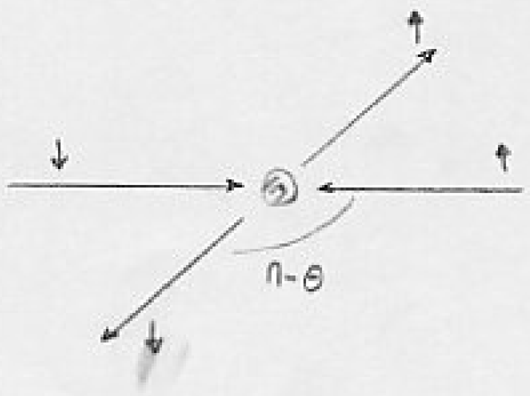
or;



IV)



or;



Total scattering probability: (spin $\frac{1}{2}$);

$$P = \sum P_i = \frac{1}{2}\left|f(\theta)-f(\pi-\theta)\right|^2 + \frac{1}{2}\left|f(\theta)\right|^2 + \frac{1}{2}\left|f(\pi-\theta)\right|^2$$
$$= \left|f(\theta)\right|^2 + \left|f(\pi-\theta)\right|^2 - \mathrm{Re}\left[f(\theta)f^*(\pi-\theta)\right]$$

For unpolarized beam of spin $=\frac{1}{2}$ electrons, with spin-indep. V, we have;

$$\chi(m_{s_1}, m_{s_2}) = \begin{cases} \left.\begin{array}{l} \alpha(1)\,\alpha(2) \\ \frac{1}{\sqrt{2}}\left(\alpha(1)\,\beta(2)+\beta(1)\,\alpha(2)\right) \\ \beta(1)\,\beta(2) \end{array}\right\} \text{spin triplet (sym.)} \\ \left.\frac{1}{\sqrt{2}}\left(\alpha(1)\,\beta(2)-\beta(1)\,\alpha(2)\right)\right\} \text{spin singlet (Anti-Sym)} \end{cases}$$

(spin part)

$$\Psi = \Psi_{space}\Psi_{spin}\Psi_{isospin}$$

Ψ must be antis-sym. for Fermions. Still some of $\Psi_{space}, \Psi_{spin}, \Psi_{isospin}$ can be sym.

Remark: He^4 nucleus is a boson (integer spin)

He^3 " " a fermion (half-int. spin)

Remark: If $\Psi = \Psi_{space}\chi_{spin}$ $\longrightarrow$

$\rightarrow$

$$\begin{cases} \Psi = \underset{\text{Anti-sym}}{\Psi_{space}}\,\underset{\text{Triplet}}{\chi_{spin}} \\ \Psi = \underset{\text{sym.}}{\Psi_{space}}\,\underset{\text{singlet}}{\chi_{spin}} \end{cases}$$

For unpolarized beam, we have the statistical contribution;

$(1\times\frac{1}{4})$ $\frac{1}{4}$: for spin-singlet $\left(\begin{array}{l}\text{spin part} = \text{Anti-sym}\\ \text{space} \;\text{"} = \text{sym.}\end{array}\right)$

$(3\times\frac{1}{4})$ $\frac{3}{4}$: " "-triplet $\left(\begin{array}{l}\text{spin part} = \text{sym.}\\ \text{space} \;\text{"} = \text{Anti-sym}\end{array}\right)$

$(S_1\cdot S_2 = 2\vec{S}^2 - \frac{3}{4}\hbar^2 I)$

Remark: $S_1\cdot S_2 = \begin{cases}\frac{\hbar^2}{4} & \text{triplet}\\ -\frac{3\hbar^2}{4} & \text{singlet}\end{cases}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left|f(\theta)+f(\pi-\theta)\right|^2 + \frac{3}{4}\left|f(\theta)-f(\pi-\theta)\right|^2 \longleftarrow \text{space parts}$$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - Re\left[f(\theta)\, f^*(\pi-\theta)\right]$$

We expect destructive <u>interference</u> at $\theta \approx \frac{\pi}{2}$

At $\theta = \frac{\pi}{2}$

$$\frac{d\sigma}{d\Omega} = |f(\pi/2)|^2 + |f(\pi/2)|^2 - |f(\tfrac{\pi}{2})|^2 = |f(\tfrac{\pi}{2})|^2$$

7-10 Symmetry Consideration in Scattering:

Suppose; U: sym. operation

$$[H_0, U] = 0 \qquad [V, U] = 0 \qquad (1)$$ V: Potential

i) If the symmetry op. is unitary;

$$UU^+ = I$$

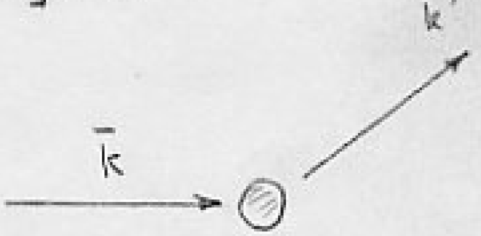
Due to (1) and;

Since; $T = V + V\dfrac{1}{E - H_0 + i\epsilon}V + \dots$

$\longrightarrow \quad UTU^+ = T \quad \longrightarrow \quad [T, U] = 0$

We define; $|\tilde{k}\rangle = U|k\rangle$

$|\tilde{k}'\rangle = U|k'\rangle$



Then; $\langle \tilde{k}' | T | \tilde{k} \rangle = \langle k' | U^+ T U | k \rangle$

$$= \langle k' | U^+ \underbrace{U T U^+}_{T} U | k \rangle = \langle k' | T | k \rangle$$

Ex. – As an example, we consider the specific case where U stands for the parity operator.

$$U = \Pi$$

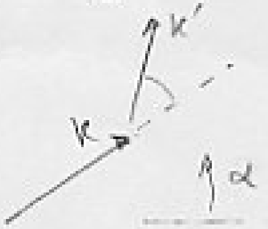
$$\Pi|K\rangle = |-K\rangle$$

If H_0 and V are invariant under parity; then

$$\langle -K'|\,T\,|-K\rangle = \langle K'|T|K\rangle \Rightarrow \Big\{$$

Also, the fact that T is diagonal in the $|E,l,m\rangle$ representation is a direct consequence of $[T, D(R)] = 0$

Notice that $\langle k'|T|k\rangle$ depends only on the *relative* orientation of $\bar{k}$ and $\bar{k}'$



ii) If the symmetry op. is *antiunitary*, (as in time-reversal) we must be more *careful*.

If $\quad [H_0, \Theta] = 0 \qquad \Theta H_0 \Theta^{-1} = H_0$

$\quad\;\; [V, \Theta] = 0 \qquad \Theta V \Theta^{-1} = V$

$$\longrightarrow \quad \Theta T \Theta^{-1} = T^{\dagger}$$

because: $\Theta T \Theta^{-1} = \Theta V \Theta^{-1} + \Theta V \Theta^{-1} \Theta \dfrac{1}{E - H_0 + i\epsilon} \Theta^{-1} \Theta V \Theta^{-1}$

$$= V + V \frac{1}{E - H_0 - i\epsilon} V = T^{\dagger}$$

We also recal;

$$\Theta|\alpha\rangle = |\tilde{\alpha}\rangle \qquad , \quad \Theta|\beta\rangle = |\tilde{\beta}\rangle$$

$$\langle\beta|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\beta}\rangle$$

Let us consider

$$|\alpha\rangle = T|k\rangle \; , \; \langle\beta| = \langle k'| \quad \rightarrow \langle\beta|\alpha\rangle = \langle k'|T|k\rangle$$

$$|\tilde{\alpha}\rangle = \Theta T|k\rangle = \Theta T\Theta^{-1}\Theta|k\rangle = T^{+}|-k\rangle$$

$$\rightarrow \langle\tilde{\alpha}| = \langle -k|T$$

$$|\tilde{\beta}\rangle = \Theta|k'\rangle = |-k'\rangle$$

$$\langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle -k|T|-k'\rangle$$

$$\langle\beta|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\beta}\rangle$$

$$\rightarrow \langle k'|T|k\rangle = \langle -k|T|-k'\rangle$$

Note that $\left\{ \begin{array}{l} k \rightleftarrows k' \\ k \rightarrow -k \; , \; k' \rightarrow -k' \end{array} \right.$

Now we combine the requirements of time-reversal and parity.

$$\left\{ \begin{array}{l} \langle -k'|T|-k\rangle \overset{\Pi}{=} \langle k'|T|k\rangle \\ \langle k'|T|k\rangle \overset{\Theta}{=} \langle -k|T|-k'\rangle \end{array} \right.$$

$$\rightarrow \langle k'|T|k\rangle \overset{\Theta}{=} \langle -k|T|-k'\rangle \overset{\Pi}{=} \langle k|T|k'\rangle \qquad (1)$$

Now, note;

$$f(k',k) = \sum_{n=1}^{\infty} f^{(n)}(k',k)$$

$$f^{(1)}(k,k') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \langle k'|V|k\rangle$$

$$f^{(2)}(k,k') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3 \langle k'|V\frac{1}{E-H_0+i\epsilon}V|k\rangle$$

$$\rightarrow f(k,k') = f(k',k)$$

which results in

$$\frac{d\sigma}{d\Omega}(k \rightarrow k') = \frac{d\sigma}{d\Omega}(k' \rightarrow k)$$ detailed balance equ.

It is more interesting to look at the analogue equ (1) when we have spin. Here we may characterize the initial <u>free-particle</u> ket by $|k, m_s\rangle$

We had $\Theta|j,m\rangle = (i)^{2m}|j,-m\rangle$ (j-int. or half-int.)

$$\langle k', m_s'|T|k, m_s\rangle \overset{\Theta}{=} (i)^{2(m_s-m_s')}\langle -k, -m_s|T|-k', -m_s'\rangle$$

$$\overset{\Pi}{=} (i)^{2(m_s-m_s')}\langle k, -m_s|T|k', -m_s'\rangle$$

For unpolarized initial states, we sum over the initial spin-states, and divide by (2S+1):

if the final polarization is not observed, we must sum over final states.

$$\overline{\frac{d\sigma}{d\Omega}}(k \to k') = \overline{\frac{d\sigma}{d\Omega}}(k' \to k)$$

The bar on the top of $\frac{d\sigma}{d\Omega}$ means:

We average over the initial spin states and sum over the final spin states

Coulomb Scattering

The Coulomb Potential is of the form;

$$V(r) = \frac{a}{r} = \frac{(z_1 e)(z_2 e)}{r}$$

The Schrödinger equ.;

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r) - E\right)\Psi(r) = 0$$

where $E = \frac{\hbar^2 k^2}{2m}$

$$\left(\nabla^2 + k^2 - \frac{2m z_1 z_2 e^2}{\hbar^2}\frac{1}{r}\right)\Psi(r) = 0$$

or $\left(\nabla^2 + k^2 - \frac{\alpha}{r}\right)\Psi(r) = 0$

where $\alpha = \frac{2m z_1 z_2 e^2}{\hbar^2}$ $\qquad m = \frac{m_1 m_2}{m_1 + m_2}$

In general; $V(r) = \frac{a}{r}$ is modified by the other interactions to:

$$V(r) = \frac{a}{r^{1+n}} \qquad (n>0) \qquad \text{for } r<R$$

Boundary conds. are diff. with those previously studied.

But due to shielding effects, we may approximate as;

$$\begin{cases} V(r) = \frac{\alpha}{r} & r<R \\ V(r) = 0 & r>R \end{cases}$$ R: effective range

Coulomb scattering can be solved exactly, Then we ignore the shielding effects.

We transfer to the parabolic coord. system by defining the following new variables;

$$\begin{cases} \xi = r+z \\ \eta = r-z \\ \varphi = \tan^{-1}\frac{y}{x} \end{cases}$$

$$(dl)^2 = \frac{\xi+\eta}{4\xi}(d\xi)^2 + \frac{\xi+\eta}{4\eta}(d\eta)^2 + \eta\xi(d\varphi)^2$$ surface element

$$dV = \frac{1}{4}(\xi+\eta)\,d\xi\,d\eta\,d\varphi$$

$$\begin{cases} \eta: \text{ ranges from } 0 \text{ to } \infty \\ \xi: \text{ " " " " "} \\ \varphi: \text{ " " " " } 2\pi \end{cases}$$

Laplacian in the parabolic coord.;

$$\nabla^2 = \left(\frac{4}{\xi+\eta}\right)\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)\right]+\frac{1}{\xi\eta}\frac{\partial^2}{\partial\varphi^2}$$

The schrödinger equ.;

$$\left(\frac{4}{\xi+\eta}\right)\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right)\right]+\frac{1}{\xi\eta}\frac{\partial^2\psi}{\partial\varphi^2}+\left[k^2-\frac{2\alpha}{\xi+\eta}\right]\psi=0$$

The equ. may be seperated by;

$$\psi = f(\xi)\, g(\eta)\, e^{im\varphi}$$

If the incident is along of the Z-dir., the only contribution comes from $m=0$;

$$\rightarrow \psi = f(\xi)\, g(\eta)$$

$$\left(\frac{4}{\xi+\eta}\right)\left\{g\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right)+f\frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right)\right\}+0+\left[k^2-\frac{2\alpha}{\xi+\eta}\right]gf=0$$

$$\left\{\frac{1}{f}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right)+\frac{1}{g}\frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right)\right\}+\left\{k^2-\frac{2\alpha}{\xi+\eta}\right\}\frac{\xi+\eta}{4}=0$$

$$\underbrace{\left(\frac{1}{f}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right)+\frac{k^2\xi}{4}\right)}_{-\beta}+\underbrace{\left(\frac{1}{g}\frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right)+\frac{k^2\eta}{4}\right)-\frac{\alpha}{2}}_{\beta}=0$$

indep. terms

$$\rightarrow \begin{cases} \frac{d}{d\xi}\left(\xi \frac{df}{d\xi}\right) + \left(\frac{1}{4}k^2\xi + B\right) f = 0 \\ \frac{d}{d\eta}\left(\eta \frac{dg}{d\eta}\right) + \left(\frac{1}{4}k^2\eta - B - \frac{\alpha}{2}\right) g = 0 \end{cases}$$

The sol. must behave like;

$$\Psi \sim e^{ikz} \quad \begin{cases} \text{as} \quad z \longrightarrow -\infty \left(\begin{cases} \eta \to \infty \\ \xi \to \infty \end{cases} \right) \\ \text{also} \quad \eta \longrightarrow \infty \quad \forall \xi \end{cases}$$

$$\longrightarrow \Psi \sim e^{ikz} = e^{ik\left(\frac{\xi-\eta}{2}\right)} = e^{ik\frac{\xi}{2}} e^{-ik\frac{\eta}{2}}$$

The cond. is satisfied if <u>B is pure imaginary</u>;

take; $B = -\frac{1}{2}ik$

The sol. of the first equ.;

$$\frac{d}{d\xi}\left(\xi \frac{df}{d\xi}\right) + \left(\frac{1}{4}k^2\xi - \frac{1}{2}ik\right) f = 0$$

is; $\quad f(\xi) = e^{ik\xi/2}$

For the second equ.;

$$\frac{d}{d\eta}\left(\eta \frac{dg}{d\eta}\right) + \left(\frac{1}{4}k^2\eta - \frac{1}{2}\alpha + \frac{1}{2}ik\right) g = 0$$

To satisfy the boundary cond. we set;

$$g(\eta) = e^{-\frac{1}{2}ik\eta} W(\eta)$$

we will look for the sols. such that;

$$W(\eta) \longrightarrow \text{const} \quad \text{as } \eta \to \infty \quad \forall z$$

Because, for example, at $z=0$ and $x^2+y^2 \longrightarrow \infty$ (i.e. $\eta \to \infty$) the incidence must be unaffected.

Substitution yields;

$$\eta \frac{d^2W}{d\eta^2} + (1-ik\eta)\frac{dW}{d\eta} - \frac{\alpha}{2} W = 0$$

This equ. is of the form;

$$x \frac{d^2F}{dx^2} + (b-x)\frac{dF}{dx} - aF = 0$$

This is satisfied by the confluent hypergeometric func. defined by the expression;

$$F(a,b,x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b)}{\Gamma(b+n)\,\Gamma(a)} \frac{x^n}{n!} \quad \text{(regular at the origin)}$$

Comparison yields;

$$a = -i\frac{\alpha}{2k} \qquad b=1 \qquad x = ik\eta \quad \left(\eta = \frac{x}{ik}\right)$$

Therefore:

$$\psi = c f(\xi) g(\eta) = c e^{ik(\frac{\xi-\eta}{2})} F(-\frac{i\alpha}{2k}, 1, ik\eta)$$

Now, again we return to the polar coord. system:

Since; $\frac{\xi-\eta}{2} = z$

and $\eta = r - z = r(1-\cos\theta)$

$$\psi^{+}_{k}(r) = c e^{ikz} F(-\frac{i\alpha}{2k}, 1, ikr(1-\cos\theta))$$

The asymptotic form of F for large x is:

$$F(a,b,x) \sim \frac{\Gamma(b)}{\Gamma(b-a)} e^{-a\ln(-x)} \left[1 + \frac{a(a+1-b)}{x} + \cdots\right] + \cdots$$

$$+ \frac{\Gamma(b)}{\Gamma(a)} e^{x+(a-b)\ln x} \left[1 + \frac{(1-a)(b-a)}{x} + \cdots\right] + \cdots$$

To evaluate this we note that:

$$\Gamma(b) = \Gamma(1) = 1$$

$$\Gamma(b-a) = \Gamma(1 + \frac{i\alpha}{2k})$$

Since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$$\Gamma(a) = \Gamma(-\frac{i\alpha}{2k}) = \frac{i\Gamma(1-\frac{i\alpha}{2k})}{\frac{\alpha}{2k}}$$

also;

$$\frac{a(a+1-b)}{x} = \frac{-\frac{\alpha^2}{4k^2}}{ikr(1-C\theta)}$$

$$\frac{(1-a)(b-a)}{x} = \frac{(1+\frac{i\alpha}{2k})(1+\frac{i\alpha}{2k})}{ikr(1-C\theta)} = \frac{(1+\frac{i\alpha}{2k})^2}{i2krS^2\frac{\theta}{2}}$$

Since; $\ln z = \ln|z| + i\arg z$

$$\rightarrow e^{-a\ln(-x)} = e^{\frac{i\alpha}{2k}\left[\ln(-ikr(1-C\theta))\right]}$$

$$\ln(-ikr(1-C\theta)) = \ln(kr(1-C\theta)) + i(-\tfrac{\pi}{2})$$

$$e^{-a\ln(-x)} = e^{\frac{\alpha\pi}{4k}}\, e^{\frac{i\alpha}{2k}\ln(kr(1-C\theta))}$$

Also;

$$e^{x+(a-b)\ln x} = e^{ikr(1-C\theta)+(-\frac{i\alpha}{2k}-1)\ln(ikr(1-C\theta))}$$

$$= e^{ikr(1-C\theta)-(\frac{i\alpha}{2k}+1)\left[\ln(kr(1-C\theta)+i\frac{\pi}{2})\right]}$$

$$= e^{ikr(1-C\theta)}\, e^{-\ln(kr(1-C\theta))}\, (-i)\, e^{\frac{-i\alpha}{2k}\ln(kr(1-C\theta))}\, e^{\frac{\alpha\pi}{4k}}$$

$$= -i\, e^{ikr}\, e^{-ikz}\, e^{-\ln(kr(1-C\theta))}\, e^{-\frac{i\alpha}{2k}\ln 2kr}\, e^{\frac{-i\alpha}{2k}\ln S^2\frac{\theta}{2}}\, e^{\frac{\alpha\pi}{4k}}$$

$$= -i\, e^{ikr}\, e^{-ikz}\, \frac{1}{2krS^2\frac{\theta}{2}}\, e^{\frac{-i\alpha}{2k}\ln 2kr}\, e^{\frac{-i\alpha}{2k}\ln S^2\frac{\theta}{2}}\, e^{\frac{\alpha\pi}{4k}}$$

Now, the wave func. can be written as:

$$\psi^{+}_{\vec{k}}(r) \sim C \frac{e^{\frac{\alpha\pi}{4k}}}{\Gamma(1+\frac{i\alpha}{2k})} \Big\{ \Big[1 - \frac{\alpha^2}{4ik^3 r(1-c\theta)} \Big] e^{i[kz+\frac{\alpha}{2k}\ln kr(1-c\theta)]} +$$

$$+ \frac{\Gamma(1+\frac{i\alpha}{2k})}{\Gamma(1-\frac{i\alpha}{2k})} \Big(-\frac{i\alpha}{2k}\Big) \Big\{ -i e^{ikr} e^{-\frac{i\alpha}{2k}\ln 2kr} e^{-\frac{i\alpha}{2k}\ln s^2\frac{\theta}{2}} \Big\} \cdot$$

$$\cdot \frac{1}{2kr\, s^2\frac{\theta}{2}} \Big[1 + \overset{\simeq 0}{\cancel{\frac{(1+\frac{i\alpha}{2k})^2}{i2kr\, s^2\frac{\theta}{2}}}} \Big] \Big\}$$

$\rightarrow$

$$\psi^{+}_{\vec{k}}(r) \sim C \frac{e^{\frac{\alpha\pi}{4k}}}{\Gamma(1+\frac{i\alpha}{2k})} \Big\{ \Big[1 - \frac{\alpha^2}{4ik^3 r(1-c\theta)} \Big] e^{i[kz+\frac{\alpha}{2k}\ln kr(1-c\theta)]} +$$

$$+ f^{c}_{k}(\theta) \frac{1}{r} e^{i[kr-\frac{\alpha}{2k}\ln 2kr]} \Big\}$$

where; $f^{c}_{k}(\theta) = -\frac{\alpha}{4k^2 s^2\frac{\theta}{2}} \frac{\Gamma(1+\frac{i\alpha}{2k})}{\Gamma(1-\frac{i\alpha}{2k})} e^{-i(\frac{\alpha}{2k})\ln s^2\frac{\theta}{2}}$

The asymptotic form is valid for scattering large r, <u>except</u> in the forward dir. ($c\theta=1$)

Differential Cross section;

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = \frac{\alpha^2}{16k^2 S^4\frac{\theta}{2}}$$

For identical particles; the wave func. must either be sym. or antisym.;

In the center of mass system;

$$\Phi(r,\theta) = \psi(r,\theta) \pm \psi(r,\pi-\theta)$$

The detector can not distinguish the scattered particles and recoiled particles;

Then;

$$\frac{d\sigma}{d\Omega} = |f(\theta) \pm f(\pi-\theta)|^2$$

For Coulomb scattering between two spinless charged particles, the two funcs. must be symmetric (+ sign.)
(spinless → bosons)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16k^2 S^4\frac{\theta}{2}}\left[1 + \underbrace{\tan^4\frac{\theta}{2} + 2\tan^2\frac{\theta}{2}\cos\left(\frac{\alpha}{2k}\ln\tan^2\frac{\theta}{2}\right)}\right]$$

The last term arises from the interfrence between the two amplitudes, and depends directly to the phase.