Chapter 4

Eigenfunctions and Eigenvalues

Consider time-dep. Schrödinger equ.:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \qquad (1)$$

Our attempt: To solve it by reducing it to a pair of <u>ordinary</u> differential equs in <u>one</u> variable.

$$\psi(x,t) = T(t)\,u(x) \qquad (2)$$

$$\rightarrow i\hbar\, u(x)\frac{dT(t)}{dt} = \left[-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)\,u(x)\right]T(t)$$

$$\rightarrow \underbrace{i\hbar\frac{dT(t)/dt}{T(t)}}_{\substack{t\text{-dep}\\ =E}} = \underbrace{\frac{-(\hbar^2/2m)(d^2u(x)/dx^2) + V(x)\,u(x)}{u(x)}}_{\substack{x\text{-dep}\\ =E}} \qquad (3)$$

$$\rightarrow \quad i\hbar\frac{dT(t)}{dt} = E\,T(t) \quad \rightarrow \quad T(t) = C e^{-iEt/\hbar} \qquad (4)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)\,u(x) = E\,u(x) \qquad (5)$$

Remark: Equ(1) describes the time-development of $\psi(x,t)$ while Equ(5) is an eigenvalue equ.

Now once again the operators;

Def: Most generally, an op. acting on a func., maps it onto another func.

Examples: $\mathcal{O} f(x) = f(x) + x^2$ $\qquad$ $\mathcal{O}$: operator

$\mathcal{O} f(x) = [f(x)]^2$ $\qquad$ $f(x)$: $f(x)$

$\mathcal{O} f(x) = f(3x^2+1)$

$\mathcal{O} f(x) = [df(x)/dx]^3$

$$\left\{ \begin{array}{l} \mathcal{O} f(x) = df(x)/dx - 2f(x) \\ \mathcal{O} f(x) = \lambda f(x) \end{array} \right. \qquad (6)$$

Linear ops: They have the property that; $\qquad$ L: Linear op.

$$L[f_1(x) + f_2(x)] = Lf_1(x) + Lf_2(x)$$ $\qquad$ c: complex number

and $$Lcf(x) = cLf(x) \qquad (7)$$

Remark: For antilinear ops: $Lcf(x) = c^* L f(x)$ (8)

Thus, in our list only the last two are linear ops.

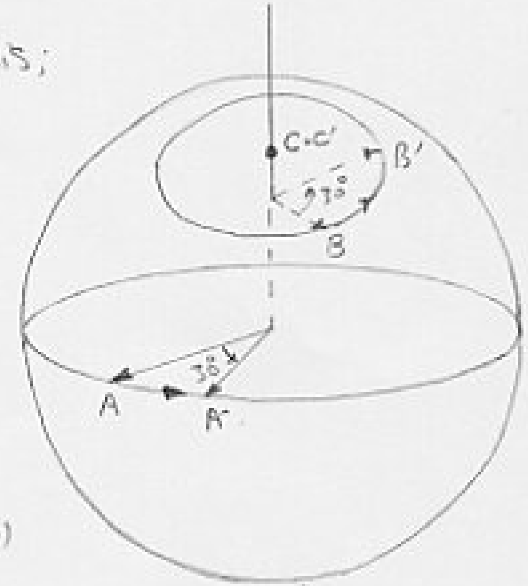
Ex.: $$Lf(x) = \frac{df(x)}{dx} - 2f(x) \qquad (9)$$

It is instructive to think of the funcs. as analogous to vectors in a 3-dim. space

$$O\,[\text{vector}] \longrightarrow [\text{another vector}]$$

In the special case that the vectors are all of unit length, an op. will transform one point on a unit sphere into another.

Ex.: O = Rotation of 30° about z-axis;



$$\begin{aligned} &OA \rightarrow A' \\ &OB \rightarrow B' \\ &OC \rightarrow C' = C \quad (\text{special class}) \end{aligned} \tag{10}$$

Equ(5) is like last item of equ(10)

$$H\,U_E(x) = E\,U_E(x) \tag{11}$$

eigenvalue   special class

The sol $U(x)$ depends on E, therefore we lable it with $E \rightarrow U_E(x)$

We shall see E,s can form $\begin{cases}\text{continuum} \\ \text{or discrete}\end{cases}$.

eigenfunc. corresponding to eigenvalue E of the op. H.

$$(2) \Rightarrow \quad \psi(x,t) = U_E(x)\, e^{-iEt/\hbar} \tag{12}$$

Since: equ (1) is a linear equ., a sum of sols. of the above form, with permissible values of E is also a sol.

$$\rightarrow \Psi(x,t) = \left(\sum + \int dE\right) C(E)\, U_E(x)\, e^{-iEt/\hbar} \qquad (13)$$

E: energy eigenvalues of the op. $H = \frac{P_{op}^2}{2m} + V(x)$

Remark: If $V = V(t)$ (explicitly) $\longrightarrow$ the seperation of the equ. would fail.

$\longrightarrow$ The energy <u>is not</u> a const. of motion.

The Eigenvalue Problem for a particle in a Box:

Consider equ. (5) with $\begin{cases} V(x) = 0 & \text{for } |x|<a \\ = \infty & \text{elsewhere} \end{cases} \qquad (14)$

<u>This implies</u> $\rightarrow$ $U(a) = 0 \quad , U(-a) = 0 \qquad (15)$

Inside the box; $\frac{d^2 U(x)}{dx^2} + \frac{2mE}{\hbar^2} U(x) = 0 \qquad (16)$

i) If $E < 0 \rightarrow \frac{d^2 U(x)}{dx^2} - K^2 U(x) = 0 \qquad (17)$

where $K^2 = \frac{2m|E|}{\hbar^2} \qquad (18)$

$\rightarrow$ General sol. $= A e^{kx} + B e^{-kx}$ (19)

With this sol. there is <u>no way</u> of satisfying the boundary conds. (15)

$\longrightarrow$ ii) $E > 0$ (must be)

(16) $\rightarrow$ $$\frac{d^2 u(x)}{dx^2} + k^2 u(x) = 0 \qquad (20)$$

where $k^2 = \frac{2mE}{\hbar^2}$ (21)

$\rightarrow$ Sols.: $\sin kx$, $\cos kx$

(15) $\longrightarrow$ $\begin{cases} \text{if } u^{(-)}(x) \sim \sin kx & \text{then } ka = n\pi \;, n=1,2,3,\dots \\ \text{" } u^{(+)}(x) \sim \cos kx & \text{" } ka = (n-\frac{1}{2})\pi \text{ "} \end{cases}$ (22)

(22) $\rightarrow$ $$E_n^{(-)} = \frac{n^2\pi^2\hbar^2}{2ma^2} \;,\quad E_n^{(+)} = \frac{[n-\frac{1}{2}]^2\pi^2\hbar^2}{2ma^2} \qquad (23)$$

$\rightarrow$ $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ when $L = 2a$ (for both cases) $n = 1,2,3,\dots$

One may show that;

$$u_n^{(-)}(x) = \frac{1}{\sqrt{a}} \sin \frac{n\pi x}{a} \;,\quad u_n^{(+)}(x) = \frac{1}{\sqrt{a}} \cos \frac{[n-\frac{1}{2}]\pi x}{a} \qquad (24)$$

The sols. have the property that;

$$\int_{-a}^{a} dx\, u_m^{(+)*}(x)\, u_n^{(+)}(x) = \int_{-a}^{a} dx\, u_m^{(-)*}(x)\, u_n^{(-)}(x) = \delta_{mn} \qquad (25)$$

$$\int_{-a}^{a} dx\, u_m^{(+)*}(x)\, u_n^{(-)}(x) = 0 \qquad (26)$$

Note that $u_1^{(+)}(x)$ is the ground state;

$$E_1^{(+)} = \frac{\pi^2 \hbar^2}{8ma^2} \qquad \text{(lowest energy)} \qquad (27)$$

Note also that; $\langle p \rangle = 0$

This can be shown by;

i) Direct calculation

or by symmetry argument:

ii) The sols. are real ($\cos kx$, $\sin kx$)

$$\text{So, } \underset{\downarrow\ \text{must be real}}{\langle p \rangle} = \underbrace{\frac{\hbar}{i}}\underbrace{\int \cdots}_{\downarrow\ \text{real}} \rightarrow \int \cdots = 0 \qquad (28)$$

$$\text{iii) } \langle p \rangle = \frac{\hbar}{i}\int_{-a}^{a} \text{even func} \frac{d}{dx}(\text{even func}) = \int_{-a}^{a} \text{odd func} = 0$$

$$\text{or } \langle p \rangle = \frac{\hbar}{i}\int_{-a}^{a} \text{odd func.} \frac{d}{dx}(\text{odd func}) = \int_{-a}^{a} \text{odd func} = 0 \qquad (29)$$

Now $\langle p^2 \rangle = ?$

This can be calculated for various sols by;

i) Direct calculation

ii) or since inside the box $p^2 = 2mE$ (30)

$$\rightarrow \quad \langle p^2 \rangle = 2m E_n^{(\pm)} \qquad (31)$$

Notice that; $\quad \underbrace{2a}_{\Delta x} \underbrace{\sqrt{\langle p^2 \rangle}}_{\Delta p} \sim 2\pi n \hbar > \hbar \qquad (32)$

$\Delta x \, \Delta p \geq \hbar$

It is the general feature that for higher eigenfuncs.

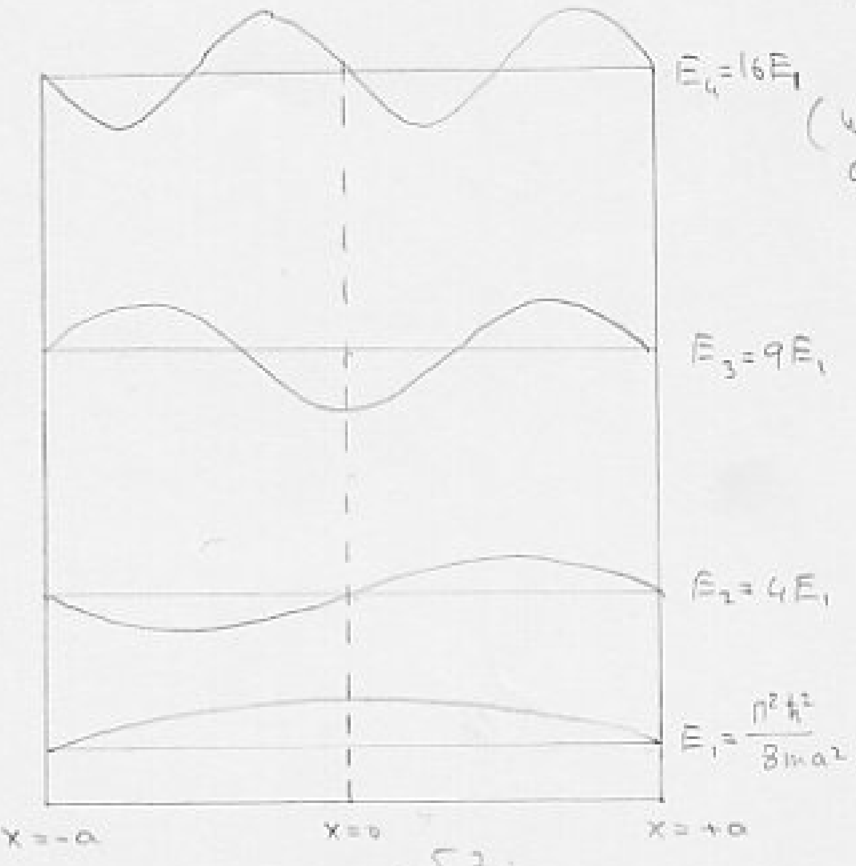
$\Delta x \Delta p$ grows with the eigenvalue.

Note also that, $\quad E \sim$ the number of nodes;

This is reasonable, since the kinetic energy is larger for a sol. with a larger curvature, a measure of which is $\frac{d^2 u}{dx^2}$.

$$-\frac{\hbar^2}{2m}\int dx\, u^*(x) \frac{d^2 u(x)}{dx^2} = 0 + \frac{\hbar^2}{2m}\int dx \frac{du^*}{dx}\frac{du}{dx} = \frac{\hbar^2}{2m}\int dx \left|\frac{du}{dx}\right|^2 \rightarrow \text{large} \qquad (33)$$

(when u has a lot of variation)



The Expansion Postulate:

An arbitrary func. $\psi(x)$, satisfying the boundary conds. $\begin{cases}\psi(a)=0\\ \psi(-a)=0\end{cases}$, can be constructed from our sols..

$$\psi(x) = \sum_{n=1}^{\infty}\left[A_n^{(+)} U_n^{(+)}(x) + A_n^{(-)} U_n^{(-)}(x)\right] \qquad \text{superposition} \qquad (34)$$

By the orthogonality conds;

$$(26) \to \int dx\, U_n^{(+)*}(x)\,\psi(x) = \sum_{m=1}^{\infty}\left[A_m^{(+)}\int U_n^{(+)*}(x)\, U_m^{(+)}(x)\,dx + A_m^{(-)}\int U_n^{(+)*}(x)\, U_m^{(-)}(x)\,dx\right] = A_n^{(+)} \qquad (35)$$

$$\to A_n^{(\pm)} = \int dx\, U_n^{(\pm)*}(x)\,\psi(x) \qquad (36)$$

$\psi(x,t) = ?$

$$(13) \to \psi(x,t) = \sum_{n=1}^{\infty}\left[A_n^{(+)} U_n^{(+)}(x)\, e^{-iE_n^{(+)}t/\hbar} + A_n^{(-)} U_n^{(-)}(x)\, e^{-iE_n^{(-)}t/\hbar}\right] \qquad (37)$$

Physical meaning of the coeffs. $A_n^{(\pm)}$:

Let us calculate $\langle H\rangle$;

we know $\quad H\, U_n^{(\pm)}(x) = E_n^{(\pm)}\, U_n^{(\pm)}(x) \qquad (38)$

$$\langle H \rangle = \int dx\, \psi^*(x) H \psi(x) =$$

$$= \int dx \left\{ \sum_{n=1}^{\infty} \left[A_n^{(+)*} u_n^{(+)*}(x) + A_n^{(-)*} u_n^{(-)*}(x) \right] \right\}$$

$$\cdot \left\{ \sum_{m=1}^{\infty} \left[E_m^{(+)} A_m^{(+)} u_m^{(+)}(x) + E_m^{(-)} A_m^{(-)} u_m^{(-)}(x) \right] \right\}$$

$$= \sum_{m=1}^{\infty} \left[E_m^{(+)} |A_m^{(+)}|^2 + E_m^{(-)} |A_m^{(-)}|^2 \right] \qquad \text{(by orthogonality)} \quad (39)$$

Similarly using: $\int dx\, \psi^*(x) \psi(x) = 1$

$$\longrightarrow \sum_{n=1}^{\infty} \left[|A_n^{(+)}|^2 + |A_n^{(-)}|^2 \right] = 1 \qquad (40)$$

Equ (39) and equ (40) strongly suggest that:

$|A^{(\pm)}|^2$ can be interpreted as the probability that a measurement of the <u>energy</u> for the arbitrary state yields the value $E_n^{(\pm)}$.

Question:

For what packet will the measurement always yield an energy (say) $E_K^{(+)}$ (an eigenvalue)?

Clearly this will be so only when:

$$|A_n^{(+)}|^2 = \delta_{nk} \quad \text{and} \quad |A_n^{(-)}|^2 = 0 \qquad (41)$$

i.e. when $\psi(x) = u_k^{(+)}(x)$

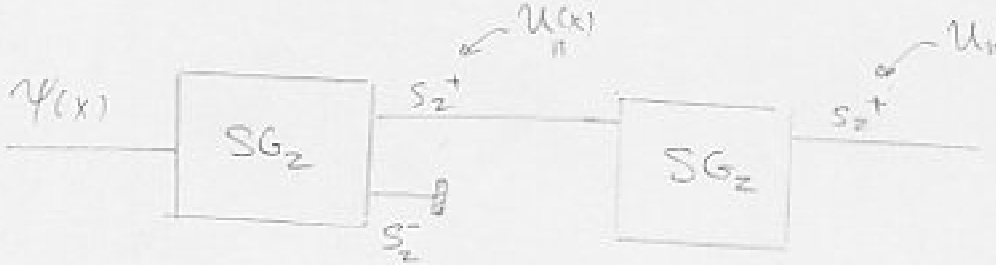
Important conclusion:

Suppose we have a general packet described by $\psi(x)$.

$$\xrightarrow[\text{results}]{\text{H-measurement}} E_n \quad \text{with probability } P(E_n) = \overbrace{\left|\int dx\, u_n^*(x)\, \psi(x)\right|^2}^{|A_n|^2} \qquad (41')$$

(where we have left off the $(\pm)$ label for generality). (See (36), (39))

Furthermore, after the measurement that has <u>yielded</u> the eigenvalue E_n, the state of the system is described by the eigen func. $u_n(x)$, since otherwise a repetition of the measurement <u>would not</u> necessarily give the same result, and the reproducibility of a measurement for a given system is essential for the measurement to have any meaning.



Parity:

If $\quad \psi(-x) = \psi(x) \quad$ then $\psi(x)$ is even

$\quad\;\; \psi(-x) = -\psi(x) \quad$ " " " odd $\qquad (42)$

Equ. (37) shows that if a wave packet is even (odd) at $t=0$, it remains so for all time.

Thus for our box which is symmetric about $x=0$, the evenness or oddness are time-indep.

Π op.: $\qquad \Pi\psi(x) = \psi(-x) \qquad (43)$

$\rightarrow \quad \Pi\psi^{(+)}(x) = \psi^{(+)}(x) \quad , \quad \Pi\psi^{(-)}(x) = -\psi^{(-)}(x) \quad$ (eigenvalue equs.) $\quad (44)$

The eigenvalues ± 1 are the only possible ones. Suppose we have:

$$\Pi u(x) = \lambda u(x) \quad \rightarrow \quad \Pi^2 u(x) = \Pi\lambda u(x) = \lambda^2 u(x) \qquad (45)$$

However $\quad \underset{\text{must be}}{\Pi^2 u(x) = u(x)} \quad$ (since two reflection should not change anything)

$$\rightarrow \lambda^2 = 1 \qquad \lambda = \pm 1 \qquad (46)$$

In our box example the funcs. $u^{(\pm)}(x)$ are simultaneously eigenfuncs of Π.

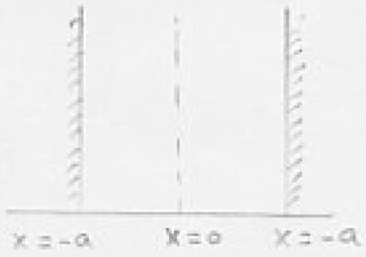
An arbitrary func. $\psi(x)$ can always be written as a sum of an even and an odd func.

$$\psi(x) = \frac{1}{2}\left[\psi(x) + \psi(-x)\right] + \frac{1}{2}\left[\psi(x) - \psi(-x)\right] \qquad (47)$$

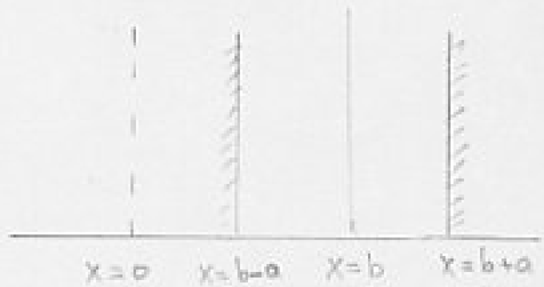
This is the general feature of the Hermitian ops: the eigenfuncs of any Hermitian op are said to form a complete set, in terms of which any func. can be expanded (H is another example).

Ex. $\langle \Pi \rangle = \text{real} \quad \forall\, \psi(x) \longrightarrow \Pi$ is Hermitian

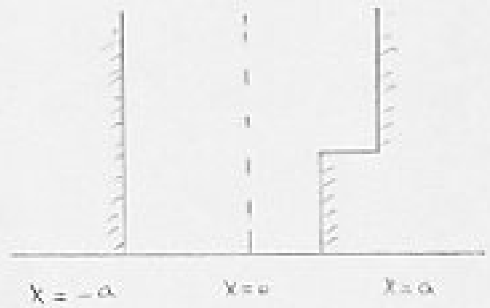
Remark:



sym. pot.



sym. pot



No symmetry under reflection (uneven)

Question:

Under what circumstances an <u>even</u> func. will <u>remain</u> even for all time?

Let $\psi(x,0) = \psi(-x,0) \equiv \psi^{(+)}(x)$ (48)

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = H\psi(x,t) \longrightarrow \Pi\left(i\hbar \frac{\partial \psi(x,t)}{\partial t}\right) = \Pi H \psi(x,t)$$

$$\longrightarrow i\hbar \frac{\partial}{\partial t} \Pi \psi(x,t) = \Pi H \psi(x,t) \quad (49)$$

Under special circumstances that $\Pi H \psi(x,t) = H \Pi \psi(x,t)$ (50)

which holds when $H(-x) = H(x)$ (51)

i.e. when $V(-x) = V(x)$ even, because $\frac{d^2}{dx^2}$ is even in

$$H = \frac{P_{op}^2}{2m} + V(x)$$

$$\longrightarrow i\hbar \frac{\partial}{\partial t}\left(\Pi \psi(x,t)\right) = H\left(\Pi \psi(x,t)\right) \quad (52)$$

Hence $\psi^{(+)}(x,t) = \frac{1}{2}\left[\psi(x,t) + \psi(-x,t)\right] = \frac{1}{2}(1+\Pi)\psi(x,t)$ (53) { Remark: use equ (47)

and $\psi^{(-)}(x,t) = \frac{1}{2}\left[\psi(x,t) - \psi(-x,t)\right] = \frac{1}{2}(1-\Pi)\psi(x,t)$

separately obey the Schrödinger equ. and <u>don't</u> mix, if the initial state is even (or odd) (due to 52).

The cond. for the time-independence of parity only holds if:

$$(\Pi H - H\Pi)\Psi(x,t) = 0 \longrightarrow [\Pi, H] = 0 \qquad (54)$$

In general: if $[A, H] = 0$ and $\frac{\partial A}{\partial t} = 0$

$\longrightarrow$ A: const of motion

Remark: If $V = V(x, t)$ $\longrightarrow$ Energy is not const. of motion

when V is t-dep., the separation of the equ. into an equ. for the time dependence and an energy eigen-value equ. is not possible.

Momentum Eigenfunc. and the Free Particle:

Let us solve the eigen-value equ for the mom. op.:

$$P_{op}\, U_p(x) = p\, U_p(x) \qquad \text{where } P_{op} = \frac{\hbar}{i}\frac{d}{dx} \qquad (55)$$

$$\longrightarrow \quad \frac{d\, U_p(x)}{dx} = \frac{ip}{\hbar} U_p(x) \qquad (56)$$

The sol. $\longrightarrow$ $U_p(x) = C e^{ipx/\hbar}$ $\qquad (57)$

p: real and continuous

$\longrightarrow$ P_{op} has continuous spectrum.

By analogy with (25) and (26);

$$\int dx\, u_{p'}^*(x)\, u_p(x) = |C|^2 \int dx\, e^{i(p-p')x/\hbar} = 2\pi |C|^2 \hbar\, \delta(p-p') \quad (58)$$

with the choice $u_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ (59)

$$\rightarrow \int_{-\infty}^{\infty} dx\, u_{p'}^*(x)\, u_p(x) = \delta(p-p') \quad (60)$$ orthogonality cond.

δ ↑ Dirac δ-func.

Analog of (34)

$$\psi(x) = \int_{-\infty}^{\infty} dp\, \varphi(p) \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \quad (61)$$

where $\varphi(p) = \int dx \left(\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\right)^* \psi(x)$ (62) (analog of (35)(36))

Acc. to the interpretation in (41'), $|\varphi(p)|^2$ gives the probability that a measurement of the momentum for an arbitrary packet $\psi(x)$ yields the eigenvalue p.

Now; for the free particle, $V(x)=0$;

$$H = \frac{p_{op}^2}{2m}$$

$$\frac{d^2 u(x)}{dx^2} + k^2 u(x) = 0 \quad (63)$$

where $K^2 = \frac{2mE}{\hbar^2}$ (6.4)

The sols. $\longrightarrow$ e^{ikx} , e^{-ikx} (6.4') or linear combination of these (for example $\cos kx$ and $\sin kx$).

The trouble with <u>all</u> of them is that they <u>are not</u> <u>square integrable</u> , Since;

$$\int_{-\infty}^{\infty} dx \, | A e^{ikx} + B e^{-ikx} |^2 \text{ diverges } \forall \text{ all } A, B$$

$\left\{ \text{For example: } B = 0 \rightarrow \int_{-\infty}^{\infty} dx \, |A e^{ikx}|^2 = |A|^2 \times | _{-\infty}^{\infty} \rightarrow \infty \right.$

There are 3-ways of getting arround this difficulty;

a) We may consider the prob. defined by (63) as the limiting case of a particle in a box with walls receding to <u>infinity</u>, that is $a \rightarrow \infty$

The sols. $u^{(-)}(x) = \frac{1}{\sqrt{a}} \sin \frac{n\pi x}{a}$ $(ka = n\pi)$ (65)

and $u^{(+)}(x) = \frac{1}{\sqrt{a}} \cos \frac{[n-\frac{1}{2}]\pi x}{a}$ $(ka = (n-\frac{1}{2})\pi)$

(even aside from the normalization factors $\frac{1}{\sqrt{a}}$) will become <u>trivial</u> $\left(\begin{matrix} \sin \rightarrow 0 \\ \cos \rightarrow 1 \end{matrix} \right)$ unless $n \rightarrow$ very large , so that;

$$\frac{n\pi}{a} = k \rightarrow \text{finite} \qquad \frac{(n-\frac{1}{2})\pi}{a} \simeq \frac{n\pi}{a} \rightarrow \text{finite}$$ (66)

We obtain the sols. $\frac{1}{\sqrt{a}}\sin kx$ and $\frac{1}{\sqrt{a}}\cos kx$. (67)

The $\frac{1}{\sqrt{a}}$ factors will <u>drop out</u> of the answer to any physical question that we may ask about the system.

Ex.: $\int_{-a}^{a}\left(\frac{1}{\sqrt{a}}\sin kx\right)^*\left(\frac{1}{\sqrt{a}}\sin kx\right)dx = \frac{1}{a}\int_{-a}^{a}\sin^2 kx\,dx = \frac{1}{a}\int_{-a}^{a}\frac{1}{2}(1-\cancel{\cos 2kx})\,dx$ odd

$= \frac{1}{a}\left[\frac{x}{2}\right]_{-a}^{a} \qquad \lim_{(a\to\infty)}\int = 1$

b) We may work with wave packets. A sol. of the form;

$$\psi(x) = e^{ikx} \qquad (68)$$

is a special case of (61) with

$$\Phi(p) = \sqrt{2\pi\hbar}\;\delta(p-k\hbar) \qquad (69)$$

(an infinitly peaked mom.-space dist.)

Suppose we replace $\Phi(p) \xrightarrow{\text{by}}$ a very sharply peaked func. $\sqrt{2\pi\hbar}\,g(p-k\hbar)$

i.e. $\delta(p-k\hbar) \longrightarrow g(p-k\hbar)$ (70)

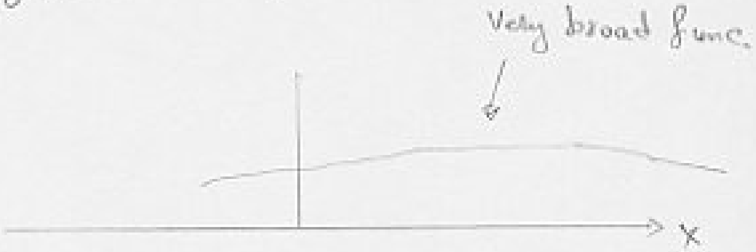
Then

$$e^{ikx} \xrightarrow[\text{by}]{\text{is replaced}} \psi(x) = \int dp\, e^{ipx/\hbar}\, g(\underbrace{p-k\hbar}_{q}) \qquad (71)$$

$$= e^{ikx}\int dq\, e^{iqx/\hbar}\, g(q)$$

i.e. $\psi(x) = e^{ikx} \cdot$ (a very broad func of x) (72)

We may make this func. so broad that it is essentially <u>const</u> over the region of physical interest.



Since $\Delta x \Delta p \simeq \hbar$

$$\rightarrow \Delta p \approx \frac{\hbar}{\text{Size of } x\text{-packet}} \qquad (73)$$

and if the denominator $\sim$ of macroscopic size

$\rightarrow \Delta p \simeq$ negligible

$\longrightarrow$ We thus satisfy the mathematical requirements without changing of the physics.

The wave packet description is actually the one that is <u>closest</u> to what really happens physically, since the way of preparing the initial state, for example, firing an electron gun, <u>can never</u>, in practice, create <u>an exact</u> momentum eigenstate.

c) The difficulty for a wave func. like e^{ikx} is that the particle is not confined to any region of space $\longrightarrow \rho(x,t) = 0$

Probability of finding particle anywhere

If we don't ask questions that involve the probability of finding the particle in any finite region no problem arise. Remark: $\rho(x,t) = |\psi(x,t)|^2 \sim \frac{1}{a}$, $a \to \infty$

One way of avoiding the normalization difficulty is to deal with the probability current (or flux);

$$j(x) = \frac{\hbar}{2mi}\left[\psi^*(x)\frac{d\psi(x)}{dx} - \frac{d\psi^*(x)}{dx}\psi(x)\right]$$

$$\psi(x) = Ce^{ipx/\hbar} \longrightarrow j(x) = |C|^2\frac{p}{m} \qquad (74)$$

$$\psi(x) = Ce^{-ipx/\hbar} \longrightarrow j(x) = -|C|^2\frac{p}{m} \qquad (75)$$

In One-dim. Case;

Suppose:
density of particles = 1 Particle/Cm
velocity $v = \frac{p}{m}$

$$\text{Flux} = J(x) = v = \frac{p}{m} \quad (76)$$

(i.e. the number of particles crossing a point x_0 per second)

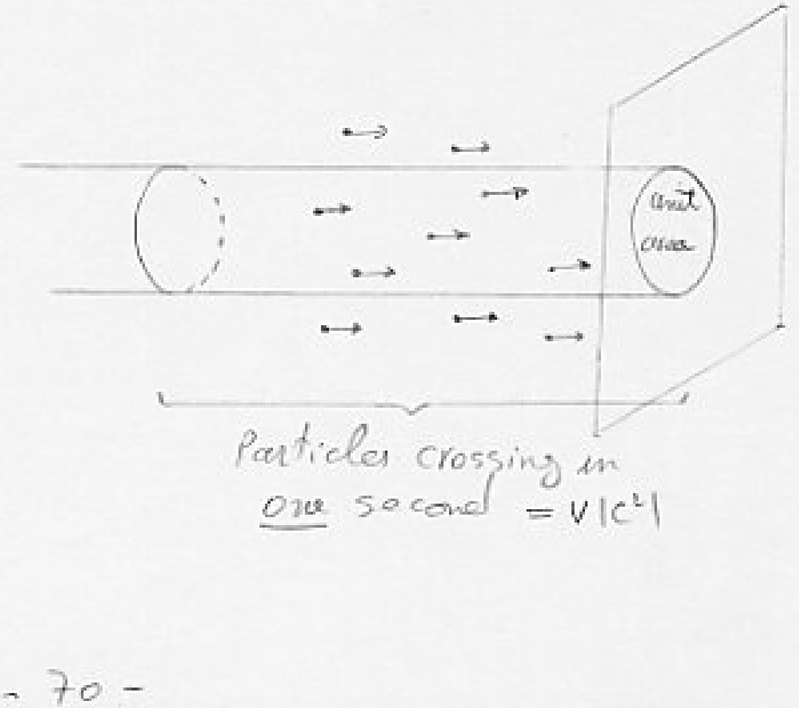
(74) or (75) and (76) $\longrightarrow |C|^2$ = density of particles per cm

Thus for example $u_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ represent particles with density $\frac{1}{2\pi\hbar}$ per cm

In 3-dim. : with $u_{\bar{p}}(\bar{r}) = C e^{i\bar{p}\cdot\bar{r}/\hbar}$

$$\rightarrow \bar{J}(x) = |C|^2 \frac{\bar{p}}{m}$$

This corresponds to a flow of particles with density $|C|^2$ per cm^3 crossing a unit area perpendicular to $\bar{p}$



The energy eigenvalue equ. (63) has two indep. sols.;

e^{ikx} and e^{-ikx}

or $\cos(kx)$ and $\sin(kx)$ (equivalently)

In contrast to the problem of a particle in a box, there are <u>two sols.</u> that have the same energy.

This happens quite frequently:

There may be <u>more than one</u> independent eigenfunc. that corresponds to the <u>same</u> eigenvalue of a Hermitian op.
(We say there is <u>degeneracy</u>).

The two sols. in our example are orthogonal;

$$\int_{-\infty}^{\infty} dx \left(e^{-ikx}\right)^{*} e^{ikx} = \int_{-\infty}^{\infty} dx\, e^{2ikx} = 0$$

Remark: $\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ikx}$

or $\displaystyle\int_{-\infty}^{\infty} dx \underbrace{\sin kx \cos kx}_{\text{odd}} = 0$ for $k \neq 0$

Question?
What distinguishes the two degenerate eigenfuncs?

For the set $\{e^{-ikx}, e^{ikx}\}$, the difference is that, they are eigenfuncs of the mom. op.:

$$P_{op}\, e^{\pm ikx} = \frac{\hbar}{i}\frac{d}{dx} e^{\pm ikx} = \pm \hbar k\, e^{\pm ikx}$$

corresponding to different eigenvalues of the mom.

Similarly the pair of $\{\cos kx, \sin kx\}$ are eigenfuncs. of the parity op. corresponding to different eigenvalues,

$$\Pi \cos kx = + \cos kx$$
$$\Pi \sin kx = - \sin kx$$

In both cases, what differentiates the degenerate eigenfuncs, is that they are simultaneous eigenfuncs of another Hermitian op.

Note that:

$$[P_{op}, H] = 0 \quad , \quad [\Pi, H] = 0 \qquad \left(H = \frac{P_{op}^2}{2m}\right)$$

Remark: $[P_{op}, \Pi] \neq 0$ $\qquad$ (because $\frac{\hbar}{i}\frac{d}{dx} \xrightarrow{\Pi} \frac{-\hbar}{i}\frac{d}{dx}$)

$\rightarrow$ They don't have simultaneous eigenfuncs.