Chapter 3

Theory of Angular Momentum

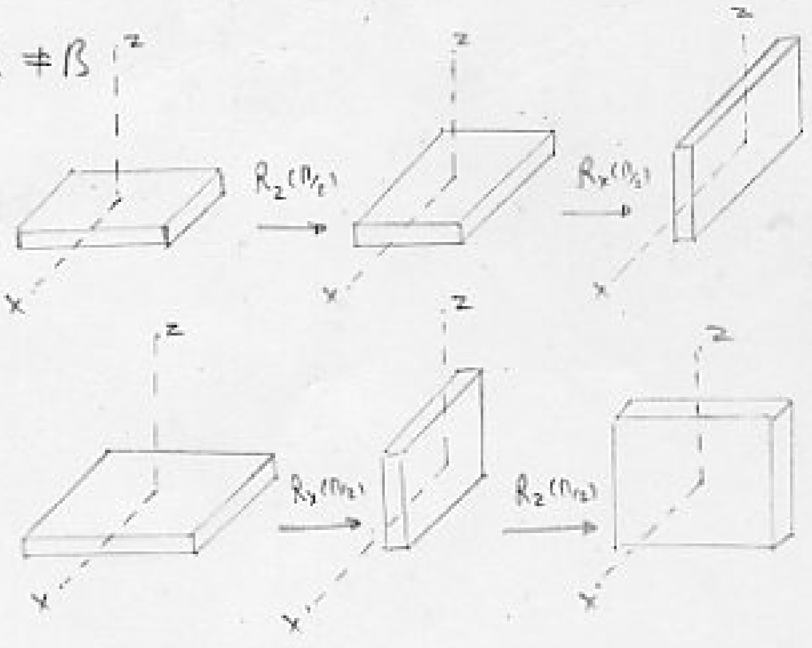
3-1 Rotations and angular Momentum Commutation Relations

Finite Versus Infinitesimal Rotations

From the elementary physics:

$$[R_\alpha(\theta), R_\alpha(\varphi)] = 0 \qquad \alpha\text{: indicate the axis}$$

while $[R_\alpha(\theta), R_\beta(\varphi)] \neq 0 \quad \alpha \neq \beta$



Ex.

$$[R_z(\tfrac{\pi}{6}), R_z(\tfrac{\pi}{3})] = 0$$

but $[R_z(\pi/2), R_x(\pi/2)] \neq 0$

In 3-dim.:

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \\ R \\ \\ \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad RR^T = R^T R = I$$

orthogonal

$$\sqrt{V_x^2+V_y^2+V_z^2} = \sqrt{V_x'^2+V_y'^2+V_z'^2} \quad \text{by the orthogonality}$$

Let us consider a rotation about the z-axis by angle φ.

The <u>convention</u> we <u>follow</u>:

The <u>rotation</u> operation affects a physical system <u>itself</u>, while the coord. axes remain <u>unchanged</u>.

φ : positive in counterclockwise in the xy-plane, as viewed from the positive z-side.

$$\rightarrow \quad R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(active rotation)}$$

For infinitesimal rotations:

$$R_z(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

likewise;

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix}, \quad R_y(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix}$$

$$R_x(\epsilon)R_y(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}, \quad R_y(\epsilon)R_x(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}$$

$$R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ϵ^n, $(n>2)$ terms are ignored

$$= R_z(\epsilon^2) - I$$

If ϵ^2 and higher terms are ignored $\longrightarrow$ The infinitesimal rotations about different axes do commute.

Ex.

We also have $\quad I = R_{\substack{\text{any}\\\text{axis}}}(0)$

$$\longrightarrow R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) = R_z(\epsilon^2) - R_{\text{any}}(0)$$

Infinitesimal Rotations in Q.M. :

Since we use <u>active rotation</u> $\longrightarrow |\alpha\rangle_R \underset{\text{different}}{\overset{\text{look}}{\neq}} |\alpha\rangle$

rotated ↑ ... original ↑

Given $R = \begin{pmatrix} .. & .. & .. \\ .. & .. & .. \\ .. & .. & .. \end{pmatrix} \xrightarrow{\text{we associate}} \mathcal{D}(R)$ op. in the appropriate ket space

such that $|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle$ ($\mathcal{D}$: Drehung = rotation)

Note that $R \xrightarrow{\text{acts on}}$ column vector (with 3-components)

where $\mathcal{D}(R) \xrightarrow{\text{"}}$ state vector (in ket space)

Matrix representation of $\mathcal{D}(R) \xrightarrow{\text{depends on}}$ the dimensionality N of the particular ket space

<u>Ex.</u> For spin $\frac{1}{2} \longrightarrow N = 2 \longrightarrow \mathcal{D}(R) = \begin{pmatrix} .. & .. \\ .. & .. \end{pmatrix}$

For spin 1 $\longrightarrow N = 3 \longrightarrow \mathcal{D}(R) = \begin{pmatrix} .. & .. & .. \\ .. & .. & .. \\ .. & .. & .. \end{pmatrix}$

To construct $\mathcal{D}(R)$, we examine first its properties under an infinitesimal rotation.

As we faced in translation and time evolution cases, the appropriate infinitesimal ops. could be written as;

$$U_\epsilon = I - iG\epsilon \qquad (\text{where } G^\dagger G = I)$$

specifically; $G \to \frac{p_x}{\hbar} \qquad \epsilon \to dx' \qquad$ (in translation)

and $\qquad G \to \frac{H}{\hbar} \qquad \epsilon \to dt \qquad$ (in time-evolution)

Now from cl. M., we know;

L: the generator of rotation (angular mom.)

Therefore $\longrightarrow$ we define J_k: angular mom. op. about kth axis

Therefore for an infinitesimal rotation;

$$G \to \frac{J_k}{\hbar}, \quad \epsilon \to d\varphi$$

$$\mathcal{D}(\hat{k}, d\varphi) = I - i\frac{J_z}{\hbar}d\varphi$$

More generally $\mathcal{D}(\hat{n}, d\varphi) = I - i\left(\frac{J\cdot\hat{n}}{\hbar}\right)d\varphi$

If $J = J^+$ $\longrightarrow \begin{cases} \mathcal{D}(\hat{n}, d\varphi)\,\mathcal{D}^+(\hat{n}, d\varphi) = I \\ \text{and } \mathcal{D}(\hat{n}, d\varphi) \to I \\ \quad \text{as } d\varphi \to 0 \end{cases}$

Remark: In Cl. M.:

$L = \bar{r} \times \bar{p}$ is the generator of rotation

In Q.M.:

J is the generator of rotation

J may be $J = L$ or $J = S$ or $J = L + S$ or ...

<u>Finite</u> rotation:

$$\mathcal{D}_z(\varphi) = \lim_{N\to\infty}\left[I - \left(\frac{J_z}{\hbar}\right)\left(\frac{\varphi}{N}\right)\right]^N = e^{-\frac{iJ_z\varphi}{\hbar}}$$

$$= I - \frac{iJ_z\varphi}{\hbar} - \frac{J_z^2\varphi^2}{2\hbar^2} + \dots$$

We remarked that:

$\forall\ R$ op. $\exists\ \mathcal{D}(R)$ (in ket space)

We further postulate that $\mathcal{D}(R)$ has the same group <u>properties</u> as R:

$$R\,I = R \longrightarrow \mathcal{D}(R)\,I = \mathcal{D}(R) \qquad \text{Identity}$$

$$R_1 R_2 = R_3 \longrightarrow \mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_3) \qquad \text{closure}$$

$$R R^{-1} = I \longrightarrow \mathcal{D}(R)\mathcal{D}^{-1}(R) = I \qquad \text{Inverses}$$

$$R^{-1} R = I \longrightarrow \mathcal{D}^{-1}(R)\mathcal{D}(R) = I$$

$$R_1(R_2 R_3) = (R_1 R_2) R_3 = R_1 R_2 R_3$$

$$\rightarrow \mathcal{D}(R_1)\left[\mathcal{D}(R_2)\mathcal{D}(R_3)\right] = \left[\mathcal{D}(R_1)\mathcal{D}(R_2)\right]\mathcal{D}(R_3) \qquad \text{Associativity}$$

$$= \mathcal{D}(R_1)\mathcal{D}(R_2)\mathcal{D}(R_3)$$

Commutation Relations

Similar to $[R_x(\epsilon), R_y(\epsilon)] = R_z(\epsilon^2) - I$

$$\rightarrow [\mathcal{D}(R_x(\epsilon)), \mathcal{D}(R_y(\epsilon))] = \mathcal{D}(R_z(\epsilon^2)) - I$$

$$\left(I - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{\hbar^2}\right)\left(I - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{\hbar^2}\right) -$$

$$- \left(I - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{\hbar^2}\right)\left(I - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{\hbar^2}\right) = \left(I - \frac{iJ_z\epsilon^2}{\hbar}\right) - I$$

Terms of order ϵ automatically drop out.

Equating terms of order ϵ^2 on both sides:

$$\rightarrow [J_x, J_y] = i\hbar J_z$$

In general $[J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k$ Fundamental commutation relations of ang. mom.

In general when the generator of infinitesimal transformations do not commute $\longrightarrow$ the corresponding group of operations is said to be non-Abelian

$\longrightarrow$ The rotation group in 3-dims. is non-Abelian

But since $[P_i, P_j] = 0 \quad \forall i, j$

$\longrightarrow$ The translation group in 3-dims. is Abelian.

In obtaining $[J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k$ we have used the following concepts:

i) J_k is the generator of of rotation about the kth axis

ii) Rotation about different axes fail to commute.

3.2 Spin $\frac{1}{2}$ Systems and Finite Rotations:

Rotation Op. for Spin $\frac{1}{2}$ $(N = 2)$

One may check that; the ops.;

$$S_x = \frac{\hbar}{2}\left\{|+\rangle\langle -| + |-\rangle\langle +|\right\}$$

$$S_y = \frac{i\hbar}{2}\left\{-|+\rangle\langle -| + |-\rangle\langle +|\right\}$$

$$S_z = \frac{\hbar}{2}\left\{|+\rangle\langle +| - |-\rangle\langle -|\right\}$$

satisfy $\quad [S_i, S_j] = i\hbar\epsilon_{ijk} S_k \qquad (J \to S)$

Consider a rotation by a finite angle φ about the z-axis

let $|\alpha\rangle$: the ket of spin $\frac{1}{2}$ before rotation

$$\to |\alpha\rangle_R = \mathcal{D}_z(\varphi)|\alpha\rangle$$

with $\mathcal{D}_z(\varphi) = e^{-\frac{iS_z\varphi}{\hbar}}$

To see this op. really rotates the physical system, let us look at its effect on $\langle S_x\rangle$.

$$\langle\alpha|S_x|\alpha\rangle \xrightarrow{\mathcal{D}} {}_R\langle\alpha|S_x|\alpha\rangle_R = \langle\alpha|\mathcal{D}_z^\dagger(\varphi)S_x\mathcal{D}_z(\varphi)|\alpha\rangle$$

$$e^{\frac{iS_z\varphi}{\hbar}} S_x e^{-\frac{iS_z\varphi}{\hbar}} = ?$$

Derivation 1 –

$$e^{\frac{iS_z\varphi}{\hbar}}\left(\frac{\hbar}{2}\right)\left[|+\rangle\langle-| + |-\rangle\langle+|\right]e^{-\frac{iS_z\varphi}{\hbar}}$$

$$= \frac{\hbar}{2}\left\{e^{i\varphi/2}|+\rangle\langle-|e^{i\varphi/2} + e^{-i\varphi/2}|-\rangle\langle+|e^{-i\varphi/2}\right\}$$

$$= \frac{\hbar}{2}\left\{\left[|+\rangle\langle-| + |-\rangle\langle+|\right]\cos\varphi + i\left[|+\rangle\langle-| - |-\rangle\langle+|\right]\sin\varphi\right\}$$

$$= S_x\cos\varphi - S_y\sin\varphi$$

Derivation 2 -

$$e^{iB\lambda} A e^{-iB\lambda} = A + i\lambda [B,A] + \left(\frac{i^2\lambda^2}{2!}\right) [B[B,A]] + \cdots + \left(\frac{i^n\lambda^n}{n!}\right)[B,[B,[\cdots[B,A]]\cdots] + \cdots$$

$B = B^+$
λ: real

$$\Rightarrow e^{\frac{iS_z\varphi}{\hbar}} S_x e^{\frac{-iS_z\varphi}{\hbar}} = S_x + \left(\frac{i\varphi}{\hbar}\right)\underbrace{[S_z, S_x]}_{i\hbar S_y}$$

$$+ \left(\frac{1}{2!}\right)\left(\frac{i\varphi}{\hbar}\right)^2 \underbrace{[S_z, \underbrace{[S_z, S_x]}_{i\hbar S_y}]}_{\hbar^2 S_x} + \left(\frac{1}{3!}\right)\left(\frac{i\varphi}{\hbar}\right)^3 \underbrace{[S_z, \underbrace{[S_z,[S_z,S_x]]}_{\hbar^2 S_x}]}_{i\hbar^3 S_y} + \cdots$$

$$= S_x \left[I - \frac{\varphi^2}{2!} + \cdots\right] - S_y\left[\varphi - \frac{\varphi^3}{3!} + \cdots\right] = S_x \cos\varphi - S_y \sin\varphi$$

This method can be used in general case with N; arbitrary.

So; $\longrightarrow \langle S_x\rangle \equiv \langle\alpha| S_x |\alpha\rangle \xrightarrow{D_z(\varphi)} {}_R\langle\alpha| S_x |\alpha\rangle_R = \langle S_x\rangle \cos\varphi - \langle S_y\rangle \sin\varphi$

Similarly: $\langle S_y\rangle \xrightarrow{D_z(\varphi)} \langle S_y\rangle \cos\varphi + \langle S_x\rangle \sin\varphi$

But $\langle S_z\rangle \xrightarrow{D_z(\varphi)} \langle S_z\rangle$

These are quite reasonable.

i.e., under $|\alpha\rangle \xrightarrow{D_z(\varphi)} |\alpha\rangle_R = D_z(\varphi)|\alpha\rangle$

$$\longrightarrow \qquad \langle S_k\rangle \longrightarrow \sum_l R_{kl}\langle S_l\rangle$$

(like a classical vector under rotation)

From derivation 2 it is clear that this property is not restricted to spin $\frac{1}{2}$; thus in general:

$$\langle J_k\rangle \longrightarrow \sum_l R_{kl}\langle J_l\rangle$$

Later we will see that relations of this kind can be further generalized to any vector op. .

A surprize!

Consider the ket $|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$

under the rotation $D_z(\varphi)$:

$$e^{-\frac{iS_z\varphi}{2}}|\alpha\rangle = e^{-\frac{i\varphi}{2}}|+\rangle\langle+|\alpha\rangle + e^{\frac{i\varphi}{2}}|-\rangle\langle-|\alpha\rangle$$

if $\varphi = 2\pi$; $\quad |\alpha\rangle \xrightarrow{D_z(2\pi)} -|\alpha\rangle$

$\varphi = 4\pi$, $\quad |\alpha\rangle \xrightarrow{D_z(4\pi)} |\alpha\rangle$

Note that the minus sign disappears for the expectation value of S.

Spin Precession Revisited

$$H = -\left(\frac{e}{mc}\right) S \cdot B = \omega S_z \qquad \text{for } \bar{B} = B\hat{z}$$

where $\omega \equiv \frac{|e|B}{mc}$

$$\mathcal{U}(t,0) = e^{-\frac{iHt}{\hbar}} = e^{\frac{-iS_z\omega t}{\hbar}} \qquad \text{time-evolution op.}$$

Comparing this with the rotation op. we see that they are the same, if we set $\varphi = \omega t$

In this manner we see immediately why this Hamiltonian causes spin precession.

Using the results of rotation;

$\varphi = \omega t$

$$\rightarrow \langle S_x \rangle_t = \langle S_x \rangle_{t=0} \cos\omega t - \langle S_y \rangle_{t=0} \sin\omega t$$

$$\langle S_y \rangle_t = \langle S_y \rangle_{t=0} \cos\omega t + \langle S_x \rangle_{t=0} \sin\omega t$$

$$\langle S_z \rangle_t = \langle S_z \rangle_{t=0}$$

(comparable with the eqns. of p 243)

At $t = \frac{2\pi}{\omega}$; the spin returns to its original dir. .

Ex.

Muon precession in an external mag. field ;

$\mu_\mu = \frac{e\hbar}{2m_\mu c}$,(the mag. mom.) of muon can be determined for exaple from hyperfine spilitting in muonium (a bound state of μ^+ a d e^-)

Knowing this we can predict the ω of the precession, so the spin precession relations for $\langle S_x\rangle_t$, $\langle S_y\rangle_t$ and $\langle S_z\rangle_t$ can be checked.

Remark : $H = -\mu \cdot B$

Spin dir. can be analyzed by taking advantage of the fact that electrons from muon decay tend to be emitted pereferentially in the dir. <u>opposite</u> to the muon spin

$\mu^+ \longrightarrow e^+ + \nu_e + \bar{\nu}_\mu$

Let us now look at the time evolution of the state ket:

$$|\alpha, t_0=0; t\rangle = e^{-\frac{i\omega t}{2}} |+\rangle\langle+|\alpha\rangle + e^{+\frac{i\omega t}{2}} |-\rangle\langle-|\alpha\rangle$$

At $t = \frac{2\pi}{\omega}$ $\qquad |\alpha, t_0=0; t\rangle_{t=\frac{2\pi}{\omega}} = -|\alpha, t_0=0, t\rangle_{t=0}$

" $\frac{4\pi}{\omega}$ $\qquad |\alpha, t_0=0; t\rangle_{t=\frac{4\pi}{\omega}} = +|\alpha, t_0=0; t\rangle_{t=0}$

$$\longrightarrow \quad \tau_{\text{precession}} = \frac{2\pi}{\omega}, \quad \tau_{\text{state ket}} = \frac{4\pi}{\omega}$$

Neutron Interferometry Experiment to Study 2π Rotations:

This experiment detects the <u>minus sign</u> in

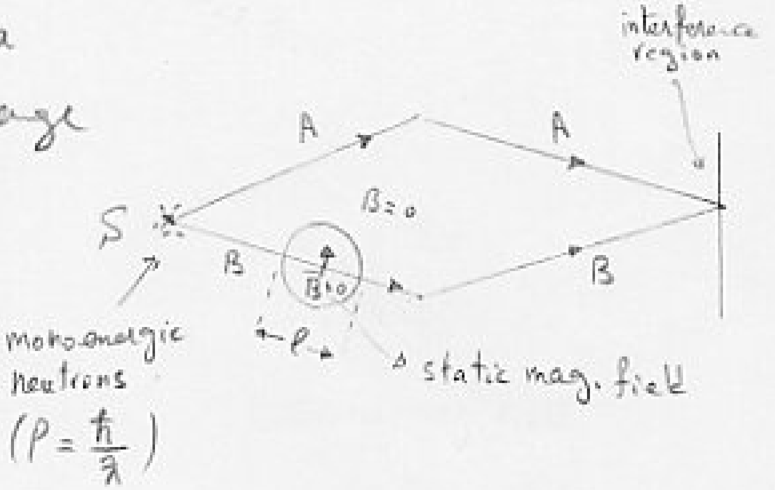
$$|\alpha\rangle \xrightarrow{D_z(2\pi)} -|\alpha\rangle$$

Quite clearly, if every state ket in the universe is multiplied by a <u>minus sign</u> $\longrightarrow$ the will be <u>no</u> observable effect.

The only way to <u>detect</u> the predicted <u>minus sign</u> is to make a <u>comparison</u> between an <u>unrotated</u> state and <u>rotated</u> state.

Neutron state ket going via path B suffers a phase change

$e^{\mp i\omega T/2}$



T: time spent in the $\vec{B}\neq 0$ region

$$\omega = \frac{g_n e B}{m_p c} \qquad (g_n \simeq -1.91) \quad \text{Precession frequency}$$

$$\mu_n = \frac{g_n e \hbar}{2 m_p c} \qquad \text{mag. mom. of neutron}$$

Now;

C_1: the amplitude of neutron in the interference region via path A

$C_2 = C_2(B=0)\, e^{\mp i\omega T/2}$: " " " " " " B

→ So the intensity observable in the interference region <u>must</u> exhibit a sinusoidal variation;

$$\cos\left(\frac{\mp\omega T}{2} + \delta\right)$$

δ: the phase difference between C_1 and $C_2(B=0)$

In practice T is fixed (the length of the region with $B \neq 0$ and also the energy of the neutrons are fixed), and the precession frequency is varied by changing the strength of $\bar{B}$.

$|\Delta\bar{B}|$ needed to produce two successive maxima is given by

$$|\Delta\bar{B}| = \frac{4\pi\hbar c}{e g_n \lambda l}$$

This relation can easily be derived using the fact that 4π rotation is needed the state ket to return to it original state with the same sign.

If, on the other hand, our description of spin $\frac{1}{2}$ systems were incorrect and the ket were to return to its original ket with the same sign under a 2π rotation $\longrightarrow$ the predicted value for ΔB would be just $\frac{1}{2}$ of the given relation.

The experiment supports our formalism.

Pauli Two-Component Formalism

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_+ \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_-$$

$$\langle +| \doteq (1,0) \equiv \chi_+^\dagger \qquad \langle -| \doteq (0,1) \equiv \chi_-^\dagger$$

$$|\alpha\rangle = |+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle \doteq \begin{pmatrix} \langle +|\alpha\rangle \\ \langle -|\alpha\rangle \end{pmatrix}$$ Two-component spinor

and

$$\langle\alpha| = \langle\alpha|+\rangle\langle +| + \langle\alpha|-\rangle\langle -| \doteq (\langle\alpha|+\rangle, \langle\alpha|-\rangle)$$

$$\chi = \begin{pmatrix} \langle +|\alpha\rangle \\ \langle -|\alpha\rangle \end{pmatrix} \equiv \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+\chi_+ + c_-\chi_-$$

$$\chi^\dagger = (\langle\alpha|+\rangle, \langle\alpha|-\rangle) = (c_+^*, c_-^*)$$

Now,

$$S_k = \begin{pmatrix} \langle +|S_k|+\rangle & \langle +|S_k|-\rangle \\ \langle -|S_k|+\rangle & \langle -|S_k|-\rangle \end{pmatrix}$$

Writing S_k in terms of base kets:

$$\rightarrow S_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_1 \qquad S_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_2$$

$$S_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_3$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices

Also;

$$\langle S_k \rangle = \langle \alpha | S_k | \alpha \rangle = \sum_{a'=+,-} \sum_{a''=+,-} \langle \alpha | a' \rangle \langle a' | S_k | a'' \rangle \langle a'' | \alpha \rangle$$

$$= \chi^{+} S_k \chi = \frac{\hbar}{2} \chi^{+} \sigma_k \chi$$

We record some properties of the Pauli matrices:

i) $\begin{cases} \sigma_i^2 = I \\ \{\sigma_i, \sigma_j\} = 0 \qquad i \neq j \end{cases} \longrightarrow \{\sigma_i, \sigma_j\} = 2\delta_{ij} \qquad (1)$

ii) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk} \sigma_k \qquad (2)$

iii) $(1)(2) \longrightarrow \sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i\sigma_3$

iv) $\sigma_i^{+} = \sigma_i$

v) $\det(\sigma_i) = -1$

vi) $\mathrm{Tr}(\sigma_i) = 0$

Now $\sigma . a = \sum_k a_k \sigma_k = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$

An important identity: $(\sigma . a)(\sigma . b) = a . b + i\sigma . (a \times b)$

Proof: $\sum_j \sigma_j a_j \sum_k \sigma_k b_k = \sum_j \sum_k \left(\frac{1}{2} \{\sigma_j, \sigma_k\} + \frac{1}{2} [\sigma_j \sigma_k] \right) a_j b_k$

$$= \sum_j \sum_k (\delta_{jk} + i\epsilon_{jkl} \sigma_l) a_j b_k = a . b + i\sigma . (a \times b)$$

Also, if the components of vector $\bar{a}$ are <u>real</u>;

$$(\sigma\cdot a)^2 = |a|^2$$

Rotations in the Two-Component Formalism:

$$e^{-\frac{iS\cdot\hat{n}\varphi}{\hbar}} \doteq e^{-\frac{i\sigma\cdot\hat{n}\varphi}{2}}$$

using $(\sigma\cdot\hat{n})^K = \begin{cases} 1 & \text{for } k \text{ even} \\ \sigma\cdot n & \text{〃 〃 odd} \end{cases}$

$$e^{-\frac{i\sigma\cdot\hat{n}\varphi}{2}} = \left[1 - \frac{(\sigma\cdot\hat{n})^2}{2!}\left(\frac{\varphi}{2}\right)^2 + \frac{(\sigma\cdot\hat{n})^4}{4!}\left(\frac{\varphi}{2}\right)^4 - \cdots\right]$$
$$-i\left[(\sigma\cdot\hat{n})\frac{\varphi}{2} - \frac{(\sigma\cdot\hat{n})^3}{3!}\left(\frac{\varphi}{2}\right)^3 + \cdots\right]$$

$$= I\cos\frac{\varphi}{2} - i\sigma\cdot\hat{n}\sin\frac{\varphi}{2}$$

$$\rightarrow e^{-\frac{i\sigma\cdot\hat{n}\varphi}{2}} = \begin{pmatrix} \cos\frac{\varphi}{2} - in_z\sin\frac{\varphi}{2} & (-in_x - n_y)\sin\frac{\varphi}{2} \\ (-in_x + n_y)\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} + in_z\sin\frac{\varphi}{2} \end{pmatrix} \qquad (1)$$

$$\chi \xrightarrow{\text{rot.}} e^{-\frac{i\sigma\cdot\hat{n}\varphi}{2}}\chi$$

On the other hand,

The σ_k's themselves <u>are to remain</u> <u>unchanged</u> under rots.

$\chi \longrightarrow e^{-\frac{i\sigma\cdot\hat{n}\varphi}{2}}\chi$, but we keep σ_k unchanged.

So strictly speaking, despite its appearance α is not to be regarded as a vector!

Rather it is $\chi^\dagger \alpha \chi$ which obeys the transformation property of a vector:

- $$\chi^\dagger \alpha_k \chi \xrightarrow{\text{rot.}} \sum_\ell R_{k\ell} \chi^\dagger \alpha_\ell \chi$$

(Note: If α_k's are kept unchanged $\longrightarrow$ the expectation values behave like a vector. Such an op. is called vector op.)

We can prove this using

$$e^{\frac{i\alpha_3 \varphi}{2}} \alpha_1 e^{-\frac{i\alpha_3 \varphi}{2}} = \alpha_1 C\varphi - \alpha_2 S\varphi$$

(See P243)

$$e^{\frac{i\alpha_3 \varphi}{2}} \alpha_2 e^{-\frac{i\alpha_3 \varphi}{2}} = \alpha_2 C\varphi + \alpha_1 S\varphi$$

$$e^{\frac{i\alpha_3 \varphi}{2}} \alpha_3 e^{-\frac{i\alpha_3 \varphi}{2}} = \alpha_3$$

Also, $e^{-\frac{i\alpha\cdot\hat{n}\varphi}{2}}\Big|_{\varphi=2\pi} = -\underline{1}$ as expected

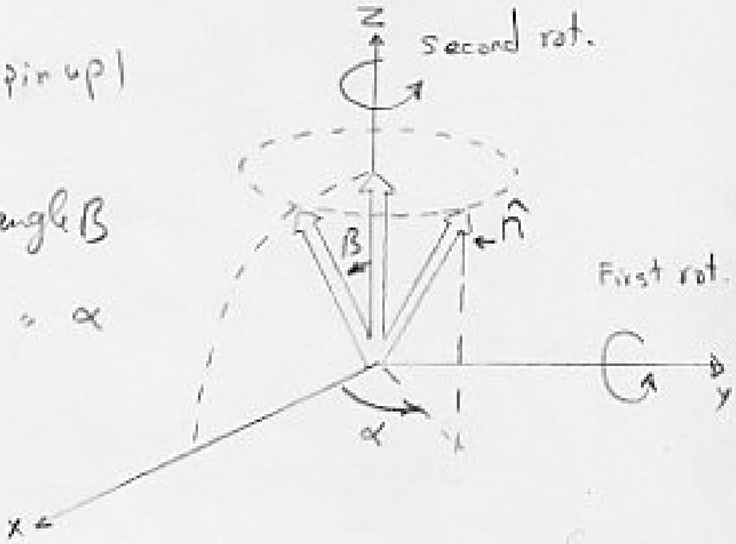
As an application of (1) let us construct χ in such a way;

$$\alpha\cdot\hat{n}\,\chi = (+1)\,\chi$$

Or equivalently;

$$(S\cdot\hat{n})\,|S\cdot\hat{n};+\rangle = \left(\frac{\hbar}{2}\right)|S\cdot\hat{n};+\rangle$$

Actually this can be solved as a straightforward eigenvalue prob., but we present an alternative method based on rot. matrix.



We start with $\begin{pmatrix}1\\0\end{pmatrix}$ state (spin up)

First rot. : about y-axis by angle β

Second " : " z-x " " α

$$\chi(\hat{n}) = e^{-i\sigma_3\frac{\alpha}{2}}\, e^{-i\sigma_2\frac{\beta}{2}} \begin{pmatrix}1\\0\end{pmatrix}$$

$$\chi = \left[\cos\left(\frac{\alpha}{2}\right) - i\sigma_3 \sin\left(\frac{\alpha}{2}\right)\right]\left[\cos\left(\frac{\beta}{2}\right) - i\sigma_2 \sin\left(\frac{\beta}{2}\right)\right]\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2} & 0\\ 0 & \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\beta/2)\, e^{-i\alpha/2}\\ \sin(\beta/2)\, e^{i\alpha/2}\end{pmatrix}$$

Ex.

Consider a rot. about y-axis of an angle $\varphi = \frac{\pi}{2}$, from initial state along the z-dir

Acc. to

$$U = e^{\frac{-i\sigma\cdot\hat{n}\varphi}{2}} = \begin{pmatrix} \cos\frac{\varphi}{2} - in_z \sin\frac{\varphi}{2} & (-in_x - n_y)\sin\frac{\varphi}{2} \\ (-in_x + n_y)\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} + in_z \sin\frac{\varphi}{2} \end{pmatrix}$$

(see P275

Then $n_y = 1$, $n_x = n_z = 0$

Remark: Note that there are 3 parameters for rot. φ, n_x, n_y, n_z together with the constraint $n_x^2 + n_y^2 + n_z^2 = 1$

$$U = e^{\frac{-i\sigma_y \pi}{4}} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\left(|+\rangle + |-\rangle\right)$$

$$\rightarrow U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\sigma_x, +\rangle$$

$$\sigma_z \xrightarrow{tr.} U\sigma_z U^{-1}$$

Since $A^{-1} = \frac{1}{|A|}[\mathrm{Cofac}\, A]^T$ $\qquad [\mathrm{Cofac}\, A]^T = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\det U = 1 \rightarrow U^{-1} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$U\sigma_z U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$U\sigma_z U^{-1} = \sigma_x$ as expected

$$\sigma_z|+\rangle = (+1)|+\rangle \xrightarrow{tr} \sigma_x|\sigma_x;+\rangle = (+1)|\sigma_x;+\rangle$$

or $\sigma\cdot\hat{n}|\sigma\cdot n, +\rangle = (+1)|\sigma\cdot n, +\rangle$ <u>in general</u>

3.3 O(3), SU(2), and Euler Rotations:

Groups:

Set;

Def. – A set is any kind of collection of entities of any sort.

Ex. – $A = \{$Kurus, Mitra, Jacqueline, Dariush$\}$

$B = \{1, 2, 5, 9, 11\}$ $C = \{$apples, oranges$\}$

$D = \phi$ = empty set $= \{\ \}$

Binary Operation;

Def. – A binary operation $*$ on a set is a rule which assigns to each ordered pair of elements of the set some elements of the set.

Def. – The binary operation $*$ on a set S is commutative iff $a * b = b * a \quad \forall\ a, b \in S$

It is associative iff $(a * b) * c = a * (b * c)$

$\forall\ a, b \in S$

Ex.

*	a	b	c
a	b	c	a
b	a	c	b
c	c	b	a

$a * b = c$, $b * a = a$

$\longrightarrow$ not commutative.

Group;

Def. - A group $< G, * >$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied;

1) The binary operation $*$ is associative.
2) There is an element e in G, such that;

$e * x = x * e = x \quad \forall x \in G$

The element e is the identity element for $*$ on G.

3) For each a in G, there is an element a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$

The element a^{-1} is the inverse of a w.r.t $*$.

Def. – A func. from a set A into a set B is <u>one-to-one</u> if each element of B has at most one element of A mapped into it, and is <u>onto</u> B if each element of B has at least one element of A mapped into it.

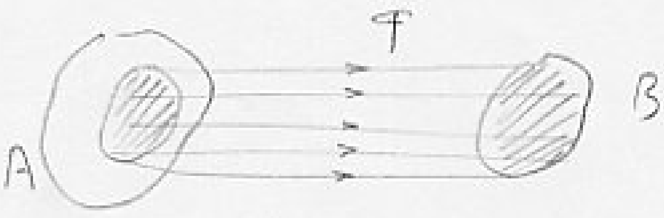
1– To show that φ is one-to-one, you show that;

$$a_1\varphi = a_2\varphi \xrightarrow{\text{implies}} a_1 = a_2$$

(not one-to-one)

2– To show that φ is onto B, we have to show;

$\forall\, b\in B$, $\exists\; a\in A$, such that $a\varphi = b$



onto

Isomorphism:

Def. - An isomorphism of a group G with a group G' is <u>one-to-one</u> func. φ, mapping G onto G' such that for all x and y in G; (one-to-one cond. $\longrightarrow$ φ^{-1} exists)

$$(x * y)\varphi = (x\varphi) *' (y\varphi)$$

The group G and G' are then isomorphic, notated by:

$$G \simeq G'$$

To show that two groups G and G' are isomorphic we,

i) Define the func. φ which gives the isomorphism of G with G'.

ii) Show that φ is a one-to-one func.

iii) Show that φ is onto G'.

iv) Show that $(x * y)\varphi = (x\varphi) *' (y\varphi)$ $\forall x, y \in G$

<u>Ex.</u>

Show that $\langle R, + \rangle$ is isomorphic to $\langle R^+, \cdot \rangle$

i) For $x \in R$ define $x\varphi = e^x$

This gives a mapping $\varphi : R \longrightarrow R^+$

ii) If $x\varphi = y\varphi \rightarrow e^x = e^y \Longrightarrow x = y$

Thus φ is one-to-one

iii) If $r \in R^+$, then $(\ln r)\varphi = e^{\ln r} = r$ where $\ln r \in R$

Thus φ is onto R^+

iv) For $x, y \in R$ we have $(x+y)\varphi = e^{x+y} = e^x e^y = (x\varphi)\cdot(y\varphi)$

There for it is an isomorphism.

Homomorphism,

'Def. - A map φ of a group G into a group G' is a homomorphism if $(a * b)\varphi = (a\varphi) *' (b\varphi)$ for all elements a and b in G

a) The general linear group

i) $GL(n,C)$: complex general linear group of regular invertiable complex matrices of deg. n.

The continuous variation of the $2n^2$ parts (i.e. the n^2 real and the n^2 imaginary parts) will generate the <u>entire</u> matrix group and hence the group is of dim. $\underline{2n^2}$ and may be characterized by $2n^2$ real parameters.

ii) $GL(n,R)$: Real general linear group with $\underline{n^2}$ parameter

clearly: $GL(n,R) \subset GL(n,C)$

b) special linear group

i) $SL(n,c)$: complex special linear group

The same as $GL(n,c)$ but with the restriction that the complex matrices $GL(n,c)$ be of determinant $\underline{+1}$

$SL(n,\mathbb{C})$ can be parametrized by $\underline{2(n^2-1)}$ parameters.

Remark: <u>Two</u> eqns arise from $|A| = +1$ for imaginary and real parts.

ii) $SL(n,\mathbb{R})$: Real special linear group

It has $\underline{n^2-1}$ parameter

It is formed by real matrices of det. $= +1$

Clearly:

$$GL(n,\mathbb{C}) \supset SL(n,\mathbb{C}) \supset SL(n,\mathbb{R})$$

and $$GL(n,\mathbb{R}) \supset SL(n,\mathbb{R})$$

The special linear group is sometimes referred to as the <u>special unimodular</u> group.

c) The Unitary group.

i) The unitary matrices A of deg. n form the elements of the n^2-parameter unitary group $\mathcal{U}(n)$ that <u>leaves</u> the Hermitian form

$$\sum_{i=1}^{n} z_i z_i^* \quad \text{invariant.}$$

Remark: There exist n^2 equs. by the unitary cond $A^+A = I$

Remark: If A is unitary $\Rightarrow |A|^*|A| = 1$

ii) The group of matrices in $GL(p+q, c)$ which leaves invariant the Hermitian form

$$-z_1 z_1^* - \cdots - z_p z_p^* + z_{p+1} z_{p+1}^* + \cdots z_{p+q} z_{p+q}^*$$

is designated by the group $\underline{U(p,q)}$

where $U(n,0) \equiv \mathcal{U}(0,n) \equiv \mathcal{U}(n)$

clearly; $GL(p+q, c) \supset \mathcal{U}(p,q)$

and $GL(n,c) \supset \mathcal{U}(n)$

d) Special Unitary group

i)

$SU(n)$: the same as Unitary matrices but the det. of the matrices is restricted to be $\underline{+1}$

The number of the parameters $= n^2 - 1$

$$SU(n) = U(n) \cap SL(n,c)$$

ii) Similarly

$$SU(P,q) = U(P,q) \cap SL(P+q, c)$$

e) The orthogonal group;

i) The group of complex orthogonal matrices $({}^{T}AA = I)$ of deg. $\underline{n}$ form a $n(n-1)$ parameter group designated as $\underline{O(n,c)}$.

Since ${}^{T}AA = I \longrightarrow |A| = \pm 1$

$\longrightarrow$ The group decomposes into two disconnected pieces and we cannot go continuously from one to the other

ii) $O(n,R)$: Real orthogonal group

The same as $O(n,c)$ but with <u>real</u> parameters

Number of parameters $= n(n-1)/2$

e) The special orthogonal group

i) $SO(n,c)$: special complex orthogonal group with $n(n-1)$ parameters

The same as $O(n,c)$, but the matrices have det. $=+1$

The matrices of $SO(n,c)$ leave <u>invariant</u> the quadratic form,

$$\sum_{i=1}^{n} z_i^2 \qquad \left(\text{not } \sum_{i=1}^{n} z_i z_i^* \right)$$

clearly;

$$SO(n,c) = SL(n,c) \cap O(n,c)$$

ii) $SO(n,R)$: Real special orthogonal group

In this case the set of the matrices which form the group have det. = +1

Again $O(n,R)$ consists of two disconnected pieces, with $SO(n,R)$ occuring as a subgroup.

Matrieces belonging to $SO(n,R)$ leave invariant the real quadratic form

$$\sum_{i=1}^{n} x_i^2$$

iii) The matrices in $SL(p+q,R)$ that leave invariant the quadratic form;

$$-\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{p+q} x_j^2$$

form the elements of the group $SO(p,q)$

g) The symplectic group

i) The symplectic group $Sp(2n,C)$ is the $2n(n+1)$ parameter group of regular complex matrices which leave invariant the nondegenerate skew-symmetric bilinear form;

$$\sum_{i=1}^{n} (x_i y_i' - x_i' y_i)$$

of two vectors

$$X = (x_1, x_2 - - x_n, x_1', x_2', - - x_n')$$

$$Y \equiv (y_1, y_2 - - - y_n, y_1', y_2', - - y_n')$$

clearly; $SP(2n,C) \subset GL(n,C)$

and the martrices <u>need not</u> to be <u>unitary</u>.

ii) $SP(2n,R)$: The same as $SP(2n,C)$ but with the real parameters.

Number of the parameters $= n(2n+1)$

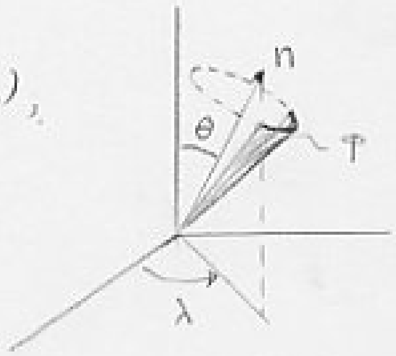
iii) The symplectic group

$$SP(2n) = U(2n) \cap SP(2n,C)$$

is Known as the unitary symplectic group.

The group as like as $SP(2n,R)$, is a $n(2n+1)$ parameter group.

Rotation and Group $O(3)$:



We need 3-parameters (3-real numbers), to characterize a general <u>rotation</u>.

$$\begin{cases} \theta: & \text{the polar angle} \\ \lambda: & \text{" azimuthal "} \\ \varphi: & \text{the rotation "} \end{cases} \quad \begin{matrix} \Big\} \text{(to determine } \hat{n}) \\ \} \text{(" " the rot. angle)} \end{matrix}$$

or equivalently by the cartesian components of $\hat{n}\varphi$ vector.

Or equivalently in term of (α, β, γ) the Euler's angles.

However, these ways of characterizing rotation <u>are not</u> so convenient from the <u>point of view</u> of studying the <u>group</u> properties of rotations.

It is much easier to work with 3×3 orthogonal matrix R

The orthogonality cond. $RR^T = R^TR = I$ gives 6-independent eqns. (since $RR^T = R^TR \longrightarrow RR^T$ is symmetric and it has 6 indep. entries)

$\longrightarrow$ There are $9-6=3$ indep. parameters in R

The set of all multiplication operations with orthogonal matrices forms a <u>group</u>.

i.e.

i) $R_1 R_2$ is another orthogonal matrix $\forall R_1, R_2 \in G$

$\rightarrow R_1 R_2$ belongs to G

The reason: $(R_1R_2)(R_1R_2)^T = R_1 \underbrace{R_2 R_2^T}_{I} R_1^T = I$

ii) The associative law holds

$$R_1(R_2R_3) = (R_1R_2)R_3$$

iii) The <u>identity</u> matrix I - physically corresponding to no rotation defined by

$$RI = IR = R$$

is a member of the class of orthogonal matrices

iv) The <u>inverse</u> matrix R^{-1} - physically corresponding to rotation in the opposite sense defined by

$$RR^{-1} = R^{-1}R = I$$

(For orthogonal matrices $A^{-1} = A^T$)

is also a member.

This is $O(3)$ group.

A general linear tr. in 2-dim.

$$Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{with 8-parameters} \quad (1)$$

Unitary cond.: $Q^{+}Q = QQ^{+} = I \xrightarrow[(2)]{\text{also}} |Q|^{*}|Q| = |Q|^2 = 1$ (magnitude of det. $= 1$)

$$\xrightarrow{\text{also}} \left\{ \begin{array}{ll} \alpha\alpha^{*} + \gamma\gamma^{*} = 1 & (\text{Real}) \\ \beta\beta^{*} + \delta\delta^{*} = 1 & (\quad " \quad) \\ \alpha^{*}\beta + \gamma^{*}\delta = 0 & (\text{complex}) \end{array} \right\} \longrightarrow 4\text{-constraints} \quad (3)$$

$8 - 4 = 4$ parameter

Additional cond. to decrease the number of the indep.-prameters to 3;

$\det(Q) = +1 \quad \rightarrow \alpha\delta - \beta\gamma = 1 \; (4) \quad$ (complex $\rightarrow$ 2-conds.)

But since the unitary property (3) already fixes the mag. of the determinat and (4) only serves to fix the phase angle. (So one of the conds. of (4) is independent) -

Matrices with $\det = +1$ are called unimodular.

Now, $(3) \rightarrow \delta = -\alpha^{*}\dfrac{\beta}{\gamma^{*}} \quad (5)$

(5) in (4) $\longrightarrow -\dfrac{\beta}{\gamma^{*}}\underbrace{(\alpha\alpha^{*} + \gamma\gamma^{*})}_{1} = 1 \rightarrow \beta = -\gamma^{*} \quad (6)$

(6) in (5) $\longrightarrow \delta = \alpha^{*} \quad \rightarrow Q = \begin{pmatrix} \alpha & \beta \\ -\beta^{*} & \alpha^{*} \end{pmatrix}$

Rotation and SU(2)

As we saw for spin $\frac{1}{2}$, an arbitrary rot. is given by:

$$e^{-\frac{i\alpha \cdot \hat{n}\,\sigma}{2}} = \begin{pmatrix} \sim & \sim \\ \sim & \sim \end{pmatrix} \qquad (1)$$

This is clearly unitary (because $\alpha_k = \alpha_k^+$)

$\rightarrow$ if $\chi^+\chi = (c_+^*, c_-^*)\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = |c_+|^2 + |c_-|^2 = 1$

then $\rightarrow$ $(\chi^+ e^{\frac{i\alpha \cdot \hat{n}\,\sigma}{2}})(e^{-\frac{i\alpha \cdot \hat{n}\,\sigma}{2}}\chi) = \chi^+\chi = |c_+|^2 + |c_-|^2 = 1$

invariant under rot.

Furthermore (equ. 1) is unimodular (det. = 1) as we show explicitly below:

$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad (2)$$ general form of the unitary unimodular matrix

where $|a|^2 + |b|^2 = 1$ (unimodular cond.) (3)

$$U^+(a,b)\,U(a,b) = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & 0 \\ 0 & |a|^2+|b|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{unitary} \qquad (4)$$

Note that the number of indep. real parameters in (2), taking into account (3), is 3

Comparing (1) and (2);

$$\mathrm{Re}(a) = \cos\frac{\varphi}{2} \qquad \mathrm{Im}(a) = -n_z \sin\frac{\varphi}{2}$$

$$\mathrm{Re}(b) = -n_y \sin\frac{\varphi}{2} \qquad \mathrm{Im}(b) = -n_x \sin\frac{\varphi}{2}$$

Conversely; it is clear that the most general unitary unimodular matrix of the form (2) can be interpreted as presenting a rotation. The two complex numbers a, and b are Cayley - Klein parameters.

Let us check the group properties of multiplication operations with unitary unimodular matrices.

$$U(a_1,b_1)\,U(a_2,b_2) = U(a_1a_2 - b_1b_2^*,\ a_1b_2 + a_2^*b_1)$$

$$|a_1a_2 - b_1b_2^*|^2 + |a_1b_2 + a_2^*b_1|^2 = (|a_1|^2+|b_1|^2)(|a_2|^2+|b_2|^2) = 1$$

(ok)

$$U^{-1}(a,b) = U(a^*, -b) \qquad \text{(exsists)} \quad \text{(ok)}$$

and so on, . .

This group is known a $SU(2)$.

In contrast $U(2)$ group (dim. = 2) has 4 indep. parameters, and can be written as

$$U = e^{i\gamma}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2+|b|^2=1 \qquad \gamma^*=\gamma \text{ (real)}$$

$SU(2) \subset U(2)$

Because we can characterize rotations using both the $O(3)$ language and the $SU(2)$ language $\longrightarrow$ one may tempted to conclude that the groups $O(3)$ and $SU(2)$ are isomorphic

i.e. there is $\quad$ the elements of $O(3)$ $\xleftrightarrow[\text{correspondence}]{\text{one-to-one}}$ the elements of $SU(2)$!

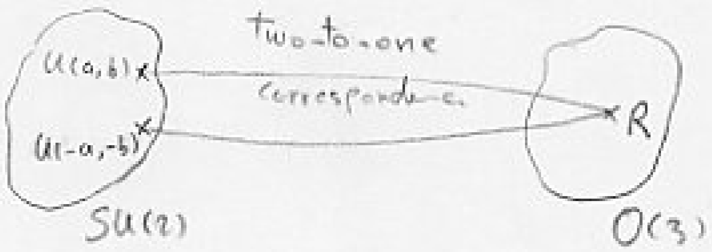
This inference is not correct!

Because consider a $\varphi = 2\pi$ and $\varphi' = 4\pi$ rotations

In $O(3)$ language $\quad R(2\pi) = I \;, R(4\pi) = I$

In $SU(2)$ " $\quad U(2\pi) = -I \;, U(4\pi) = I \qquad \left(-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

More generally:



One can say, however, that the two groups are locally isomorphic.

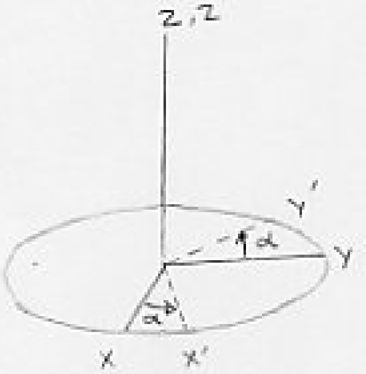
Euler Rotations

Rotate the rigid body counterclockwise;

i) First about the z-axis by angle α

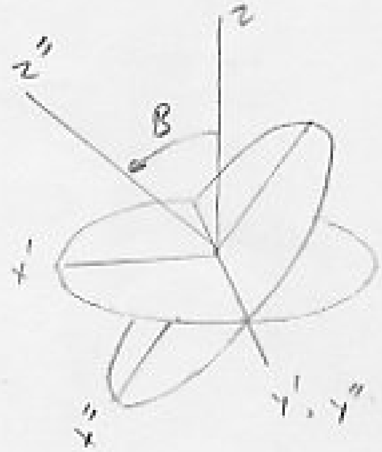
Y : space-fixed y-axis

Y' : body-" y-axis (i)



ii) Second, about y'-axis by angle β

iii) Third, about z''-axis by angle γ



$$R(\alpha,\beta,\gamma) \equiv R_{z''}(\gamma)\, R_{y'}(\beta)\, R_{z}(\alpha)$$

R : 3x3 orthogonal matrices

In most text books in Cl. M. the second rotation is performed about x-axis, rather than about y-axis.

This conversion is to be avoided in Q.M for a reason that will become appearant in a moment.

In this rot. $R_{y'}$ and $R_{z''}$ are rotations about body-fixed axis. This is not convenient in Q.M.

But there is a simple relation for $R_{y'}(\beta)$ in terms of rotations in space-fixed axes.

$$R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha)$$

To prove this, look the orientation of x, y, z-axis when we apply the right hand side ops., and compare them by the effect of left hand side (we get the same result)

Remark: Alternative Proof:

Let $\{|a',\alpha\rangle\}$: a complete set prepared in the coord. sys. rotated through α about z-axis

and $\{|a'\rangle\}$: the corresponding set as prepared in the original frame.

we have, $\quad |a',\alpha\rangle = e^{-\frac{i\alpha J_z}{\hbar}} |a'\rangle$

Also $\quad \langle a',\alpha| J_{y'} |a'',\alpha\rangle = \langle a'| J_y |a''\rangle$

Hence $\longrightarrow \quad J_{y'} = e^{-\frac{i\alpha J_z}{\hbar}}\, J_y\, e^{\frac{i\alpha J_z}{\hbar}}$

and of course

$$e^{-\frac{i\beta J_{y'}}{\hbar}} = e^{-\frac{i\alpha J_z}{\hbar}} e^{-\frac{i\beta J_y}{\hbar}} e^{\frac{i\alpha J_z}{\hbar}}$$

Similarly:

$$R_{z''}(\gamma) = R_{y'}(\beta) R_z(\gamma) R_{y'}(\beta)$$

$$\rightarrow R(\alpha,\beta,\gamma) = R_{z''}(\gamma) R_{y'}(\beta) R_z(\alpha) =$$

$$= R_{y'}(\beta) R_z(\gamma) R^{-1}_{y'}(\beta) R_{y'}(\beta) R_z(\alpha) =$$

$$= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\gamma) R_z(\alpha) = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

$$R(\alpha,\beta,\gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

where all matrices refer to fixed-axis rots. -

For spin $\frac{1}{2}$:

$$D(\alpha,\beta,\gamma) = D_z(\alpha) D_y(\beta) D_z(\gamma)$$

corresponding rot ops. in the ket space of the spin $\frac{1}{2}$ sys.

$$e^{-\frac{i\sigma_3\alpha}{2}} e^{-\frac{i\sigma_2\beta}{2}} e^{-\frac{i\sigma_3\gamma}{2}} =$$

$$= \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\gamma}{2}} & 0 \\ 0 & e^{\frac{i\gamma}{2}} \end{pmatrix} =$$

$$D(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-\frac{i(\alpha+\gamma)}{2}} \cos\frac{\beta}{2} & -e^{-\frac{i(\alpha-\gamma)}{2}} \sin\frac{\beta}{2} \\ e^{\frac{i(\alpha-\gamma)}{2}} \sin\frac{\beta}{2} & e^{\frac{i(\alpha+\gamma)}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

This is of the unitary, unimodular form. (See P 254/2)

Conversely, the most general 2×2 unitary unimodular matrix can be written in this Euler angle form.

Note; The second rot. matrix is purely real, (as we prefer in Q.M.). Instead if we had chosen the second rot. axis to be x-axis (as in the Cl. M.), this would not have been the case.

The matrix elements:

$$D^{(1/2)}_{m'm}(\alpha,\beta,\gamma) = \langle j=\tfrac{1}{2}, m' \,|\, D(\alpha,\beta,\gamma) \,|\, j=\tfrac{1}{2}, m \rangle$$

Remark: Note that the rot. parameters α, β, γ, can be expressed in terms of $\hat{n}$ (two parameters) and φ (rot. angle).

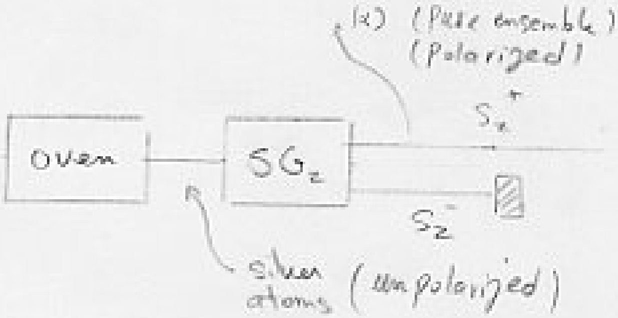
3-4 Density Operators and Pure Versus Mixed Ensembles:

Polarized versus Unpolarized Beams;

The formalism of Q.M. developed so far makes statistical predictions on an ensemble, that is, a collection, of identically prepared physical systems.

In such an ensemble $\longrightarrow$ The same $|\alpha\rangle$ characterizes all members

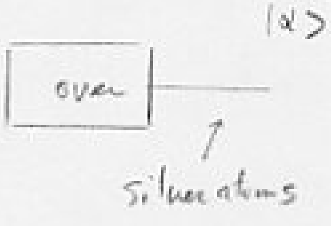
Ex.



How to describe Q. mechanically an ensemble of physical systems for which some, say, 60% are characterized by $|\alpha\rangle$ and remaining 40% by $|B\rangle$?

To show that formalism of Q.M. developed so far is not capable to solve the probs such as the mentioned prob. look the following example;

Ex.



Silver atoms comming directly out the oven have random spin orientations.

$|\alpha\rangle = C_+|+\rangle + C_-|-\rangle$

Clearly this equ. <u>is not</u> capable to describe a collection of atoms with <u>random spin</u> orientaion.

This ket has a <u>definite</u> spin orientation ($\hat{n}$-dir. with polar and azimuthal angles β and α, resp.)

$$\text{Sine } \chi(\hat{n}) = \begin{pmatrix} \cos(\frac{\beta}{2})\, e^{-i\alpha/2} \\ \sin(\frac{\beta}{2})\, e^{i\alpha/2} \end{pmatrix} \longrightarrow \frac{C_+}{C_-} = \frac{\cos\beta/2}{e^{i\alpha}\sin\frac{\beta}{2}}$$

(α, and β can be obtained from) this equ.

An ensemble of silver atoms with completely <u>random</u> spin orientation can be <u>viewed</u> as a collection of silver atoms in which:

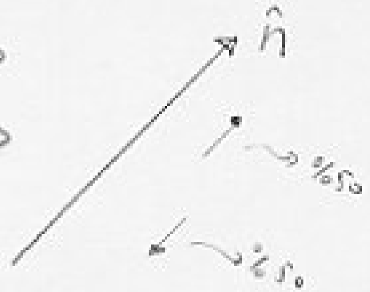
%50 of the members of the ensemble are characterized by $|+\rangle$

%50 " " " " " " " $|-\rangle$

i.e. $\quad W_+ = 0.5 \quad W_- = 0.5$ $\quad$ fractional population or probability weight

That is, $\begin{cases} \%50 : |S_x, +\rangle \\ \%50 : |S_x, -\rangle \end{cases}$, or $\begin{cases} \%50 : |S_z, +\rangle \\ \%50 : |S_z, -\rangle \end{cases}$

or in general $\begin{cases} \%50 \ |S \cdot n; +\rangle \\ \%50 \ |S \cdot n; -\rangle \end{cases}$ (since there is no preferred dir.)



Note that we introduce two real number w_+, and w_-.

There is no information on the relative phase between the spin-up and spin-down Ket (incoherent mixture of spin-up and spin-down states).

Remark: w_+ and w_- are different from $|C_+|^2$ and $|C_-|^2$

Ex. $w_+ = 0.5$ and $w_- = 0.5$ for random silver atoms are different from $|C_+|^2 = 0.5$ and $|C_-|^2 = 0.5$ in coherent linear superposition (for example $\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$). The phase relation between $|+\rangle$ and $|-\rangle$ contains information on the spin orientation in xy-plane, in this case in the positive x-dir. ($e^{-i\varphi/2}\frac{1}{\sqrt{2}}|+\rangle + e^{i\varphi/2}\frac{1}{\sqrt{2}}|-\rangle$: a state in the xy-plane)

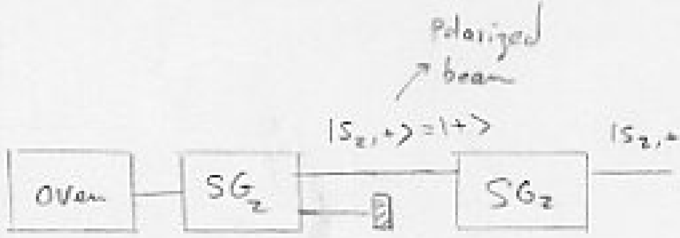
Ex. Males $\to |+\rangle$ %50, Females $\to |-\rangle$ %50, but whoever heard of a human referred to as a coherent linear superposition of male and female with a particular hase relation? ($|\alpha\rangle = \sqrt{0.7}\,|+\rangle + \sqrt{0.3}\,|-\rangle$ Zeki Muren)
(or $|\alpha\rangle = e^{-i\frac{\Omega}{2}}\frac{1}{\sqrt{2}}|+\rangle + e^{i\frac{\Omega}{2}}\frac{1}{\sqrt{2}}|-\rangle$)

The difference between polarized and unpolaried beam:

i) Unpolarized beam

$\hat{n}$: any dir.

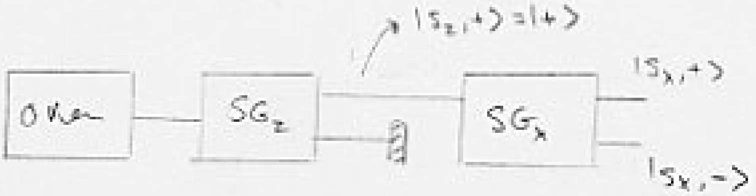
ii) Polarized beam



Relative intensity $= \frac{1}{1} = 1$

$$|S_x,+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

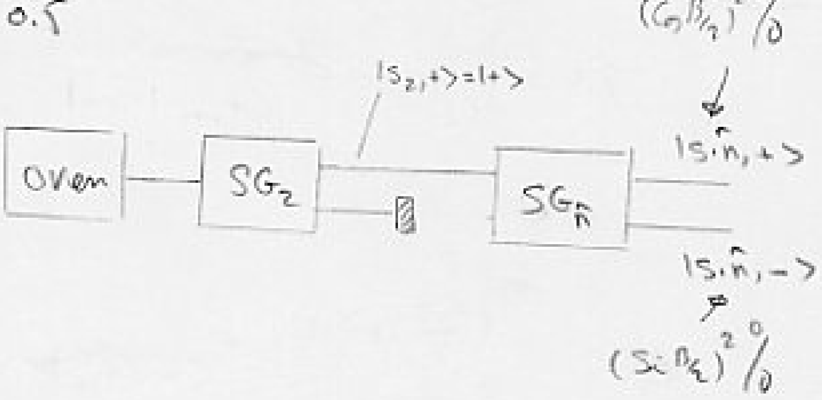
$$|S_x,-\rangle = \frac{1}{\sqrt{2}}\left(-|+\rangle + |-\rangle\right)$$



$$\text{Rel. Int. for } |S_x;+\rangle = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{1} = 0.5 \qquad (\text{Rel. In.} = \text{Relative Intensity})$$

$$\text{Rel. Int. for } |S_x;-\rangle = \frac{\left(-\frac{1}{\sqrt{2}}\right)^2}{1} = 0.5$$

And in general;



$$|S\cdot\hat{n},+\rangle = \begin{pmatrix} \cos(\beta/2)\, e^{-i\alpha/2} \\ \sin(\beta/2)\, e^{i\alpha/2} \end{pmatrix}$$

$$|S\cdot\hat{n},-\rangle = \begin{pmatrix} \cos\left(\frac{\beta+\pi}{2}\right) e^{-i\alpha/2} \\ \sin\left(\frac{\beta+\pi}{2}\right) e^{i\alpha/2} \end{pmatrix} = \begin{pmatrix} -\sin(\beta/2)\, e^{-i\alpha/2} \\ \cos(\beta/2)\, e^{i\alpha/2} \end{pmatrix}$$

$$|\langle S_z,+|S\cdot\hat{n},+\rangle|^2 = \cos^2\beta/2 \quad , \quad |\langle S_z,+|S\cdot n,-\rangle|^2 = \sin^2\beta/2$$

$$\text{Total probability} = \cos^2\beta/2 + \sin^2\beta/2 = 1$$

$$\text{Rel. Int. for } |S\cdot\hat{n},+\rangle = \frac{\cos^2\beta/2}{1} = \cos^2\beta/2$$

$$\text{" \quad " \quad , } |S\cdot\hat{n},-\rangle = \frac{\sin^2\beta/2}{1} = \sin^2\beta/2$$

Note: In two dim. case:

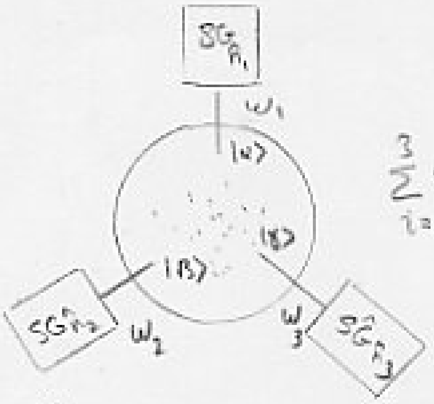
Pure ensemble		Mixed ensemble		Random ensemble
%100 from $|\alpha\rangle$	← extereme	%a from $|\alpha\rangle$ %(100−a) " $|\beta\rangle$	extereme →	%50 from $|\alpha\rangle$ %50 " $|\beta\rangle$

Ensemble Averages and Density operators:

(J. Von Neumann 1927)

Pure ensemble: Collection of physical systems with <u>all</u> members characterized by the <u>same</u> ket $|\alpha\rangle$

Mixed ensemble: A fraction of the members with relative population w_1 are characterized by $|\alpha^{(1)}\rangle$, some other fraction with rel. pop. w_2 by $|\alpha^{(2)}\rangle$ and so on...



Of course; $\sum_{i}^{n} w_i = 1$

$|\alpha^{(i)}\rangle$ and $|\alpha^{(j)}\rangle$ <u>need not</u> be orthogonal.

<u>Ex.</u> $|\alpha^{(1)}\rangle = |+\rangle$, $|\alpha^{(2)}\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

Remark: n <u>need not</u> coincide with N (the dim. of the ketspace)

<u>Ex.</u> $|\alpha^{(1)}\rangle = |S_z,+\rangle$ 30% , $|\alpha^{(2)}\rangle = |S_x,+\rangle$ 50%

$|\alpha^{(3)}\rangle = |S_y,-\rangle$ 20%

Suppose we make a measurement on a mixed ensemble of some observable A.

The average measured value of A when a large number of measurements are carried out, is given by

$$[A] \equiv \sum_i w_i \langle \alpha^{(i)} | A | \alpha^{(i)} \rangle = \sum_i \sum_{a'} w_i \langle \alpha^{(i)} | A | a' \rangle \langle a' | \alpha^{(i)} \rangle$$

$$= \sum_i \sum_{a'} w_i \underbrace{|\langle a' | \alpha^{(i)} \rangle|^2}_{|c_i|^2} a' \qquad \text{ensemble average}$$

$|c_i|^2 = |\langle a' | \alpha^{(i)} \rangle|^2$: The probability of state $|\alpha^{(i)}\rangle$ to be found in $|a'\rangle$ state

w_i : The probability of finding a Q. mechanical state $|\alpha^{(i)}\rangle$ to be found in the ensemble.

Also;

$$[A] = \sum_i w_i \sum_{b'} \sum_{b''} \langle \alpha^{(i)} | b' \rangle \langle b' | A | b'' \rangle \langle b'' | \alpha^{(i)} \rangle$$

$$= \sum_{b'}^{N} \sum_{b''}^{N} \left(\sum_i^{n} w_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \right) \langle b' | A | b'' \rangle$$

Define; $\rho \equiv \sum_i w_i |\alpha^{(i)}\rangle \langle \alpha^{(i)}|$ density op.

The matrix elements of ρ;

$$\langle b'' | \rho | b' \rangle = \sum_i w_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle$$

ρ contains all the physically significant information about the ensemble.

Also,
$$[A] = \sum_{b'}\sum_{b''} \langle b''|\rho|b'\rangle\langle b'|A|b''\rangle$$
$$= \sum_{b''} \langle b''|\rho A|b''\rangle = \mathrm{Tr}(\rho A)$$

Since the trace is indep. of the representation, we may calculate it in a <u>convenient</u> basis.

Properties of ρ:

i) $\rho = \sum w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$ is Hermitian

ii) $\mathrm{Tr}(\rho) = \sum_i \sum_{b'} w_i \langle b'|\alpha^{(i)}\rangle\langle\alpha^{(i)}|b'\rangle = \sum_i w_i \langle\alpha^{(i)}|\alpha^{(i)}\rangle$

$= \sum_i w_i = 1$ normalization cond.

For $N=2$ (spin $\frac{1}{2}$):

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{+} = \begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix}$$

if A is hermitian $\rightarrow A = A^{+}$

$$\rightarrow \begin{cases} a = a^{*} & \text{real} \\ d = d^{*} & \text{"} \\ b = c^{*} \end{cases} \rightarrow \text{4-indep parameters}$$

$$\rightarrow \rho = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \qquad a, d : \text{real}$$

with the normalization cond. $\rightarrow$ there are only 3-indep parameters

The 3-numbers needed are $\begin{cases} [S_x] \\ [S_y] \\ [S_z] \end{cases}$

Having these <u>3</u> we may <u>construct</u> ρ.

Note that,
$$\rho = \sum_i^n w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$$

n: any number.

But still 3-numbers suffice to construct ρ (in spin $\frac{1}{2}$)!

$\xrightarrow[\text{suggests}]{\text{Strongly}}$ A <u>mixed</u> ensemble can be decomposed into <u>pure</u> ensembles in many diff. ways.

Pure ensemble:

$$\begin{cases} W_i = 1 & \text{for } i = n \\ W_i = 0 & \text{otherwise.} \end{cases}$$

$$\rho = |\alpha^{(n)}\rangle\langle\alpha^{(n)}| \qquad \text{pure ensemble}$$

Clearly $\rho^2_{pure} = \rho$ $\longrightarrow$ $\rho_{pure}(\rho_{pure} - 1) = 0$

$Tr(\rho^2) = Tr(\rho)$ Since $Tr(\rho) = 1$ $\longrightarrow$ $Tr(\rho^2) = 1$

Also

$$\rho_{pure}(\alpha^{(n)})\,|\alpha^{(i)}\rangle = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases} |\alpha^{(i)}\rangle$$ eigenvalue prob.

$$\langle \alpha^{(i)} | \rho_{pure} | \alpha^{(j)} \rangle = \langle \alpha^{(i)} | \alpha^{(n)} \rangle \langle \alpha^{(n)} | \alpha^{(j)} \rangle =$$

$$= \sum_{a'} \langle \alpha^{(i)} | a' \rangle \langle a' | \alpha^{(n)} \rangle \sum_{a''} \langle \alpha^{(n)} | a'' \rangle \langle a'' | \alpha^{(j)} \rangle$$

$$= \sum_{a'} C^*_{a'} \delta_{ia'} C_{a'} \delta_{na'} \sum_{a''} C^*_{a''} \delta_{na''} C_{a''} \delta_{ja''}$$

$$= \sum_{a'} |C_{a'}|^2 \delta_{in} \sum_{a''} |C_{a''}|^2 \delta_{nj} = \delta_{in}\delta_{nj} = \delta_{ij}$$

Note: $\{|a'\rangle\}$ diagonalizes ρ. $|\alpha^{(i)}\rangle$ is in terms of $\{|a'\rangle\}$ basis

$$\rho \doteq \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \\ & & & & & & & & 0 \end{pmatrix} \longleftarrow n \qquad |\alpha^{(n)}\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ |\alpha^{(n)}\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Also, $Tr(\rho^2_{pure}) = 1$ but $Tr(\rho^2_{general}) \leq 1$

(it can be shown).

Given a density op. let us construct the corresponding density matrix.

Recall $$|\alpha\rangle\langle\alpha| = \sum_{b'}\sum_{b''} |b'\rangle\langle b'|\alpha\rangle\langle\alpha|b''\rangle\langle b''|$$

Ex. A completely polarized beam with S_z^+.

$$\rightarrow \rho = |+\rangle\langle+| \doteq \begin{pmatrix}1\\0\end{pmatrix}(1, 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \quad \text{(pure)}$$

Ex. A completely polarized beam with $S_x^\pm$.

$$\rightarrow \rho = |S_x;\pm\rangle\langle S_x;\pm| = \left(\frac{1}{\sqrt{2}}\right)\left(\pm|+\rangle + |-\rangle\right)\left(\frac{1}{\sqrt{2}}\right)\left(\pm\langle+| + \langle-|\right)$$

$$= \frac{1}{2}\left[|+\rangle\langle+| \pm |+\rangle\langle-| \pm |-\rangle\langle+| + |-\rangle\langle-|\right] =$$

$$= \frac{1}{2}\left[\begin{pmatrix}1\\0\end{pmatrix}(1,0) \pm \begin{pmatrix}1\\0\end{pmatrix}(0,1) \pm \begin{pmatrix}0\\1\end{pmatrix}(1,0) + \begin{pmatrix}0\\1\end{pmatrix}(0,1)\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \pm \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \pm \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix}1 & \pm1\\\pm1 & 1\end{pmatrix} \quad \text{(pure)}$$

Ex. An unpolarized beam;

This can be regarded as an incoherent mixture of a spin-up ensemble and a spin-down ensemble with equal weights (50% each);

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \doteq \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left(\begin{array}{c}\text{unpolarized} \\ \underline{\text{not pure}}\end{array}\right)$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

So as we have remarked earlier, the same ensemble can also be regarded as an incoherent mixture of an S_x^+ and S_x^- ensembles with equal weights.

Since in this case $\rho = \frac{1}{2} I \longrightarrow \rho S_k = \frac{1}{2} S_k$

$\rightarrow \mathrm{Tr}(\rho S_k) = 0$ (remember $\mathrm{Tr}(S_k) = 0$)

Using $[A] = \mathrm{Tr}(\rho A) \rightarrow [S] = 0$

This is reasonable, because there should be no preferred spin dir. in a completely random ensemble of spin $\frac{1}{2}$ system.

Ex. Partialy polarized beam:

$$\begin{cases} 75\% & |S_z;+\rangle \\ 25\% & |S_x;+\rangle \end{cases} \quad \text{mixture.} \qquad \begin{cases} W(S_z^+) = 0.75 \\ W(S_x^+) = 0.25 \end{cases}$$

$$\rightarrow \rho \doteq \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

$$\rightarrow [S_x] = \frac{\hbar}{8}, \quad [S_y] = 0, \quad [S_z] = \frac{3\hbar}{8}$$

Time Evolution of Ensembles:

$$\rho(t) = ?$$

suppose $\rho(t_0) = \sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$

If the ensemble is to be left undisturbed, we cannot change the fractional population w_i -

The only change is in the states:

$$|\alpha^{(i)}\rangle \xrightarrow{\text{time-evolution}} |\alpha^{(i)}, t_0, t\rangle$$

Now,

$$i\hbar \frac{\partial}{\partial t} |\alpha^{(i)}, t_0, t\rangle = H |\alpha^{(i)}, t_0, t\rangle$$

$$\rightarrow \quad \langle\alpha^{(i)}, t_0, t| \left(-i\hbar \overleftarrow{\frac{\partial}{\partial t}}\right) = \langle\alpha^{(i)}, t_0, t| H$$

$$\rightarrow \begin{cases} i\hbar\left(\frac{\partial}{\partial t}|\alpha^{(i)},t_0,t\rangle\right)\langle\alpha^{(i)},t_0,t| = \left(H|\alpha^{(i)},t_0,t\rangle\right)\langle\alpha^{(i)},t_0,t| \\ |\alpha^{(i)},t_0,t\rangle\left(\langle\alpha^{(i)},t_0,t|\left(-i\hbar\frac{\overleftarrow{\partial}}{\partial t}\right)\right) = |\alpha^{(i)},t_0,t\rangle\left(\langle\alpha^{(i)},t_0,t|H\right) \end{cases}$$

$$\rightarrow i\hbar\frac{\partial\rho}{\partial t} = \sum_i w_i\left(H|\alpha^{(i)},t_0,t\rangle\langle\alpha^{(i)},t_0,t| - |\alpha^{(i)},t_0,t\rangle\langle\alpha^{(i)},t_0,t|H\right)$$

$$\rightarrow i\hbar\frac{\partial\rho}{\partial t} = \underset{\rho}{-}[\rho, H]$$

This looks like the Heisenberg equ. of motion except than the sign is wrong!

There is no prob., because ρ is not a dynamical observable in the Heisenberg pict. -

On the contrary, ρ is built up of the Schrödinger-pict. state kets and state bras which evolve in time acc. to the Schrödinger equ.

This equ. can be regarded as the Q. mechanical analogue of Liouville's theorem in cl. statistical M.,

$$\frac{\partial\rho_{cl.}}{\partial t} = -\left[\rho_{cl.}, H\right]_{cl.}$$

where $\rho_{cl.}$: the density for the representative points in the phase space.

The classical analogue of the relation for $[A]$ is,

$$A_{\text{average}} = \frac{\int \rho_{cl.}\, A(q,p)\, d\Gamma_{p,q}}{\int \rho_{cl.}\, d\Gamma_{p,q}}$$

$d\Gamma_{p,q}$: vol. element in phase space.

Remark: A pure cl. state is one represented by a single moving point in phase space $(q_1 \cdots q_f, p_1 \cdots p_f)$ at each instant of time.

A cl. statistical state, on the other hand, is described by our nonnegative density func. $\rho_{cl.}(q_1 \cdots q_f, p_1 \cdots p_f)$, such that the probability that a system is found in the interval $dq_1 \cdots dp_f$ is $\rho_{cl.}\, dq_1 \cdots dp_f$.

Continuum Generalization:

Let's consider a continuous base, say $\{|x'\rangle\}$;

$$[A] = \int d^3x' \int d^3x'' \langle x''|\rho|x'\rangle \langle x'|A|x''\rangle$$

The density matrix:

$$\langle x''|\rho|x'\rangle = \langle x''|\left(\sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|\right)|x'\rangle$$

$$= \sum_i w_i\, \psi_i(x'')\, \psi_i^*(x')$$

For $x'=x''$: $\langle x'|\rho|x'\rangle = \sum_i w_i |\psi_i(x')|^2$

In continuum cases, too, it is important to keep in mind, that the <u>same</u> ensemble can be de<u>composed</u> in different ways into <u>pure</u> ensembles.

<u>Ex.</u>

Realistic beam of particles $= \sum w_i e^{ik_i \cdot x}$ plane waves

or " $= \sum w_i' e^{-\frac{x^2}{a^2}}$ wave packets

Quantum Statistical Mechanics:

The density matrix of a <u>completely random</u> ensemble look like

$$\rho \doteq \frac{1}{N}\begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix} \quad (1)$$

(in any representation)

Note: N is the dim of the ket-space (because for random case we have to consider all of the alternatives)

This follows from the fact that all states corresponding to the base kets with respect to which the density matrix is written are equally populated.

In contrast in the basis where ρ is diagonal, we have

$$\rho \doteq \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} \quad (2)$$

for the matrix representation of the density op. for a pure ensemble.

Both satisfying $\mathrm{tr}(\rho) = 1$ (normalization requirement)

Let us construct a quantity that characterizes this dramatic difference:

$$\sigma = -\mathrm{Tr}(\rho \ln \rho)$$

The meaning of this eqn. is quite unambiguous if we use the basis in which ρ is diagonal

$$\sigma = -\sum_k \rho_{kk}^{diag} \ln \rho_{kk}^{diag}$$

Remark: $\mathrm{Tr}(AB) = \sum_i \sum_j a_{ij} b_{ji}$

$\longrightarrow \mathrm{Tr}(AB) = \sum_k a_{kk} b_{kk}$ if A and B are diagonal

Since $\quad 0 \leq \rho_{kk}^{diag} \leq 1 \quad (\rightarrow -\infty < \ln \rho_{kk}^{dig} < 0 \,)$

For completely random ensemble (equ. 1) we have

$$\alpha_{random} = -\sum_{k=1}^{N} \frac{1}{N} \ln\left(\frac{1}{N}\right) = \ln N$$

In contrast since for a pure ensemble (equ. 2) we have

$$\left\{ \begin{array}{ll} \text{either} & \rho_{kk}^{diag} = 0 \\ \text{or} & \ln \rho_{kk}^{diag} = 0 \end{array} \right. \longrightarrow \alpha_{pure} = 0$$

$\rightarrow$ α is a measure of disorder.

In pure ensemble: we have max. amount of order because all members are characterized by the same Q. mechanical state ket.

$\alpha = 0$

In completely random ensemble: All Q. mechanical states are equally likely

$\alpha = \ln N$ (we will show later this is the max. possible value, for α subject to the normalization cond. $\sum_k \rho_{kk} = 1$)

In thermodynamics the entropy measures disorder

So, σ is related to the entropy S;

$$S = k\sigma$$

k: universal const. identifiable with the Boltzmann const.

↑

Def. of entropy in Q. Statistical M.

Our aim; $\rho = ?$ for an ensemble in thermal equilibrium

In thermal equilibrium $\frac{\partial \rho}{\partial t} = 0$

$$i\hbar \frac{\partial \rho}{\partial t} = -[\rho, H] \longrightarrow [\rho, H] = 0$$

→ ρ and H can be simultaneously diagonalized.

So the base kets used in writing $\sigma = -\sum_k \rho_{kk}^{diag} \ln \rho_{kk}^{diag}$ may be taken to be energy eigenkets.

With this choice ⟶ ρ_{kk}: stands for fractional population for an energy eigenstate with energy eigenvalue E_k

Basic assumption:

Nature tends to <u>maximize</u> σ subject to the constraint that the ensemble average of the Hamiltonian has a <u>prescribed</u> value.

$$\delta\sigma = 0 \qquad (3) \quad \text{(maximizing requirement)}$$

$$[H] = \mathrm{Tr}(\rho H) = U \quad (4) \qquad U\text{: prescribed value}$$

In addition $\sum_k \rho_{kk} = 1$ (5) normalization cond.

$$\begin{cases} \rightarrow \delta[H] = \sum_k \delta\rho_{kk} E_k = 0 & \text{constraint} \quad (6) \\ \mathrm{Tr}(\rho) = 1 \rightarrow \delta\,\mathrm{Tr}(\rho) = \sum_k \delta\rho_{kk} = 0 & \text{〃} \quad (7) \end{cases}$$

$$\delta\sigma = \delta\left[-\sum_k \rho_{kk} \ln \rho_{kk}\right] = 0$$

$$\rightarrow \sum_k \left[\delta\rho_{kk} \ln \rho_{kk} + \rho_{kk}\,\delta \ln \rho_{kk}\right] = 0$$

$$\sum_k \delta\rho_{kk}(\ln \rho_{kk} + 1) = 0 \qquad (8)$$

Now we use the method of Lagrange <u>multiplier</u> method to include the constraints;

$$
\begin{array}{ll}
(6) \rightarrow & \left\{ \begin{array}{l} \beta \sum \delta_{\rho_{kk}} E_k = 0 \\ \gamma \sum_k \delta_{\rho_{kk}} = 0 \end{array} \right. \qquad (9) \\
(7) \rightarrow &
\end{array}
$$

$$
(8)(9) \longrightarrow \quad \sum_k \delta\rho_{kk} \left[(\ln \rho_{kk} + 1) + \beta E_k + \gamma \right] = 0 \qquad (10)
$$

For an arbitrary variation this is possible if:

$$
\ln \rho_{kk} + 1 + \beta E_k + \gamma = 0 \longrightarrow \rho_{kk} = e^{-\beta E_k - \gamma - 1}
$$

γ can be eliminated using (5)

$$
\sum_k \rho_{kk} = \sum_k e^{-\beta E_k - \gamma - 1} = e^{-\gamma - 1} \sum_k e^{-\beta E_k} = 1
$$

$$
e^{-\gamma-1} = \frac{1}{\sum_k e^{-\beta E_k}} \longrightarrow \rho_{kk} = \frac{e^{-\beta E_k}}{\sum_{\ell}^{N} e^{-\beta E_\ell}} \qquad (11)
$$

which directly gives the fractional population for an energy eigenstate with eigenvalue E_k -

Note: It is to be understood throughout that the sum is over distinct eigenstates; if there is a degeneracy we must sum over states with the same energy eigenvalue.

The density matrix element (11) is appropriate for what is known in statistical mechanics as a canonical ensemble.

If we maximize σ without the internal energy constraint (6) we get;

$$\rho_{kk} = \frac{1}{N} \qquad \text{(indep. of } k\text{)}$$

This is the density matrix element for completely random ensemble.

Indeed;

$$(11) \longrightarrow \lim_{\beta \to 0} \rho_{kk} = \frac{1}{N} \quad \text{(high temperature limit)}$$

i.e. Completely random ensemble can be regarded as the $\beta \to 0$ limit of a canonical ensemble.

Now; the denominator of (11);

$$Z = \sum_{k=1}^{N} e^{-\beta E_k} \qquad \text{partition func. in S.M.}$$

Also since $\mathrm{Tr}(A)$ = the sum of the eigenvalues of A

$$\rightarrow \quad Z = \mathrm{Tr}\,(e^{-\beta H})$$

Knowing ρ_{kk} given in the energy basis, we can write the density op. as;

$$\rho = \frac{e^{-\beta H}}{Z}$$

$$\rightarrow \quad [A] = Tr(\rho A) = \frac{Tr(\rho^{-\beta H} A)}{Z}$$

$$= \frac{\left(\sum_{k=1}^{N} \langle A \rangle_k \, e^{-\beta E_k}\right)}{\sum_{k=1}^{N} e^{-\beta E_k}}$$

For the internal energy per constituent,

$$U = Tr(\rho H) = \frac{\left(\sum_{k=1}^{N} E_k \, e^{-\beta E_k}\right)}{\sum_{k=1}^{N} e^{-\beta E_k}} = -\frac{\partial}{\partial \beta}(\ln Z)$$

$$\beta = \frac{1}{KT} \qquad K: \text{Boltzmann const.}$$

We saw;

In high temp. limit, $\beta \to 0$,

A canonical ensemble $\xrightarrow{\text{becomes}}$ a completely random ensemble in which all energy eigenstates are equally populated.

In low temp. limit, $\beta \to \infty$, $\left(\rho_{kk} \Big|_{k=l} = \frac{e^{-\beta E_l}}{0+\cdots 0 + e^{-\beta E_l} + 0 \cdots + 0} = 1, \ \rho_{kk} \Big|_{k\neq l} = 0 \right)$

(11) tells us, that a canonical ensemble $\xrightarrow{\text{becomes}}$ a pure ensemble when only the <u>ground state</u> is populated.

<u>Ex.</u>

A canonical ensemble made up of spin $\frac{1}{2}$ system, each with a mag. mom. $\frac{e\hbar}{2m_e c}$ subjected to a <u>uniform</u> mag. field in the z-dir.

$$H = -\left(\frac{e}{m_e c}\right) S \cdot B = \omega S_z \quad , \ \omega \equiv \frac{|e|B}{m_e c}$$

Since $[H, S_z] = 0 \longrightarrow \rho$ for this canonical ensemble is <u>diagonal</u> in the S_z basis.

$$\rightarrow \rho \doteq \frac{\begin{pmatrix} e^{-\beta\hbar\omega/2} & 0 \\ 0 & e^{\beta\hbar\omega/2} \end{pmatrix}}{Z} \quad , \quad Z = e^{-\beta\hbar\omega/2} + e^{\beta\hbar\omega/2}$$

$[S_x] = \mathrm{Tr}(\rho S_x) = 0$, $[S_y] = \mathrm{Tr}(\rho S_y) = 0$

$[S_z] = \mathrm{Tr}(\rho S_z) = -\frac{\hbar}{2} \tanh\left(\frac{\beta\hbar\omega}{2}\right)$

$[\mu_z] = \frac{e}{m_e c}[S_z]$

Also from $[\mu_z] = \chi B \rightarrow \chi = \left(\frac{|e|\hbar}{2m_e c B}\right) \tanh\left(\frac{\beta\hbar\omega}{2}\right)$

χ : susceptibility

3-5 Eigenvalues and Eigenstates of Angular Momentum.

We now study more general angular mom. states

$(N = 2 \longrightarrow N = \text{arbitrary})$

Commutation Relations and the Ladder OPS.;

we derived;
$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k \qquad (1)$$

J_ℓ: the generator of infintesimal rot.

Now $\quad (1) \longrightarrow [J^2, J_k] = 0 \quad (k=1,2,3) \qquad (2)$

where $\quad J^2 = J_x J_x + J_y J_y + J_z J_z$

Proof : $[J_x J_x + J_y J_y + J_z J_z, J_z] = J_x [J_x, J_z] + [J_x, J_z] J_x$
$$+ J_y [J_y, J_z] + [J_y, J_z] J_y + 0$$

$$= J_x(-i\hbar J_y) + (-i\hbar J_y) J_x + J_y (i\hbar J_x) + (i\hbar J_x) J_y = 0$$

The proof for $k=1$, and $k=2$ follow by cyclic permutation

$(1 \to 2 \to 3 \to 1)$

Since $\begin{cases} [J_i, J_j] \neq 0 & \text{for } i \neq j \\ [J_i, J^2] = 0 \end{cases}$

→ We can choose only one of them to be the observable to be diagonalized simultaneously with J^2.

We choose J_z.

$$\begin{cases} J^2 |a,b\rangle = a|a,b\rangle \\ J_z |a,b\rangle = b|a,b\rangle \end{cases} \qquad a = ? \quad b = ? \qquad (3)$$

Consider the ladder ops.;

$$J_\pm \equiv J_x \pm i J_y \qquad (4)$$

where $[J_+, J_-] = 2\hbar J_z$, and $[J_z, J_\pm] = \pm \hbar J_\pm$ (5)

The Physical meaning of $J_\pm$;

$$\begin{aligned} J_z \left(J_\pm |a,b\rangle \right) &= \left([J_z, J_\pm] + J_\pm J_z \right) |a,b\rangle \\ &= (b \pm \hbar) \left(J_\pm |a,b\rangle \right) \end{aligned} \qquad (6)$$

→ $J_\pm |a,b\rangle$ is the eigenket of J_z with eigenvalue increased or decreased by $\hbar$.

Now;

$$[x, \tau(dx')] = dx' I$$

$$[x_i, \tau(\bar{l})] = [x_i, e^{-i\frac{p.l}{\hbar}}] = l_i\, e^{-i\frac{p.l}{\hbar}} = l_i\, \tau(\bar{l}) \qquad (7)$$

where we have used $[x_i, F(p)] = i\hbar \dfrac{\partial F}{\partial p_i}$

Also we had

$$[N, a^+] = a^+$$
$$[N, a\ ] = -a \qquad (8)$$

We see that (5), (7) and (8) have a similar structure,

i) $\tau(l)$ changes the eigenvalue of x op. by $\bar{l}$

ii) a^+ increases " " " N " " units

iii) J_+ changes " " " J_z " " $\hbar$

But $J^2(J_\pm|a,b\rangle) = J_\pm J^2|a,b\rangle = a(J_\pm|a,b\rangle)$

$\rightarrow$ $J_\pm$ does not change the eigenvalue of J^2

$\rightarrow$ $J_\pm|a,b\rangle$: Simultaneous eigenkets of J^2 and J_z with the eigenvalues a, and $b\pm\hbar$

We may write: $J_\pm|a,b\rangle = c_\pm |a, b\pm\hbar\rangle \qquad (9)$

Eigenvalues of J^2 and J_z

$$(J_+)^n |a,b\rangle \sim |a, b+n\hbar\rangle \qquad (1)$$

But there is a limitation for n.

There is a upper limit for $b' = n+\hbar$ which in turn restricts n.

That is $\quad a \geq b^2 \qquad (2)$

Proof: $\quad J^2 - J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{2}(J_+J_+^\dagger + J_+^\dagger J_+)$

Since $\quad (\langle a,b| J_+)(J_+^\dagger |a,b\rangle) \geq 0$

$\quad (\langle a,b| J_+^\dagger)(J_+ |a,b\rangle) \geq 0$

$\rightarrow \quad \langle a,b| (J^2 - J_z^2) |a,b\rangle \geq 0 \quad \rightarrow a \geq b^2$

$\rightarrow$ There exists a, b_{max}, such that $\quad J_+ |a, b_{max}\rangle = 0 \qquad (3)$

$(3) \rightarrow J_-J_+ |a, b_{max}\rangle = 0 \qquad (4)$

But, $\quad J_-J_+ = J_x^2 + J_y^2 - i(J_yJ_x - J_xJ_y) = J^2 - J_z^2 - \hbar J_z \qquad (5)$

$\rightarrow \quad (J^2 - J_z^2 - \hbar J_z) |a, b_{max}\rangle = 0 \qquad (6)$

$\rightarrow (a - b_{max}^2 - \hbar b_{max}) |a, b_{max}\rangle = 0 \qquad (7)$

since $|a, b_{max}\rangle \neq 0$ (not null ket)

$$\rightarrow a - b_{max}^2 - \hbar b_{max} = 0 \quad \rightarrow a = b_{max}(b_{max}+\hbar) \qquad (8)$$

In a similar manner;

$$J_-|a, b_{min}\rangle = 0$$

$$J_+J_- = J^2 - J_z^2 - \hbar J_z$$

$$\rightarrow a = b_{min}(b_{min}-\hbar) \qquad (9)$$

$$\rightarrow b_{min} \leq b \leq b_{max}$$

$$(2), (8), (9) \rightarrow b_{min} = -b_{max} \qquad (10)$$

$$\rightarrow -b_{max} < b < b_{max} \qquad (11)$$

Remark: $a \geq 0$ and $b_{max}(b_{max}+\hbar) = b_{min}(b_{min}-\hbar)$

Suppose from $|a, b_{min}\rangle \xrightarrow{(J_+)^n} |a, b_{max}\rangle$ n: integer

$$\rightarrow b_{max} = b_{min} + n\hbar \quad \rightarrow b_{max} = \frac{n}{2}\hbar \qquad (12)$$

Define $j \equiv \frac{n}{2}$ $\longrightarrow$ $\begin{cases} b_{max} = j\hbar \\ b_{min} = -j\hbar \end{cases}$

$$a = \hbar^2 j(j+1)$$

Let us also define; $b \equiv m\hbar$

If $\begin{cases} j: \text{integer} \longrightarrow \text{all } m: \text{integer} \\ j: \text{half-}\text{''} \longrightarrow \text{'' '' : half-''} \end{cases}$

Remark: Remember n is integer (because it is the number of steps.)

$$m = \underbrace{-j,\ -j+1,\ \cdots\ \cdots j-1,\ j}_{2j+1 \text{ states}} \qquad \text{allowed values}$$

$$|a, b\rangle \longrightarrow |j, m\rangle$$

$$\begin{aligned} J^2 |j, m\rangle &= j(j+1)\hbar^2 |j, m\rangle \\ J_z |j, m\rangle &= m\hbar |j, m\rangle \end{aligned} \qquad (13)$$

Note that, the quantization of angular momentum, manifested in (13), is the <u>direct</u> consequence of ang. mom. commutation relations, (which in turn, follow from the <u>properties</u> of rotations, together with the def. of J_k as the <u>generator</u> of rotation).

Matrix Elements of Angular Momentum Operators:

Assume: $|j\,m\rangle$ normalized

$$\langle j', m' | J^2 | j, m\rangle = j(j+1)\hbar^2\, \delta_{j'j}\, \delta_{m'm} \qquad (1)$$

$$\langle j', m' | J_z | j, m\rangle = m\hbar\, \delta_{j'j}\, \delta_{m'm} \qquad (2)$$

$$\langle j' m' | J_\pm | j, m\rangle = ?$$

First consider $\langle j\,m | J_+^\dagger J_+ | j\,m\rangle = \langle j\,m | (J^2 - J_z^2 - \hbar J_z) | j\,m\rangle$

$$= \hbar^2 \left[j(j+1) - m^2 - m \right] \qquad (3)$$

Now, $J_+ | j, m\rangle \sim | j, m+1\rangle \longrightarrow J_+ | j, m\rangle = C_{jm}^+ | j, m+1\rangle$

$$\langle j, m | J_+^\dagger J_+ | j, m\rangle = |C_{jm}^+|^2 \langle j, m+1 | j, m+1\rangle = |C_{jm}^+|^2 \qquad (4)$$

$(3)(4) \rightarrow \quad |C_{jm}^+|^2 = \hbar^2 \left[j(j+1) - m(m+1) \right] = \hbar^2 (j-m)(j+m+1)$

$$C_{jm}^+ = e^{i\varphi} \sqrt{(j-m)(j+m+1)}\;\hbar$$

It is customary to choose C_{jm}^+ to be <u>real</u> and <u>positive</u>.

$\rightarrow \quad \varphi = 0 \qquad C_{jm}^+ = \sqrt{(j-m)(j+m+1)}\;\hbar$

$$J_+ | j, m\rangle = \sqrt{(j-m)(j+m+1)}\;\hbar\, | j, m+1\rangle$$

similarly; $J_- | j, m\rangle = \sqrt{(j+m)(j-m+1)}\;\hbar\, | j, m-1\rangle$

$\rightarrow \quad \langle j', m' | J_\pm | j, m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\;\hbar\, \delta_{j'j}\, \delta_{m', m\pm 1}$

Ref. QC174.12.T48 Q.M. (T.K. Thankappan)

Irreducible Tensor

Consider a cartezian tensor of rank 2 under rotation;

$$T_{ij} \xrightarrow{(\alpha,\beta,\gamma)} T'_{ij} = \sum_{kl} r_{ik}\, r_{jl}\, T_{kl} \qquad (T' = R^{-1} T R)$$

similarity to

T: nonsingular

R: Orthoonal rot. matrix

Out of 9 components of T, we can form, 3 group of linear combinations of the components;

$$S = \frac{1}{3} \sum_i T_{ii}$$

antisym. tensor (rank 3, 27 elements)

$$V_k = \frac{1}{2}(T_{ij} - T_{ji}) = \frac{1}{2}\epsilon_{ijk} T_{ij}$$

i, j, k cyclic (antisymmetric)

(1)

$$A_{ij} = \frac{1}{2}(T_{ij} + T_{ji} - 2S\delta_{ij})$$

Such that, $T_{ij} = A_{ij} + \epsilon_{ijk} V_k + S\delta_{ij}$ (2)

The peculiarity of the above 3 group, is that, each of them transforms, under rotations, independently of the other two.

i) Since $Tr(T') = Tr(R^{-1} T R) = Tr(R^{-1} R T) = Tr(T)$

S, is invariant under rot. $\longrightarrow$ S is scalar (tensor with rank = 0)

ii) V_1, V_2, V_3 : 3-indep. components of an antisymmetric, second rank tensor:

$\longrightarrow$ V_1, V_2, V_3 transform like a vector (tensor rank of 1).

iii) A : Traceless, symmetric tensor of rank 2 $\longrightarrow$

it has 5-indep. components which transform among themselves under rotations. ($A_{ij} = A_{ji}$, $\Sigma A_{ii} = 0$)

Thus, under rots. each of the 3-groups has a status, that is independent of the other two (Each of them transforms under rots. independently of the other subgroups)

i.e. Each of them is a tensor.

These tensors (their elements) can not be decomposed into smaller subgroups $\longrightarrow$ They are called irreducible tensors.

On the other hand, a tensor like T, whose components or linear combinations of the components, can be divided into two or more groups, which transform under rots. among themselves, is a reducible tensor.

The <u>spherical</u> tensors are nothing but the <u>irreducible</u> tensors that result from the grouping of the components of the general (cartesian) tensor as explained above.

Of course the components given by (1) are not the spherical components that transform like the components of the <u>spherical harmonics</u>.

The spherical components of V and A:

Note that;

$$\begin{cases} R_1^{\pm 1} = \mp \dfrac{x \pm iy}{\sqrt{2}} = \mp \dfrac{1}{\sqrt{2}}(r\, S\theta\, C\varphi \pm i r\, S\theta\, S\varphi) \\ = \mp \dfrac{r}{\sqrt{2}} S\theta\, (C\varphi \pm i S\varphi) = \mp \dfrac{1}{\sqrt{2}} r\, S\theta\, e^{\pm i\varphi} \\ R_1^0 = z = r\, C\theta \end{cases}$$

also $$Y_1^q = \begin{cases} Y_1^{\pm 1} = \mp\sqrt{\dfrac{3}{8\pi}}\, S\theta\, e^{\pm i\varphi} \\ Y_1^0 = \sqrt{\dfrac{3}{4\pi}}\, C\theta \end{cases} \qquad \begin{cases} r Y_1^0 = \sqrt{\dfrac{3}{4\pi}}\, z \\ r Y_1^{\pm 1} = \sqrt{\dfrac{3}{4\pi}}\left(\mp \dfrac{x \pm iy}{\sqrt{2}}\right) \end{cases}$$

$$\rightarrow R_1^q = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^q(\theta,\varphi)$$

we conclude, $$\begin{cases} T_1^{\pm 1} = V_{\pm 1} = \mp \dfrac{V_1 \pm i V_2}{\sqrt{2}} = \mp \dfrac{V_x \pm i V_y}{\sqrt{2}} \\ T_1^0 = V_0 = V_z \end{cases}$$

Similarly:
$$\begin{cases} r^2 Y_2^0 = \sqrt{\frac{5}{16\pi}}\,(2z^2 - x^2 - y^2) \\ r^2 Y_2^{\pm1} = \sqrt{\frac{5}{16\pi}}\,\left(\mp\sqrt{6}\,(xz \pm iyz)\right) \\ r^2 Y_2^{\pm2} = \sqrt{\frac{5}{16\pi}}\,\left(\sqrt{\frac{3}{2}}\,(x^2 - y^2 \pm 2ixy)\right) \end{cases}$$

$$\begin{cases} Y_2^0 = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1) \\ Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{\pm i\varphi} \\ Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{\pm 2i\varphi} \end{cases}$$

$$\rightarrow \begin{cases} T_2^0 = 2A_{33} - A_{11} - A_{22} \\ T_2^{\pm1} = \mp\sqrt{6}\,(A_{13} \pm iA_{23}) \\ T_2^{\pm2} = \sqrt{\frac{3}{2}}\,(A_{11} - A_{22} \pm 2iA_{12}) \end{cases}$$

Remark: Def.: Similitary tr. $A \to S^{-1}AS$ (S: nonsingular)

Def.: If S is unitary, the tr. is called unitary.

$Tr(A') = Tr(S^{-1}AS) = Tr(S^{-1}SA) = Tr(A)$

$\det A' = \det(S^{-1}AS) = \det(S^{-1}S)\cdot\det(A) = \det A$

If $S = U$

$A'^{+} = (U^{-1}AU)^{+} = U^{+}A(U^{-1})^{+} = U^{-1}AU = A'$ when $A^{+} = A$ (A: Hermitian)

$A'^{+} = (U^{-1}AU)^{+} = U^{+}A(U^{-1})^{+} = U^{-1}A^{-1}U = (U^{-1}AU)^{-1} = A'^{-1}$ when $A^{+} = A^{-1}$ (A: unitary)

Representation of the Rotation Operator

Having obtained the matrix elements of J_z and $J_\pm$, we are now in a position to study the matrix elements of the rotation op. $D(R)$.

$$D^{(j)}_{m'm}(R) = \langle j,m' | D(R) | j,m\rangle = \langle j,m'| e^{-\frac{i\vec{J}\cdot\hat{n}\phi}{\hbar}} |j,m\rangle \qquad (1)$$

↑ Wigner funcs.

Since $$J^2\left(D(R)|j,m\rangle\right) = D(R)J^2|j,m\rangle = j(j+1)\hbar^2\left(D(R)|j,m\rangle\right) \qquad (2)$$

→ The effect of $D(R)$ on $|j,m\rangle$ does not change j-value

→ $j' \underline{\underline{\text{must}}}\ j$ in (1)

Remark: $[J^2, D(R)]=0$

because $[J^2, J_k]=0 \rightarrow [J^2, f(J_k)]=0$

Conclusion: Rotations cannot change the j-value.

The $(2j+1)\times(2j+1)$ matrix formed by $D^{(j)}_{m'm}(R)$ is reffered to as the $(2j+1)$-dimensional irreducible representation of the rotation op $D(R)$.

Ref.: Merzbacher

Equivalence Transformation:

A change of the basis $\xrightarrow{\text{changes}}$ the matrices of a representation

$$D'(a) = S^{-1} D(a) S \qquad (1)$$

Two representations which can be transformed into each other by a similarity tr. S, are not really different, and are called equivalent.

Tr. S in (1) is known as an equivalence tr.

Two representations are inequivalent, if there is no tr. S which will take one into the other (for example $j=2$ and $j=3$)

For an arbitrary given representation it is frequently possible to derive simpler representation by choosing a special basis in which all matrices of the representation simultaneously break up into a number of submatrices arrayed along the diagonal;

D: reducible

D_i: irreducible if it can not break up more

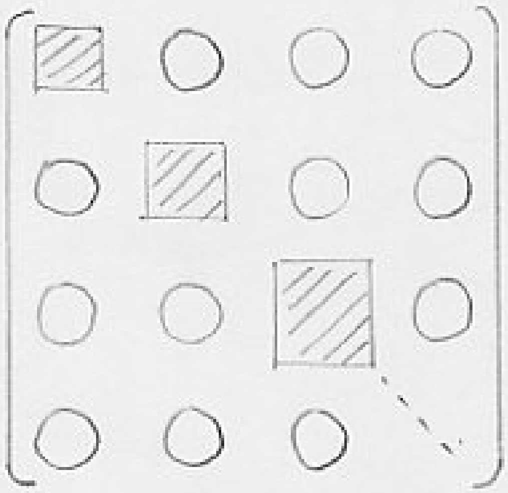
$$D(a) = \begin{pmatrix} D_1(a) & 0 & 0 & \cdots \\ 0 & D_2(a) & 0 & \cdots \\ 0 & 0 & D_3(a) & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

$n = n_1 + n_2 + \cdots$

Dim. of D ; Dim. of D_1

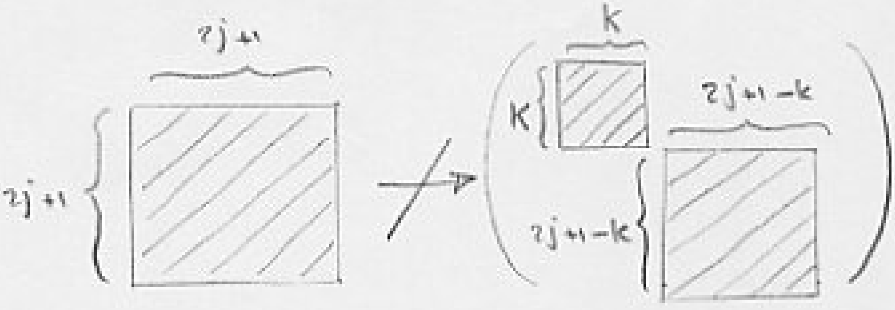
$$D = \begin{pmatrix} \cdot & & & & 0 \\ & \vdots & & & \\ & & \vdots & & \\ & & & \vdots & \\ 0 & & & & \ddots \end{pmatrix}$$

This means that the matrix which corresponds to an arbitrary rot. op. in ket space not necessarily characterized by a single j-value, can, with a suitable choice of basis, be brought to block-diagonal form,



where each shaded square is a $(2j+1)\times(2j+1)$ square matrix formed by $D^{(j)}_{m'm}$ with some definite value of j. (3)

Furthermore, each square matrix itself cannot be broken into smaller blocks.



It can be proved that decomposition (3) is unique.

Remark:

$$
J^2 = \hbar^2 \begin{pmatrix}
0 & & & & \\
& \begin{matrix} 3/4 & 0 \\ 0 & 3/4 \end{matrix} & & & \\
& & \begin{matrix} 2 & & 0 \\ & 2 & \\ 0 & & 2 \end{matrix} & & \\
& & & \begin{matrix} 15/4 & & & 0 \\ & 15/4 & & \\ & & 15/4 & \\ 0 & & & 15/4 \end{matrix} & \\
& & & & \begin{matrix} 6 & & 0 \\ & 6 & \\ 0 & & \ddots \end{matrix}
\end{pmatrix}
, \quad
J_z = \hbar \begin{pmatrix}
0 & & & & & \\
& \begin{matrix} 1/2 & 0 \\ 0 & -1/2 \end{matrix} & & & & \\
& & \begin{matrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{matrix} & & & \\
& & & \begin{matrix} 3/2 & & & 0 \\ & 1/2 & & \\ & & -1/2 & \\ 0 & & & -3/2 \end{matrix} & & \\
& & & & \begin{matrix} 2 & & & & 0 \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & 2 \end{matrix} & \\
& & & & & \begin{matrix} 5/2 & & 0 \\ & 5/2 & \\ 0 & & \ddots \end{matrix}
\end{pmatrix}
$$

$$
J_+ = \hbar \begin{pmatrix}
0 & & & & & \\
& \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & & & & \\
& & \begin{matrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{matrix} & & & \\
& & & \begin{matrix} 0 & \sqrt{3} & 0 & 0 \\ & 0 & 2 & 0 \\ & & 0 & \sqrt{3} \\ 0 & & & 0 \end{matrix} & & \\
& & & & \begin{matrix} 0 & 2 & & & \\ & 0 & \sqrt{6} & & \\ & & 0 & \sqrt{6} & \\ & & & 0 & 2 \\ & & & & 0 \end{matrix} & \\
& & & & & \begin{matrix} 0 & \sqrt{5} & & \\ & 0 & 2\sqrt{2} & \\ & & 0 & \ddots \end{matrix}
\end{pmatrix}
$$

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & & & & & \\ & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & & 0 \\ & & \begin{matrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{matrix} & & & \\ & & & \begin{matrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{matrix} & & \\ & 0 & & & \begin{matrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{matrix} & \\ & & & & & \ddots \end{pmatrix}$$

$$J_y = \frac{i\hbar}{2}\begin{pmatrix} 0 & & & & & \\ & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & & & 0 \\ & & \begin{matrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{matrix} & & & \\ & & & \begin{matrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{matrix} & & \\ & 0 & & & \begin{matrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{matrix} & \\ & & & & & \ddots \end{pmatrix}$$

The rotation matrices of <u>definite</u> j <u>form a group</u>.

i) First, $D^{(j)}(\hat{n},0)=I$ $\quad (2j+1)\times(2j+1)$ matrix

This is the identity op. $\in$ Group.

ii) Second $\quad D^{(j)^{-1}}(\hat{n},\varphi) \quad D^{(j)}(\hat{n},-\varphi)$

The <u>inverse</u> is also a member.

iii) Third, the product of any two members is <u>also</u> a member;

Explicitly: $$\sum_{m'} D^{(j)}_{m m'}(R_1)\, D^{(j)}_{m'm}(R_2) = D^{(j)}_{m''m}(\underbrace{R_1R_2}_{\text{a single rot.}})$$

(can be proved by explicit forms of $D^{(j)}_{m'm}$'s)

Since $\begin{cases} \text{(i) The rot. op. is unitary} \\ \text{(ii) The base is orthonormal} \end{cases} \longrightarrow$ The rot. matrices are <u>unitary</u>.

Explicitly, from; $D^{(j)}_{m'm}(R) = \langle j,m'| e^{-\frac{i\mathbf{J}\cdot\hat{n}\varphi}{\hbar}} |j,m\rangle$

$$D_{m'm}(R^{-1}) = \langle j,m'| e^{+\frac{i\mathbf{J}\cdot\hat{n}\varphi}{\hbar}} |j,m\rangle = \langle j,m| e^{-\frac{i\mathbf{J}\cdot\hat{n}\varphi}{\hbar}} |j,m'\rangle^{*}$$

$$= D^{*}_{mm'}(R) \qquad \text{complex transposed}$$

Note that;
$$\sum_m D_{m'm}(R^{-1}) D_{mm''}(R) = \delta_{m'm''}$$
$$\to \sum_m \langle jm'| e^{\frac{iJ\cdot n\varphi}{\hbar}} |jm\rangle\langle jm| e^{-\frac{iJ\cdot n\varphi}{\hbar}} |jm''\rangle = \sum_m \langle jm| e^{-\frac{iJ\cdot n\varphi}{\hbar}} |jm'\rangle^{*} \langle jm| e^{-\frac{iJ\cdot n\varphi}{\hbar}} |jm''\rangle$$
$$= \sum_m D^{*}_{mm'}(R) D_{mm''}(R) = \delta_{m'm''}$$

Now consider a rot.;

$$|j,m\rangle \xrightarrow{D} D(R)\,|j,m\rangle$$

$$D(R)\,|j,m\rangle = \sum_{m'} |j,m'\rangle\langle j,m'|\,D(R)\,|j,m\rangle = \sum_{m'} |j,m'\rangle\, D^{(j)}_{m'm}(R)$$

Remark: $D(R)$ connects only states with the same j -

$\rightarrow D^{(j)}_{m'm}(R)$: The amplitude of the rotated state to be found in $|j,m'\rangle$ (after rot.)

Now, $D(\alpha,\beta,\gamma) = D_z(\alpha)\, D_y(\beta)\, D_z(\gamma)$

$$D^{(j)}_{m'm}(\alpha,\beta,\gamma) = \langle j,m'|\, e^{-\frac{iJ_z\alpha}{\hbar}}\, e^{-\frac{iJ_y\beta}{\hbar}}\, e^{-\frac{iJ_z\gamma}{\hbar}}\, |j,m\rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \langle j,m'|\, e^{-\frac{iJ_y\beta}{\hbar}}\, |j,m'\rangle$$

Define; $$d^{(j)}_{m'm}(\beta) \equiv \langle j,m'|\, e^{-\frac{iJ_y\beta}{\hbar}}\, |j,m\rangle$$

Let us turn to some examples;

For spin $\frac{1}{2}$, we have already obtained;

$$d^{(1/2)} = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

The next simplest case is $j=1$;

$$J_y = \frac{J_+ - J_-}{2i}$$

Using $\langle j', m' | J_\pm | j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \hbar\, \delta_{j'j}\, \delta_{m', m\pm 1}$

$$\rightarrow \quad J_y^{(j=1)} = \frac{\hbar}{2} \begin{array}{cc} \begin{pmatrix} 0 & -\sqrt{2}\, i & 0 \\ \sqrt{2}\, i & 0 & -\sqrt{2}\, i \\ 0 & \sqrt{2}\, i & 0 \end{pmatrix} & \begin{array}{l} m=1 \\ m=0 \\ m=-1 \end{array} \\ \begin{array}{ccc} m=1 & m=0 & m=-1 \end{array} & \end{array}$$

Unlike the case $j=\frac{1}{2}$, $[J_y^{(j=1)}]^2$ is <u>indep.</u> of I and $J_y^{(j=1)}$.

But; $\left(\dfrac{J_y^{(j=1)}}{\hbar} \right)^3 = \dfrac{J_y^{(j=1)}}{\hbar}$

So, for $j=1$ <u>only</u>; it is <u>legitimate</u> to replace

$$e^{\left(\frac{-i J_y \beta}{\hbar}\right)} \longrightarrow I - \left(\frac{J_y}{\hbar}\right)^2 (1 - \cos\beta) - i \left(\frac{J_y}{\hbar}\right) \sin\beta$$

$$\rightarrow \quad d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}} \sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}} \sin\beta & \cos\beta & -\frac{1}{\sqrt{2}} \sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}} \sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

This method is time-consuming for <u>large</u> j.

The usefulness of group theoretical consideration:

A symmetry operation $\xrightarrow[\text{leave}]{\text{must}}$ the Schrödinger equ. invariant.

so that $\longrightarrow$ the energies of the system are unaltered.

$$[H, U(a)] = 0 \quad \forall a \qquad (1)$$

the criterion for the invariance of the Schrödinger equ. under the operations of the group.

a: the element of the group

Now, if there is n-fold degeneracy;

$$H \Psi_k = E \Psi_k \qquad K = 1, 2, \cdots n \qquad (2)$$

(1)(2) $\rightarrow$ $H(U(a)\Psi_k) = U(a) H \Psi_k = E(U(a)\Psi_k)$ (3)

$\rightarrow$ $U(a)\Psi_k$: eigenstate of H with the same eigenvalue E

$$\longrightarrow \quad U(a)\Psi_k = \sum_{j=1}^{n} \Psi_j \, D_{jk}(a) \qquad (4)$$

$D_{jk}(a)$: Complex coeffs which depend on the group element.

$$U(b)U(a)\Psi_k = \sum_{j=1}^{n} U(b)\Psi_j D_{jk}(a) = \sum_{j=1}^{n}\sum_{\ell=1}^{n} \Psi_\ell D_{\ell j}(b) D_{jk}(a) \quad (5)$$

But, also; $$U(ba)\Psi_k = \sum_{\ell=1}^{n} \Psi_\ell D_{\ell k}(ba) \quad (6)$$

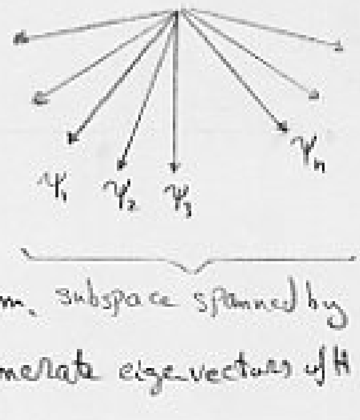
Since $U(ba) = U(b)U(a)$ (assumption)

$(5)(6) \longrightarrow$ $$D_{\ell k}(ba) = \sum_{j=1}^{n} D_{\ell j}(b) D_{jk}(a) \quad (7)$$

This is the central eqn. of the theory.

(7) shows that the coeffs D_{ij} define a unitary representation of the symmetry group.

$\Psi \in$ (n-dim. subspace) $\xrightarrow{U}$ $\Psi' \in$ (the same subspace).



i.e. $\longrightarrow$ The symmetry operations leave the subspace invariant.

Since any representation D of the symmetry group $\xrightarrow[\text{characterized}]{\text{can be}}$ by the irreducible representations which it contains

$\longrightarrow$ The stationary states of the system can be classified by the irreducible representations to which the eigenvectors of H belong.

A partial determination of these eigenvectors can be accomplished thereby.

The lables of the irreducible representations to which an energy eigenvalue belongs, are the Q. numbers of the stationary state.

Ex.

$$[D(R), H] = 0 \quad \longrightarrow \quad \begin{cases} [J, H] = 0 \\ [J^2, H] = 0 \end{cases} \qquad (8)$$

All $D(R)|n, j, m\rangle$ have the same energy.

$$D(R)|n, j, m\rangle = \sum_{m'} |n, j, m'\rangle D^{(j)}_{mm'}(R) \qquad (9)$$

Changing the rotation parameter R, we get different linear combinations of $|n, j, m'\rangle$.

$$
\left\{
\begin{aligned}
&H\left(D(R)|n,j,m\rangle\right) = E\left(D(R)|n,j,m\rangle\right) \\
&H\left(D(R')|n,j,m\rangle\right) = E\left(D(R')|n,j,m\rangle\right) \\
&\vdots \\
&H\sum_{m'}|n,j,m'\rangle D_{m'm}(R) = E\sum_{m'}|n,j,m'\rangle D_{m'm}(R) \\
&H\sum_{m'}|n,j,m'\rangle D_{m',m}(R') = E\sum_{m'}|n,j,m'\rangle D_{m'm}(R') \\
&\vdots
\end{aligned}
\right. \tag{10}
$$

→ All $|n,j,m\rangle$ with different m have the same energy.

→ $(2j+1)$ fold degeneracy

$$
D^{(j)}(R) = \begin{pmatrix} D^{(j)}_{m'm}(R) \end{pmatrix} = \begin{pmatrix} \langle n,j,m=-j|D^{(j)}(R)|j,n,m=-j\rangle & \cdots \\ \vdots & \end{pmatrix}
$$

irreducible representation

E (or E_j) $\in D^{(j)}(R)$

$(2j+1$ component$)$ subspace

$(2j+1)$ number of $|n,j,m\rangle$ $\in$ //

The lables $m=-j \ldots m=+j$ of the representation are the <u>Q. numbers</u> of the stationary state.

$$J \quad \cdots\ -4\ \ -3\ \ \underbrace{-2\ \ \underbrace{-1\ \ 0\ \ 1}_{\text{subspace}}\ \ 2}_{\text{subspace}}\ \ 3\ \ 4\ \cdots$$

There is a;

$$\text{Q.M} \xleftrightarrow[\text{relation between}]{\text{mutual}} \text{Group Theory}$$

i) The eigen funcs. of (2) $\xrightarrow{\text{generate}}$ representations of the symmetry groups of the system described by H

Conversely;

ii) A knowledge of the appropriate symmetry groups and their irreducible representations $\xrightarrow[\text{considerably}]{\text{can aid}}$ in the sol. of the Schrödinger equ. for a <u>complex</u> system

If all symmetries of a system are recognized $\xrightarrow[\text{can be shed}]{\text{much light}}$ on the eigenvalue spectrum and on the nature of the eigenstates.

Ex.

Strong ints. satisfy certain general symmetry principles, such as invariance under,

i) rotations
ii) Lorentz trs.
iii) Charge conjucations
iv) interchange of identical particles
v) rotations in isospin space
vi) operations of SU(3) trs. in a 3-dim complex vector space (at least approximately)

By constructing all irreducible representations of the groups which correspond to these symmetries,

$\longrightarrow$ We may obtain some of the basic Q. numbers and selection rules for the system without committing ourselves with regard to the ultimate form of a complete dynamical theory governing elementary particles.

3.6. Orbital Angular Momentum:

We introduced the concept of angular momentum by defining it to be the generator of an infinitesimal rot.

There is another way to approach the subject of ang. mom. when $S=0$ (or can be ignored).

$$J \longrightarrow L = x \wedge p \qquad \text{for a single particle.}$$

We will explore the connection between the two approaches.

Orbital Ang. Mom. as Rot. Generator:

Note that: $[L_i, L_j] = i\epsilon_{ijk}\hbar L_k$

Because; $[L_x, L_y] = [yP_z - zP_y\ ,\ zP_x - xP_z]$

$$= [yP_z, zP_x] + [zP_y, xP_z] + 0 + 0$$

$$= yP_x[P_z, z] + P_y x[z, P_z] = i\hbar(xP_y - yP_x) = i\hbar L_z$$

and so on, ...

Also,

$$[L_x, y] = [yP_z - zP_y, y] = -z[P_y, y] = i\hbar z$$

$$[L_x, P_y] = [yP_z - zP_y, P_y] = [y, P_y]P_z = i\hbar P_z$$

$$[L_x, x] = 0 \quad , \quad [L_x, P_x] = 0$$

$$[L^2, L_z] = 0$$

Now, let us examine, whether $\left(I - i\left(\frac{\delta\varphi}{\hbar}\right)L_z\right)$ can be interpreted as the infinitesimal rot. op. about the z-axis by angle $\delta\varphi$?

$$I - i\left(\frac{\delta\varphi}{\hbar}\right)L_z = I - i\left(\frac{\delta\varphi}{\hbar}\right)(xP_y - yP_x)$$

$$\left[I - i\left(\frac{\delta\varphi}{\hbar}\right)L_z\right]|x', y', z'\rangle = \left[I - i\left(\frac{P_y}{\hbar}\right)(\delta\varphi\, x') + i\left(\frac{P_x}{\hbar}\right)(\delta\varphi\, y')\right] |x', y', z'\rangle$$

$$= |x' - y'\delta\varphi, \; y' + x'\delta\varphi, \; z'\rangle \qquad (1)$$

where we have used the property of the translation op. $\mathcal{T}(dx')$,

$$\mathcal{T}(dx') = I - i\,\mathbf{P}\cdot dx'/\hbar \qquad \mathcal{T}(dx')|x'\rangle = |x' + dx'\rangle$$

$\longrightarrow$ If $\mathbf{P}$ generates translation $\xrightarrow{\text{then}}$ $\mathbf{L}$ generates rot.

Now, consider,

$\langle x', y', z'|\alpha\rangle$: the wave func. of a spinless particle.

Remark:
$$\langle x',y',z'|(I-i(\tfrac{\delta\varphi}{\hbar})L_z \longleftrightarrow (I+i(\tfrac{\delta\varphi}{\hbar})L_z|x',y',z')$$

The rotated wave func.;

$$\langle x',y',z'|(I-i(\tfrac{\delta\varphi}{\hbar})L_z)|\alpha\rangle = \langle x'+y'\delta\varphi\,,\,y'-x'\delta\varphi\,,\,z'|\alpha\rangle \quad (2)$$

In spherical coord.; $\langle x',y',z'|\alpha\rangle \rightarrow \langle r,\theta,\varphi|\alpha\rangle$

and

$$\langle r,\theta,\varphi|(I-i(\tfrac{\delta\varphi}{\hbar})L_z)|\alpha\rangle = \langle r,\theta,\varphi-\delta\varphi|\alpha\rangle$$

$$= \langle r,\theta,\varphi|\alpha\rangle - \delta\varphi\frac{\partial}{\partial\varphi}\langle r,\theta,\varphi|\alpha\rangle \quad (3)$$

Because $\langle r,\theta,\varphi|$ is an arbitrary position ket;

$$\longrightarrow \langle \bar{x}'|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle x'|\alpha\rangle \quad (4)$$

This result can also be obtained by using the position representation of the momentum op.

Next consider a rot. about x-axis by angle $\delta\varphi_x$;

$$\langle x',y',z'|(I-i(\tfrac{\delta\varphi_x}{\hbar})L_x)|\alpha\rangle = \langle x',y'+z'\delta\varphi_x\,,\,z'-y'\delta\varphi_x|\alpha\rangle \quad (5)$$

By expressing x', y', and z' in spherical coord.;
(We may also expand the Taylor series of (5) and afterward express it in spherical coord.)

$$\langle\bar{x}'|L_x|\alpha\rangle = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\langle\bar{x}'|\alpha\rangle \quad (6)$$

Likewise; $$\langle \bar{x}'|L_y|\alpha\rangle = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\langle\bar{x}'|\alpha\rangle \qquad (7)$$

Using the last two equs.;

$$\langle \bar{x}'|L_\pm|\alpha\rangle = -i\hbar\, e^{\pm i\varphi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)\langle\bar{x}'|\alpha\rangle \qquad (8)$$

And by the use of $L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+)$

$$\langle \bar{x}'|L^2|\alpha\rangle = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\langle\bar{x}'|\alpha\rangle \qquad (9)$$

Alternative approach;

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} \qquad (10)$$

where, $$\begin{cases} \hat{r} = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k} \\ \hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} \\ \hat{\theta} = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k} \end{cases} \qquad (11)$$

Remark: From the equ. for ∇ it is clear that the 3-spherical polar components of the momentum op. $\frac{\hbar}{i}\nabla$, <u>unlike</u> its Cartesian components, <u>do not</u> commute.

Now $$L = \bar{x}\wedge\bar{p} = r\hat{r}\wedge\frac{\hbar}{i}\nabla = \frac{\hbar}{i}\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \qquad (12)$$

$$(12) \longrightarrow [L, f(r)] = 0$$

$$(11)(12) \rightarrow \begin{cases} L_x = \frac{\hbar}{i}\left(-S\varphi \frac{\partial}{\partial\theta} - C\varphi \cot g\,\theta \frac{\partial}{\partial\varphi}\right) \\ L_y = \frac{\hbar}{i}\left(C\varphi \frac{\partial}{\partial\theta} - S\varphi \cot g\,\theta \frac{\partial}{\partial\varphi}\right) \\ L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \end{cases}$$

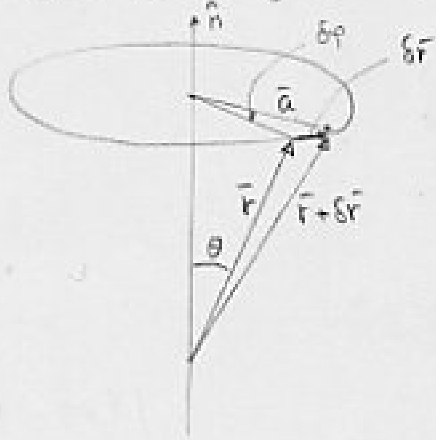
$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left[\frac{1}{S^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{S\theta}\frac{\partial}{\partial\theta}\left(S\theta\frac{\partial}{\partial\theta}\right)\right]$$

Note that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{S^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{S\theta}\frac{\partial}{\partial\theta}\left(S\theta\frac{\partial}{\partial\theta}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{L^2}{r^2} \qquad (13)$$

Theorem: Ang. mom. is the generator of infinitesimal rot. (rigid rot.) (classical)



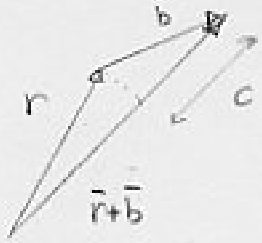
Proof: $f(r)$: differentiable func.

If $\bar{r} \to \bar{r} + \bar{b} \longrightarrow f(\bar{r}) \to F(\bar{r})$

such that $F(\bar{r}+\bar{b}) = f(\bar{r})$

for example; $f(r) = r$

under $\bar{r} \to \bar{r}+\bar{b} \longrightarrow r \to r + c$



$F(\bar{r}) = r + c \qquad F(\bar{r}+\bar{b}) = \bar{r} \longrightarrow F(\bar{r}+\bar{b}) = f(\bar{r})$

$\rightarrow F(\bar{r}) = f(\bar{r} - \bar{b})$

Now for an infinitesimal displacement;

$\bar{b} = \delta \bar{r}$

$F(\bar{r}) = f(\bar{r} - \delta\bar{r}) = f(r) - \delta\bar{r} \cdot \nabla f(\bar{r}) + \cdots$

If the displacement $\delta\bar{r}$ is a rotation by an angle φ about $\hat{n}$;

$|\delta\bar{r}| = a\,\delta\varphi = r \sin\theta\, \delta\varphi \qquad |\delta\bar{r}| = |(\delta\varphi\,\hat{n}) \times \bar{r}| = |\delta\bar{\varphi} \times \bar{r}|$

$\delta\bar{r} = \delta\bar{\varphi} \times \bar{r}$

$\delta f(\bar{r}) = F(\bar{r}) - f(\bar{r}) = f(r-a) - f(\bar{r}) = -(\delta\bar{\varphi} \times r) \cdot \nabla f(r)$

$= -\delta\bar{\varphi} \times (r \times \nabla f(r)) = -\frac{i}{\hbar}\, \delta\bar{\varphi} \cdot (r \times \frac{\hbar}{i} \nabla) f = -\frac{i}{\hbar} \delta\bar{\varphi} \cdot \bar{L} f$

$\rightarrow \delta f = -\frac{i}{\hbar} \delta\varphi \cdot L f$

<u>Ex.</u> $\quad f(\bar{r}) = V(r) g(\bar{r})$

Under a rot. $\longrightarrow \delta f(\bar{r}) = \delta(V(r) g(\bar{r})) = V(r)\, \delta g(\bar{r})$

$\delta f(\bar{r}) = -\frac{i}{\hbar} \delta\bar{\varphi} \cdot L(Vg) \; (1), \quad V(r)[\delta g(r)] = V[-\frac{i}{\hbar} \delta\bar{\varphi} \cdot L g] \; (2)$

$\xrightarrow{(1)(2)} L V(r) - V(r) L = 0 \qquad [L, V(r)] = 0$

Kinetic Energy and Ang. Mom.;

a) In cl. M. case;

$$\bar{L}^2 = (\bar{r}\times\bar{p})^2 = \epsilon_{ijk}\,\epsilon_{ilm}\, r_j p_k r_l p_m = (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\, r_j p_k r_l p_m$$

$$= r_j p_k r_j p_k - r_j p_k r_k p_j = r_j r_j p_k p_k - r_j p_j r_k p_k = r^2 p^2 - (r\cdot p)^2$$

$$\rightarrow p^2 = \frac{L^2}{r^2} - \left(\frac{r\cdot p}{r}\right)^2$$

expressing Kinetic energy in terms of a const. of motion L^2 and the radial component of momentum

b) Q.M. case;

$$\bar{L}^2 = (\bar{r}\times\bar{p})\cdot(\bar{r}\times\bar{p}) = \epsilon_{ijk}\,\epsilon_{ilm}\, r_j p_k r_l p_m$$

$$= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\, r_j p_k r_l p_m = \underbrace{r_j p_k r_j p_k}_{\mathrm{I}} - \underbrace{r_j p_k r_k p_j}_{\mathrm{II}}$$

$$\mathrm{I} = r_j p_k r_j p_k = r_j(r_j p_k - i\hbar\,\delta_{jk})p_k = r_j r_j p_k p_k - i\hbar\,\delta_{jk} r_j p_k = r^2p^2 - i\hbar\,\bar{r}\cdot\bar{p}$$

$$\mathrm{II} = -r_j p_k r_k p_j = -r_j p_k(p_j r_k + i\hbar\,\delta_{jk}) = -r_j p_j p_k r_k - i\hbar\, r_j p_j$$

$$= -r_j p_j(r_k p_k - i\hbar\,\delta_{kk}) - i\hbar\, r_j p_j = -r_j p_j(r_k p_k - 3i\hbar) - i\hbar\, r_j p_j$$

$$= -(r\cdot p)^2 + 3i\hbar(r\cdot p) - i\hbar(r\cdot p) = -(r\cdot p)^2 + 2i\hbar(r\cdot p)$$

$$\rightarrow L^2 = r^2p^2 - (r\cdot p)^2 + i\hbar(\bar{r}\cdot\bar{p})$$

$$\bar{r} = (r, 0, 0)$$

$$\bar{p} = \frac{\hbar}{i}\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}, \frac{1}{r S\theta}\frac{\partial}{\partial \varphi}\right)$$

$$\bar{r}\cdot\bar{p} = \frac{\hbar}{i} r \frac{\partial}{\partial r} \longrightarrow (r\cdot p)^2 = -\hbar^2\left(r\frac{\partial}{\partial r}\right)^2$$

$$\rightarrow L^2 = r^2 p^2 - \left\{-\hbar^2\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right)\right\} + \hbar^2 r \frac{\partial}{\partial r}$$

$$L^2 = r^2p^2 + \hbar^2 r^2 \frac{\partial^2}{\partial r^2} + \hbar^2 r \frac{\partial}{\partial r} + \hbar^2 r \frac{\partial}{\partial r} = r^2p^2 + \hbar^2 r^2 \frac{\partial^2}{\partial r^2} + 2\hbar^2 r \frac{\partial}{\partial r}$$

$$\rightarrow L^2 = r^2p^2 + \hbar^2 \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \rightarrow p^2 = \frac{L^2}{r^2} - \frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

$$T = \frac{p^2}{2\mu} = -\frac{\hbar^2}{2\mu}\nabla^2 = \frac{L^2}{2\mu r^2} - \frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

Now $[L, \text{any radial derivative}] = 0$

also $[L, L^2] = 0$

$\rightarrow [L, T] = 0$

And if V in $H = T + V$ is central, (i.e. $V = V(r)$)

$\rightarrow [L, H] = 0 \longrightarrow L$: const. of motion

Alternative approach:

$$\langle x' | x \cdot p | \alpha \rangle = x' \cdot \langle x' | p | \alpha \rangle = x' \cdot \left(-i\hbar \nabla' \langle x' | \alpha \rangle \right)$$

$$= -i\hbar r' \frac{\partial}{\partial r'} \langle x' | \alpha \rangle$$

Likewise; $\langle x' | (x \cdot p)^2 | \alpha \rangle = \int d^3x'' \, \langle x' | (x \cdot p) | x'' \rangle \langle x'' | (x \cdot p) | \alpha \rangle$

$$= \int d^3x'' \left(-i\hbar r' \frac{\partial}{\partial r'} \langle x' | x'' \rangle \right) \left(-i\hbar r'' \frac{\partial}{\partial r''} \langle x'' | \alpha \rangle \right)$$

$$= \int d^3x''_{\varphi''} \left(-i\hbar r' \frac{\partial}{\partial r'_{\varphi'}} \delta^3(x' - x'') \right) \left(-i\hbar r'' \frac{\partial}{\partial r''} \langle x'' | \alpha \rangle \right)$$

$$= -\hbar^2 r' \frac{\partial}{\partial r'} \left(r' \frac{\partial}{\partial r'} \langle x' | \alpha \rangle \right) = -\hbar^2 \left(r'^2 \frac{\partial^2}{\partial r'^2} \langle x' | \alpha \rangle + r' \frac{\partial}{\partial r'} \langle x' | \alpha \rangle \right)$$

Now using. $L^2 = x^2 p^2 - (x \cdot p)^2 + i\hbar x \cdot p$

$$\rightarrow \quad \langle x' | L^2 | \alpha \rangle = r'^2 \langle x' | p^2 | \alpha \rangle + \hbar^2 \left(r'^2 \frac{\partial^2}{\partial r'^2} \langle x' | \alpha \rangle + 2r' \frac{\partial}{\partial r'} \langle x' | \alpha \rangle \right)$$

$$\rightarrow \frac{1}{2\mu} \langle x' | p^2 | \alpha \rangle = -\left(\frac{\hbar^2}{2\mu} \right) \nabla'^2 \langle x' | \alpha \rangle =$$

$$= -\left(\frac{\hbar^2}{2\mu} \right) \left(\frac{\partial^2}{\partial r'^2} \langle x' | \alpha \rangle + \frac{2}{r'} \frac{\partial}{\partial r'} \langle x' | \alpha \rangle - \frac{1}{\hbar^2 r'^2} \langle x' | L^2 | \alpha \rangle \right)$$

Some useful formulae:

$$\int_{-\infty}^{\infty} f(x) \frac{d^n}{dx^n} \delta(x - x_0) \, dx = 0 \; (-)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n} f(x) \, \delta(x - x_0) = (-1)^n f^{(n)}(x_0)$$

$$\frac{d\,\theta(x - x_0)}{dx} = \delta(x - x_0) \qquad \delta^3(x - x_0) = \frac{1}{r^2} \frac{1}{S\theta} \delta(r - r_0) \, \delta(\theta - \theta_0) \, \delta(\varphi - \varphi_0)$$

Spherical Harmonics:

Consider a spinless particle in a spherical symmetrical pot.

$\longrightarrow$ The wave equ. is known to be separable in spherical coords, and the energy eigenfuncs. can be written as:

$$\langle x' | n, l, m \rangle = R_{nl}(r)\, Y_l^m(\theta, \varphi)$$

n : Some Q. number other than l and m, for example, the radial Q. number for bound state prob. or the energy for free-particle spherical wave.

For spherically symmetric H;

$$[H, L_z] = 0 \quad [H, L^2] = 0$$

$\rightarrow$ The energy eigenkets are expected to be the eigenkets of L_z and L^2

$$H|n, l, m\rangle = E_n |n, l, m\rangle \ , \ L^2 |n, l, m\rangle = l(l+1)\hbar^2 |n, l, m\rangle$$

$$L_z |n, l, m\rangle = m\hbar\, |n, l, m\rangle$$

Angular dependence is common for all probs. with spherical symmetry

$$\langle \hat{n} | l, m \rangle = Y_l^m(\theta, \varphi) = Y_l^m(\hat{n})$$

$|\hat{n}\rangle$: direction eigenket

$\longrightarrow$ $Y_\ell^m(\theta,\varphi)$: The amplitude for a state characterized by ℓ and m to be found in the dir. $\hat{n}$.

Now;
$$L_z|\ell,m\rangle = m\hbar|\ell,m\rangle$$

$$\longrightarrow \langle\hat{n}|L_z|\ell,m\rangle = m\hbar\langle\hat{n}|\ell,m\rangle$$

Since $\langle x'|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle x'|\alpha\rangle$

$$-i\hbar\frac{\partial}{\partial\varphi}\langle\hat{n}|\ell,m\rangle = m\hbar\langle\hat{n}|\ell,m\rangle$$ ($|\hat{x}\rangle$ is replaced by $|\hat{n}\rangle$)

$$\longrightarrow \quad -i\hbar\frac{\partial}{\partial\varphi}Y_\ell^m(\theta,\varphi) = m\hbar\, Y_\ell^m(\theta,\varphi)$$

$\longrightarrow$ φ-dependence of $Y_\ell^m(\theta,\varphi) \sim e^{im\varphi}$

Likewise;
$$L^2|\ell,m\rangle = \ell(\ell+1)\hbar^2|\ell,m\rangle$$

$$\langle\hat{n}|L^2|\ell,m\rangle = \ell(\ell+1)\hbar^2\langle\hat{n}|\ell,m\rangle$$

Using; $\langle x'|L^2|\alpha\rangle = -\hbar^2\left[\frac{1}{S^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{S\theta}\frac{\partial}{\partial\theta}\left(S\theta\frac{\partial}{\partial\theta}\right)\right]\langle x'|\alpha\rangle$

$$\rightarrow \left[\frac{1}{S\theta}\frac{\partial}{\partial\theta}\left(S\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{S^2\theta}\frac{\partial^2}{\partial\varphi^2} + \ell(\ell+1)\right]Y_\ell^m(\hat{n}) = 0$$

A partial diff. equ. for $Y_\ell^m(\hat{n})$

The $\langle l', m' | l, m \rangle = \delta_{ll'} \delta_{mm'}$ orthogonality cond.

leads to $\longrightarrow$ $$\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\, Y_{l'}^{m'*}(\theta,\varphi)\, Y_l^m(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

where we have used the completeness relation;

$$\int d\Omega_{\hat{n}}\, |\hat{n}\rangle\langle\hat{n}| = I$$

Now, $Y_l^m(\theta,\varphi) = ?$

$$L_+ |l,l\rangle = 0$$

Using $\langle x' | L_\pm | \alpha \rangle = -i\hbar e^{\pm i\varphi}\left(\pm i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\varphi} \right) \langle x' | \alpha \rangle$

$$\rightarrow \quad -i\hbar e^{i\varphi}\left(i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\varphi} \right) \langle \hat{n} | l,l \rangle = 0$$

Since φ-dep. $\sim e^{il\varphi}$

The partial diff. equ. is satisfied by:

$$\langle \hat{n} | l,l \rangle = Y_l^l(\theta,\varphi) = c_l\, e^{il\varphi} \sin^l\theta \qquad (1)$$

$$c_l = \left[\frac{(-1)^l}{2^l l!} \right] \sqrt{\frac{(2l+1)(2l)!}{4\pi}}$$ by normalization cond.

Starting with (1), we can use;

$$\langle \hat{n}|\ell, m-1\rangle = \frac{\langle \hat{n}|L_-|\ell,m\rangle}{\sqrt{(\ell+m)(\ell-m+1)}\,\hbar} = \frac{1}{\sqrt{(\ell+m)(\ell-m+1)}}\, e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\langle \hat{n}|\ell,m\rangle \quad (2)$$

successively to obtain all Y_ℓ^m with fixed ℓ.

$$\longrightarrow Y_\ell^m(\theta,\varphi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}\, e^{im\varphi}\frac{1}{\sin^m\theta}\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell}$$

Also, $\quad Y_\ell^{-m}(\theta,\varphi) = (-1)^m\left[Y_\ell^m(\theta,\varphi)\right]^*$

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi}$$

$$P_\ell^m(x) = (-)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$$

$$\int_{-1}^{1} P_{\ell'}^m(x)\,P_\ell^m(x)\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'} \qquad \text{orthogonality}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\theta,\varphi)\,Y_\ell^m(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

completeness

From angular mom. commutation relations alone, it might not appear obvious why l cannot be half integer.

There are several arguments:

i) If l: half-integer $\longrightarrow$ m: half-integer

$\longrightarrow e^{im(2\pi)} = -1 \longrightarrow$ the wave func. aquires a minus sign

$\longrightarrow$ The wave func. would not be single valued.

$\longrightarrow$ The expansion of the state ket in terms of position eigenkets would not be unique.

If $L = \bar{r} \times \bar{p}$ is to be identified as the generator of rot.

Then $\longrightarrow$ the wave func. must aquire a plus sign under a 2π rot..

This follows from the fact that

$$\psi \xrightarrow{(2\pi)-\text{rot.}} +\psi$$

$$\langle x' | e^{\frac{-iL_z(2\pi)}{\hbar}} | \alpha \rangle = \langle x' \cos 2\pi + y' \sin 2\pi ,\ y' \cos 2\pi - x' \sin 2\pi ,\ z' | \alpha \rangle$$

$$= \langle x' | \alpha \rangle \longrightarrow m: \text{integer} \rightarrow l: \text{integer}$$

$$\langle x'| e^{-\frac{i}{\hbar}L_z(2\pi)} \longleftrightarrow e^{+\frac{i}{\hbar}L_z(2\pi)} |x'\rangle \qquad R_z^{-1}(2\pi)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x'\cos 2\pi + y'\sin 2\pi \\ -x'\sin 2\pi + y'\cos 2\pi \\ z' \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii) Suppose l : half-integer

To be specific let $l = m = \frac{1}{2}$

Acc. to $\langle \hat{n}|l,l\rangle = Y_l^l(\theta,\varphi) = c_l e^{il\varphi} \sin^l\theta$

$$\rightarrow Y_{\frac{1}{2}}^{\frac{1}{2}}(\theta,\varphi) = c_{\frac{1}{2}} e^{i\varphi/2} \sqrt{\sin\theta} \qquad (3)$$

$$(2) \rightarrow Y_{\frac{1}{2}}^{-\frac{1}{2}}(\theta,\varphi) = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right)\left(c_{\frac{1}{2}} e^{i\varphi/2}\sqrt{\sin\theta}\right)$$

$$= -c_{\frac{1}{2}} e^{-i\varphi/2} \cot\theta \sqrt{\sin\theta} \qquad (4)$$

This expression is not permissible because it is singular at $\theta = 0, \pi$

What is worse; from the partial diff. equ.;

$$\langle\hat{n}|L_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -i\hbar e^{-i\varphi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)\langle\hat{n}|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0$$

$$\rightarrow Y_{1/2}^{-1/2}(\theta,\varphi) = c'_{\frac{1}{2}} e^{-i\varphi/2}\sqrt{\sin\theta}$$ in contradiction with (4)

Spherical Harmonics as Rotation Matrices: (connection between Them)

We wish to find $\mathcal{D}(R)$, such that;

$$|\hat{n}\rangle = \mathcal{D}(R)|\hat{z}\rangle$$

First rot.: about y-axis by angle of θ

Second rot.: " z- " " " : φ

$$\rightarrow \mathcal{D}(R) = \mathcal{D}(\alpha=\varphi, \beta=\theta, \gamma=0)$$

$$|\hat{n}\rangle = \sum_{l}\sum_{m} \mathcal{D}(R)|l,m\rangle\langle l,m|\hat{z}\rangle$$

$$\rightarrow \langle l,m'|\hat{n}\rangle = \sum_{m} \mathcal{D}^{(l)}_{m'm}(\alpha=\varphi, \beta=\theta, \gamma=0)\langle l,m|\hat{z}\rangle \quad (5)$$

when $\langle l,m|\hat{z}\rangle = Y_l^{m*}(\theta=0, \varphi=\text{undetermined})$

At $\theta = 0$; $Y_l^m = 0$ for $m \neq 0$

This can also be seen directly from the fact that;

$$\underset{\rightarrow (xp_y - yp_x)}{L_z}|\hat{z}\rangle = 0\,|\hat{z}\rangle$$

$|\hat{z}\rangle$: eigenket of L_z with eigenvalue 0.

Remark: L_z is the generator of infintesimal rot. about z-axis.

$$\left(I - \frac{i}{\hbar}\delta\varphi L_z\right)|\hat{z}\rangle = |\hat{z}\rangle \rightarrow L_z|\hat{z}\rangle = 0$$

$$\begin{cases} \langle \hat{z} | L_z | l, m \rangle = m\hbar \langle \hat{z} | l m \rangle \\ \langle \hat{z} | L_z | l, m \rangle = 0 \end{cases}$$

$$\rightarrow m\hbar \langle \hat{z} | l m \rangle = 0 \longrightarrow \begin{cases} m \neq 0 \rightarrow \langle \hat{z} | l m \rangle = 0 \\ m = 0 \rightarrow \langle \hat{z} | l m \rangle \neq 0 \end{cases}$$

$$\langle l, m | \hat{z} \rangle = Y_l^{m^*}(\theta = 0, \varphi = \text{undetermined})\, \delta_{m0}$$

$$= \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta) \Big|_{\cos\theta = 1} \delta_{m0} = \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0}$$

$$(5) \rightarrow Y_l^{m'^*}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}}\, D^{(l)}_{m'0}(\alpha = \varphi, \beta = \theta, \gamma = 0)$$

$$\rightarrow D^{(l)}_{m0}(\alpha, \beta, \gamma = 0) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^{m^*}(\theta, \varphi) \Big|_{\substack{\theta = \beta \\ \varphi = \alpha}}$$

Notice that:

$$d^{(l)}_{00}(\beta) \Big|_{\beta = \theta} = P_l(\cos\theta)$$

3.7. Formal Theory of Angular Momentum Addition:

Suppose J_1 and J_2 belong to different subspaces; $\begin{cases} \text{i - Different particles} \\ \text{ii - Different in nature like } L \text{ and } S \end{cases}$

$$[J_{1i}, J_{1j}] = i\hbar\,\epsilon_{ijk} J_{1k} \quad , \quad [J_{2i}, J_{2j}] = i\hbar\,\epsilon_{ijk} J_{2k} \qquad (1)$$

However, $[J_{1k}, J_{2l}] = 0$

A common rot. of the composite system is represented by the direct product of the rot. operators for each subsystem;

$$\left(I_1 - \tfrac{i}{\hbar} J_1\cdot\hat{n}\,\delta\varphi\right) \otimes \left(I_2 - \tfrac{i}{\hbar} J_2\cdot\hat{n}\,\delta\varphi\right) = I_1 \otimes I_2 - \tfrac{i}{\hbar}\left(J_1 \otimes I_2 + I_1 \otimes J_2\right)\cdot\hat{n}\,\delta\varphi$$

For finite rot., $\qquad (3)$

$$D_1(R) \otimes D_2(R) = e^{-\frac{i}{\hbar} J_1\cdot\hat{n}\varphi} \otimes e^{-\frac{i}{\hbar} J_2\cdot\hat{n}\varphi} \underset{\uparrow}{=} e^{-\frac{i}{\hbar} J\cdot\hat{n}\varphi} = D(R)$$

where $J = J_1 \otimes I_2 + I_1 \otimes J_2$ $\qquad$ if $[J_1, J_2] = 0$ $\qquad (4)$

Ex. If M_1 and P_1 are ops. in space 1 and N_2 and Q_2 are ops. in space 2, Prove the identity: $M_1 P_1 \otimes N_2 Q_2 = (M_1 \otimes N_2)(P_1 \otimes Q_2)$.

$(1)(2) \to [J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k \qquad (\to J \text{ is ang. mom.})$

Physically this is reasonable, because J is the generator for the entire system.

As for the choice of the base kets we have two options

Option A: The ops. J_1^2, J_2^2, J_{1z} and J_{2z} commute with each other;

$\longrightarrow |j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle$ (their simultaneous eigenkets)

$$J_1^2 \, | j_1 j_2 m_1 m_2 \rangle = j_1(j_1+1)\hbar^2 \, | j_1 j_2 m_1 m_2 \rangle$$
$$J_{1z} \, | \quad " \quad \rangle = m_1 \hbar \, | \quad " \quad \rangle$$
$$J_2^2 \, | \quad " \quad \rangle = j_2(j_2+1)\hbar^2 \, | \quad " \quad \rangle \qquad (5)$$
$$J_{2z} \, | \quad " \quad \rangle = m_2 \hbar \, | \quad " \quad \rangle$$

Option B: The ops. J^2, J_1^2, J_2^2 and J_z mutually commute.

In particular $[J^2, J_1^2] = [J^2, J_2^2] = 0$

This can readily be seen by writing J^2 as:

$$J^2 = J_1^2 + J_2^2 + 2 J_{1z} J_{2z} + J_{1+} J_{2-} + J_{1-} J_{2+}$$

We use $| j_1 j_2 , j m \rangle$ to denote the base kets of option B:

$$J_1^2 \, | j_1 j_2 , j m \rangle = j_1(j_1+1)\hbar^2 \, | j_1 j_2 , j m \rangle$$
$$J_2^2 \, | \quad " \quad \rangle = j_2(j_2+1)\hbar^2 \, | \quad " \quad \rangle$$
$$J^2 \, | \quad " \quad \rangle = j(j+1)\hbar^2 \, | \quad " \quad \rangle \qquad (6)$$
$$J_z \, | \quad " \quad \rangle = m\hbar \, | \quad " \quad \rangle$$

$$| j_1 j_2 ; j m \rangle \equiv | j m \rangle$$

It is very important to note that even though. $[J^2, J_z] = 0$

but $\quad [J^2, J_{1z}] \neq 0 \quad , \quad [J^2, J_{2z}] \neq 0$

This means that we can not add J^2 to the set of ops. of option A.

Likewise, we can not add J_{1z} and/or J_{2z} to the set of ops. of option B.

So; we have two possible sets of base kets corresponding to the two maximal sets of mutually compatible observables,

In the subspace of the simultaneous eigenvectors of J_1^2 and J_2^2 with eigenvalues j_1 and j_2 respectively we can thus write the tr. equ.;

$$|j_1 j_2; jm\rangle = \underbrace{\sum_{m_1}\sum_{m_2} |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|}_{I}\underbrace{j_1 j_2; jm\rangle}_{\text{Clebsch-Gordan Coeff.}} \qquad (7)$$

These coeffs. vanish unless

$$m = m_1 + m_2$$

Proof: $$(J_z - J_{1z} - J_{2z})\,|j_1 j_2; jm\rangle = 0$$

$$\rightarrow\ (m - m_1 - m_2)\,\langle j_1 j_2\, m_1 m_2 | j_1 j_2; jm\rangle = 0$$

which proves our assertion.

Now we apply $J_\pm$ on $|jm\rangle = \sum_{m_1,m_2} |m_1 m_2\rangle\langle m_1 m_2|jm\rangle$

where $|jm\rangle \equiv |j_1 j_2 ; jm\rangle$ and $|j_1 j_2 , m_1 m_2\rangle \equiv |m_1 m_2\rangle$

we obtain readily the following recursion ~~recursion~~ relations for the Cl. coeffs;

$$\sqrt{(j\pm m)(j\mp m+1)}\ (m_1 m_2/j, m\mp 1) = \sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}\ (m_1\pm 1, m_2|jm)$$
$$+\sqrt{(j_2\mp m_2)(j_2\pm m_2+1)}\ (m_1, m_2\pm 1|jm) \qquad (8)$$

To appreciate the usefulness of these equs, let us set $\begin{cases} m_1 = j_1 \\ m = j \end{cases}$ in the upper part of this equ.

$$m = m_1 + m_2 \rightarrow \underset{(m-1)}{(j-1)} = \underset{j_1}{m_1} + m_2 \rightarrow j-1 = j_1 + m_2$$

$$\rightarrow m_2 = j - j_1 - 1$$

$$\rightarrow \sqrt{2j}\ (j_1, j-j_1-1/j\, j-1) = \sqrt{(j_2-j+j_1+1)(j_2+j-j_1)}\ (j_1, j-j_1/jj) \qquad (9)$$

Let also set $\begin{cases} m_1 = j_1 \\ m = j-1 \end{cases}$ in the lower part;

$$\rightarrow \underset{(m+1)}{(j-1)+1} = \underset{j_1}{m_1} + m_2 \rightarrow (j-1)+1 = j_1 + m_2 \rightarrow m_2 = j - j_1$$

$$\rightarrow \sqrt{2j}\ (j_1, j-j_1/jj) = \sqrt{2j_1}\ (j_1-1, j-j_1/j, j-1) +$$
$$+\sqrt{(j_2+j-j_1)(j_2-j+j_1+1)}\ (j_1, j-j_1-1/j, j-1)$$

(10)

Acc. to (9) if;

$(j_1, j-j_1 / jj)$ is known $\longrightarrow (j_1, j-j_1-1 / j, j-1)$ can be determined

$\longrightarrow$ can be used to compute $(j_1-1, j-j_1 / j, j-1)$ from these two coeffs.

Continuing in this manner, the recursion relation (8) can be used to give for fixed values of j_1, j_2 and j all the Clebsch-Gordan coeffs. in terms of just one of them namely;

$$(j_1 j_2 j_1\ j-j_1 / j_1 j_2 j\ j) \qquad (11)$$

The absolute value of this coeff. is determined by normalization.

This coeff. is nonzero only if;

$$-j_2 \leq \underbrace{j-j_1}_{m_2} \leq j_2$$

$$\rightarrow \quad j_1 - j_2 \leq j \leq j_1 + j_2$$

But we could equally well have expressed all these Clebsch-Gordan coeffs. in terms of

$$(j_1 j_2\ j-j_2, j_2 / j_1 j_2 j j) \qquad (12)$$

$$-j_1 \leq \underbrace{j-j_2}_{m_1} \leq j_1 \quad \rightarrow \quad j_2 - j_1 \leq j \leq j_1 + j_2$$

Hence the ang. mom. Q numbers must satisfy the so called triangular cond.;

$$|j_1 - j_2| \leq j \leq j_1 + j_2 \qquad (13)$$

Now, the dimensionality of the space spanned by $\{|j_1 j_2 m_1 m_2\rangle\}$ is

$$N = (2j_1+1)(2j_2+1)$$

The dimensionality of the space spanned by $\{|j_1 j_2 ; j m\rangle\}$ is

$$N = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = \tfrac{1}{2}\left[\{2(j_1-j_2)+1\} + \{2(j_1+j_2)+1\}\right](2j_2+1) \qquad (14)$$

$$= (2j_1+1)(2j_2+1) \qquad \text{(the same)}$$

the number of terms

The two basis are the eigenkets of Hermitian ops. and they are orthonormal $\longrightarrow$ The Clebsch-Gordan coeffs constitute a unitary matrix.

From recursion relation (8) it is clear that all Clebsch-Gordan coeffs. are real numbers if one of them (say, (11)), is chosen real.

A real unitary matrix is orthogonal; so we have the orthogonality cond.;

$$\sum_j \sum_m \langle m_1 m_2 | j m\rangle \langle m_1' m_2' | j m\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \qquad (15)$$

Proof: $\langle m_1' m_2' | m_1 m_2 \rangle = \sum_j \sum_m \langle m_1' m_2' | jm \rangle \langle jm | m_1 m_2 \rangle$

$\rightarrow \delta_{m_1 m_1'} \delta_{m_2 m_2'} = \sum_j \sum_m \langle m_1' m_2' | jm \rangle \langle m_1 m_2 | jm \rangle^*$

$\rightarrow \sum_j \sum_m \langle m_1' m_2' | jm \rangle \langle m_1 m_2 | jm \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \begin{pmatrix} j_1 = j_1' \\ j_2 = j_2' \end{pmatrix}$

Where we have used the orthonormality of $\{|j_1 j_2, m_1 m_2\rangle\}$ and reality of the Clebsch-G. coeffs.

Similarly;

$\langle j'm' | jm \rangle = \sum_{m_1} \sum_{m_2} \langle j'm' | \underbrace{m_1 m_2 \rangle \langle m_1 m_2}_{\text{complete set}} | jm \rangle$

Remark: for $j_1 \neq j_1'$, $j_2 \neq j_2'$ the Clebsch-G coeffs. vanish

$\rightarrow \sum_{m_1} \sum_{m_2} \langle m_1 m_2 | jm \rangle \langle m_1 m_2 | j'm' \rangle = \delta_{jj'} \delta_{mm'}$ (16)

$(j_1 = j_1', j_2 = j_2')$

Also we have

$|m_1 m_2\rangle = \sum_j \sum_m \underbrace{|jm\rangle \langle jm|}_{\text{complete set}} m_1 m_2 \rangle$ (17)

(8) (15) determine all Clebsch-G coeffs. except for a sign - (because a ket like $|jm\rangle$ is the eigenket of ang. mom. op. but its phase is undetermined).

Convention: We remove this arbitrariness by choosing the coeff (11) to be positive and real.

Remark: $\langle j_1 m_1 j_2 m_2 | j m\rangle = (-)^{j_1 - j_2 - m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$

Wigner's 3-j symbol

Direct product space: Ref. Merzbacher

Consider: two distinguishable particles (say electron and proton);

Particle 1: with complete set of dynamical variables

Particle 2: " " " " "

$$[A_1, B_2] = 0 \qquad \{A\} \in \text{Particle } 1$$
$$\{B\} \in \text{ " } 2$$

The direct product space is defined as the space spanned by the basis vectors

$$|A_1', B_2'\rangle = |A_1'\rangle |B_2'\rangle$$

where $\{|A_1'\rangle\} \in$ Particle 1 $\qquad \{|B_2'\rangle\} \in$ Particle 2

$\downarrow$ dim. $= n_1$ $\qquad$ $\downarrow$ dim. $= n_2$

$n_1 \times n_2$: dim of $\{|A_1', B_2'\rangle\}$

Any op. which pertains to only one of the two factor spaces is regarded as acting as an identity op. with respect to the other.

More generally, if $M_1 \in$ space 1, $N_2 \in$ space 2;

such that:

$$M_1 |A_1'\rangle = \sum_{A_1''} |A_1''\rangle\langle A_1''|M_1|A_1'\rangle$$

$$N_2 |B_2'\rangle = \sum_{B_2''} |B_2''\rangle\langle B_2''|N_2|B_2'\rangle$$

We define the direct product operator $M_1 \otimes N_2$ by the eqn.:

$$(M_1 \otimes N_2)|A_1' B_2'\rangle = \sum_{A_1''}\sum_{B_2''} |A_1'' B_2''\rangle\langle A_1''|M_1|A_1'\rangle\langle B_2''|N_2|B_2'\rangle$$

Hence $M_1 \otimes N_2$ is represented by a matrix which is said to be the direct product of two matrices representing M_1 and N_2 separately and which is defined by

$$\langle A_1'' B_2''|(M_1 \otimes N_2)|A_1' B_2'\rangle = \langle A_1''|M_1|A_1'\rangle\langle B_2''|N_2|B_2'\rangle$$

Remark: The names, Tensor, or, Kronecker or outer product are also used for direct product.

Direct sum: Def. $A \oplus B = \begin{pmatrix} A & O_1 \\ O_2 & B \end{pmatrix}$. If A is $(m \times n)$ and B is $(p \times q)$, the null matrix O_1 is $(m \times q)$ and O_2 is $(p \times n)$.

Ex. $\{|A_1'\rangle\} = \{|S_{1z},+\rangle = |\alpha\rangle = \begin{pmatrix}1\\0\end{pmatrix}_1, |S_{1z},-\rangle = |\beta\rangle = \begin{pmatrix}0\\1\end{pmatrix}_1\}$

$\{|B_2'\rangle\} = \{|S_{2z},+\rangle = |\gamma\rangle = \begin{pmatrix}1\\0\end{pmatrix}_2, |S_{2z},-\rangle = |\delta\rangle = \begin{pmatrix}0\\1\end{pmatrix}_2\}$

$[A_1, B_2] = 0$

$$M_1 = \begin{pmatrix} \langle\alpha|M_1|\alpha\rangle & \langle\alpha|M_1|\beta\rangle \\ \langle\beta|M_1|\alpha\rangle & \langle\beta|M_1|\beta\rangle \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$N_2 = \begin{pmatrix} \langle\gamma|N_2|\gamma\rangle & \langle\gamma|N_2|\delta\rangle \\ \langle\delta|N_2|\gamma\rangle & \langle\delta|N_2|\delta\rangle \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

$$\underbrace{\langle A_1'' B_2'' | M_1 \otimes N_2 | A_1' B_2' \rangle}_{(n_1\times n_1)\,(n_2\times n_2)\text{ number}} = \underbrace{\langle A_1'' | M_1 | A_1' \rangle}_{n_1\times n_1\text{ number}} \underbrace{\langle B_2'' | N_2 | B_2' \rangle}_{n_2\times n_2\text{ number}}$$

$$M_1 \otimes N_2 = \begin{pmatrix} M_{11}\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} & M_{12}\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \\ M_{21}\begin{pmatrix} & // & \end{pmatrix} & M_{22}\begin{pmatrix} & // & \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11}N_{11} & M_{11}N_{12} & M_{12}N_{11} & M_{12}N_{12} \\ M_{11}N_{21} & M_{11}N_{22} & M_{12}N_{21} & M_{12}N_{22} \\ M_{21}N_{11} & M_{21}N_{12} & M_{22}N_{11} & M_{22}N_{12} \\ M_{21}N_{21} & M_{21}N_{22} & M_{22}N_{21} & M_{22}N_{22} \end{pmatrix}$$

Remark:

$C = M_1 \otimes N_2$

$C_{ik:jl} = M_{ij} N_{kl}$

Thus if M_1 is $(m_1 \times m_1')$ and N_2 is $(n_2 \times n_2')$ $\rightarrow$ C is $(m_1 n_2 \times m_1' n_2')$ matrix

Now $|A'_1, B'_2\rangle = |A'_1\rangle \otimes |B'_2\rangle$

$|a\rangle = |\alpha, \gamma\rangle = |\alpha\rangle|\gamma\rangle = \begin{pmatrix} 1|\gamma\rangle \\ 0|\gamma\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |b\rangle = |\alpha, \delta\rangle = |\alpha\rangle|\delta\rangle = \begin{pmatrix} 1|\delta\rangle \\ 0|\delta\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|c\rangle = |\beta, \gamma\rangle = |\beta\rangle|\gamma\rangle = \begin{pmatrix} 0|\gamma\rangle \\ 1|\gamma\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |d\rangle = |\beta, \delta\rangle = |\beta\rangle|\delta\rangle = \begin{pmatrix} 0|\delta\rangle \\ 1|\delta\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$M_1 \otimes N_2 = \begin{pmatrix} \langle a| M_1 \otimes N_2 |a\rangle & - & - & - \\ \langle b| M_1 \otimes N_2 |a\rangle & - & - & - \\ \vdots & & & \\ - & - & - & \langle d| M_1 \otimes N_2 |d\rangle \end{pmatrix}$$

$$= \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}$$

A check:

$$C_{34} = \langle c| M_1 \otimes N_1 |d\rangle = (0\ 0\ 1\ 0) \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= (0\ 0\ 1\ 0) \begin{pmatrix} C_{14} \\ C_{24} \\ C_{34} \\ C_{44} \end{pmatrix} = C_{34}$$

Ex.

$$S_{1z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{spin } \tfrac{1}{2} \qquad S_{2z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{spin } 1$$

$$[S_{1z}, S_{2z}] = 0$$

$$\{|A_1'\rangle\} = \left\{ |S_{1z},+\rangle = |\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |S_{1z},-\rangle = |\beta\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\{|B_2'\rangle\} = \left\{ |S_{2z},+\rangle = |\gamma\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ |S_{2z},0\rangle = |\delta\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ |S_{2z},-\rangle = |\lambda\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\underbrace{\langle A_1'' B_2'' | S_{1z} \otimes S_{2z} | A_1' B_2' \rangle}_{(2\times 2)(3\times 3)\ \text{number}} = \underbrace{\langle A_1'' | S_{1z} | A_1' \rangle}_{2\times 2\ \text{number}} \underbrace{\langle B_2'' | S_{2z} | B_2' \rangle}_{3\times 3\ \text{number}}$$

$$S_{1z} \otimes S_{2z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & 0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ 0 \begin{pmatrix} & '' & \end{pmatrix} & -1 \begin{pmatrix} & '' & \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad |\alpha\rangle|\gamma\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |\alpha\rangle|\delta\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|\alpha\rangle|\lambda\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |\beta\rangle|\gamma\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\beta\rangle|\delta\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\beta\rangle|\lambda\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\text{Ex.}}$ $\quad S_{1z} \otimes S_{2z} \; |S_{1z}+ \, ; S_{2z} -\rangle = ?$

$$(M_1 \otimes N_2) \, |A_1', B_2'\rangle = \sum_{A_1''} \sum_{B_2''} |A_1'', B_2''\rangle \langle A_1''|M_1|A_1'\rangle \langle B_2''|N_2|B_2'\rangle$$

$$(S_{1z} \otimes S_{2z}) \, |S_{1z}+ \, ; S_{2z}-\rangle \equiv (S_{1z} \otimes S_{2z}) \, |+\rangle_1 \, |-\rangle_2$$

$$= |+\rangle_1 |+\rangle_2 \; \underbrace{{}_1\langle +|S_{1z}|+\rangle_1}_{M_{11}} \; \underbrace{{}_2\langle +|S_{2z}|-\rangle_2}_{N_{12}}$$

$$+ \; |+\rangle_1 |-\rangle_2 \; \underbrace{{}_1\langle +|S_{1z}|+\rangle_1}_{M_{11}} \; \underbrace{{}_2\langle -|S_{2z}|-\rangle_2}_{N_{22}}$$

$$+ \; |-\rangle_1 |+\rangle_2 \; \underbrace{{}_1\langle -|S_{1z}|+\rangle_1}_{M_{21}} \; \underbrace{{}_2\langle +|S_{2z}|-\rangle_2}_{N_{12}}$$

$$+ \; |-\rangle_1 |-\rangle_2 \; \underbrace{{}_1\langle -|S_{1z}|+\rangle_1}_{M_{21}} \; \underbrace{{}_2\langle -|S_{2z}|-\rangle_2}_{N_{22}}$$

$$= 0 + \left(\tfrac{\hbar}{2}\right)\left(-\tfrac{\hbar}{2}\right) |+\rangle_1 |-\rangle_2 + 0 + 0$$

Now, $\quad |+\rangle_1 \, |-\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} M_{11}N_{11} \cdots & \\ & \\ & \\ & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11}N_{12} \\ M_{11}N_{22} \\ M_{21}N_{12} \\ M_{21}N_{22} \end{pmatrix} = \left(-\tfrac{\hbar}{2}\right)\left(\tfrac{\hbar}{2}\right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

<u>Ex.</u> $S_{1y} \otimes S_{2z} | s_{1z}+, s_{2z}- \rangle = ?$

$$S_y = \frac{\hbar}{2}\left[-i \left(|+\rangle\langle -| \right) + i \left(|-\rangle\langle +| \right) \right]$$

$$S_z = \frac{\hbar}{2}\left[\left(|+\rangle\langle +| \right) - \left(|-\rangle\langle -| \right) \right]$$

$$(S_{1y} \otimes S_{2z}) | s_{1z}+ ; s_{2z}- \rangle = (S_{1z} \otimes S_{2z}) |+\rangle_1 |-\rangle_2$$

$$= |+\rangle_1 |+\rangle_2 \; \underbrace{{}_1\langle +| S_{1y} |+\rangle_1}_{M_{11}} \; \underbrace{{}_2\langle +| S_{2z} |-\rangle_2}_{N_{12}}$$

$$+ |+\rangle_1 |-\rangle_2 \; \underbrace{{}_1\langle +| S_{1y} |+\rangle_1}_{M_{11}} \; \underbrace{{}_2\langle -| S_{2z} |-\rangle_2}_{N_{22}}$$

$$+ |-\rangle_1 |+\rangle_2 \; \underbrace{{}_1\langle -| S_{1y} |+\rangle_1}_{M_{21}} \; \underbrace{{}_2\langle +| S_{2z} |-\rangle_2}_{N_{12}}$$

$$+ |-\rangle_1 |-\rangle_2 \; \underbrace{{}_1\langle -| S_{1y} |+\rangle_1}_{M_{21}} \; \underbrace{{}_2\langle -| S_{2z} |-\rangle_2}_{N_{22}}$$

$$= |+\rangle_1 |+\rangle_2 (0)(0) + |+\rangle_1 |-\rangle_2 (0)\left(-\frac{\hbar}{2}\right) +$$

$$+ |-\rangle_1 |+\rangle_2 \left(\frac{\hbar}{2} i\right)(0) + |-\rangle_1 |-\rangle_2 \left(\frac{\hbar}{2} i\right)\left(-\frac{\hbar}{2}\right) = -i \frac{\hbar^2}{4} |-\rangle_1 |-\rangle_2$$

Now $|+\rangle_1 |-\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} M_{11} N_{11} & - & - & \\ \vdots & & & \\ \vdots & & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} N_{12} \\ M_{11} N_{22} \\ M_{21} N_{12} \\ M_{21} N_{22} \end{pmatrix} = \left(\frac{\hbar}{2} i\right)\left(-\frac{\hbar}{2}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Addition of Angular Momentum:

Simple Examples of Ang. Mom. Addition;

A realistic description of a particle with spin must take into account both
$\begin{cases} \text{i) the space deg. of freedom} \\ \text{ii) " internal " " "} \end{cases}$

For spin $\frac{1}{2}$ $\quad \underbrace{|x', \pm\rangle}_{\text{base ket}} = \underbrace{|x'\rangle}_{\text{infinite dim.}} \otimes \underbrace{|\pm\rangle}_{\text{2-dim.}}$

where all $[M_1, N_2] = 0$ $\quad M_1$ acting in the space spanned by $\{|x'\rangle\}$

N_2 " " " " " $\{|\pm\rangle\}$

The rot. op. still takes the form $e^{-\frac{i \bar{J}\cdot\hat{n}\varphi}{\hbar}}$ but; $\bar{J} = \bar{L} + \bar{S}$

More accurately: $\quad J = L \otimes \underbrace{I_s}_{\text{identity op in the spin space}} + \underbrace{I_L}_{\text{identity op in the infinite-dim. position ket space}} \otimes S$

In fact

$$\overset{orb}{D}(R) \otimes \overset{spin}{D}(R) = e^{-\frac{i}{\hbar}L\cdot\hat{n}\varphi} \otimes e^{-\frac{i}{\hbar}S\cdot\hat{n}\varphi} \underset{[L,S]=0}{=} e^{-\frac{i}{\hbar}J\cdot\hat{n}\varphi} = D(R)$$

$$\psi = \begin{pmatrix} \psi_+(x') \\ \psi_-(x') \end{pmatrix} \qquad \text{where } \langle x', \pm|\alpha\rangle = \psi_\pm(x')$$

$|\Psi_\pm(x')|^2$: the probability density for a particle to be found at x' with spin up and down.

Instead of $|x'\rangle$ as the base kets for the space part, we may use $|n,l,m\rangle$: eigenkets of L^2 and L_z

For the spin part we may use $|\pm\rangle$: eigenkets of S^2 and S_z

$$\longrightarrow \{ |n,l,s,m_l m_s\rangle = |n l m_l\rangle |s m_s\rangle \} \quad \text{base kets}$$

These are simultaneous eigenkets of L^2, L_z, S^2 and S_z.

Alternatively we can choose our base kets to be simultaneous eigenkets of J^2, L^2, S^2 and J_z.

$$\left\{ |l,s,jm\rangle = \sum_{m_l} \sum_{m_s} |l s m_l m_s\rangle \langle l s m_l m_s | l s, j m\rangle \right\}$$

base kets

We may express an arbitrary state in terms of each of the mentioned bases.

Ex. Two spin $\frac{1}{2}$ system with the orbital deg. of freedom suppressed.

$$S = S_1 + S_2 \qquad (\text{i.e. } S_1 \otimes I_2 + I_1 \otimes S_2) \qquad (1)$$

We have $\quad [S_{1i}, S_{2j}] = 0 \qquad i, j = 1,2,3 \qquad (2)$

But $\quad [S_{1i}, S_{1j}] = i\hbar \epsilon_{ijk} S_{1k} \quad \cdots \qquad (3)$

(1)(2)(3) $\rightarrow \quad [S_i, S_j] = i\hbar \epsilon_{ijk} S_k \qquad (4)$

Option A: Simultaneous eigenkets of S_1^2, S_2^2, S_{1z} and S_{2z};

$\{|s_1 s_2, m_{s1} m_{s2}\rangle\}$ base

$$\{ \quad \} = \{ |++\rangle, |+,-\rangle, |-,+\rangle, |--\rangle \} \qquad (5)$$

Option B: Simultaneous eigenkets of S_1^2, S_2^2, S^2 and S_z

$$\{|S, m_S\rangle\} = \{ |S=1, m_S=1\rangle, |S=1, m_S=0\rangle, |S=1, m_S=-1\rangle, |S=0, m_S=0\rangle \}$$

$$\left.\begin{aligned}
&|S=1, m=1\rangle = |++\rangle && (a)\\
&|S=1, m=0\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) && (b)\\
&|S=1, m=-1\rangle = |--\rangle && (c)\\
&|S=0, m=0\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) && (d)
\end{aligned}\right\} \quad (6)$$

Remark: $S_-|S=1, m_s=1\rangle = (S_{1-}+S_{2-})|++\rangle$

$$\rightarrow \sqrt{(1+1)(1-1+1)}\,|S=1, m_s=0\rangle = \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\;|-+\rangle$$
$$+\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\;|+-\rangle$$

Clebsch-Gordan coeffs.

$$\rightarrow |S=1, m_s=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$

Continuing in this way and using orthogonality of the base kets we can find the other members.

A point: Write the 4×4 matrix corresponding to

$$S^2 = S_1^2 + S_2^2 + 2S_1\cdot S_2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

using $\{m_1 m_2\} \equiv \{|j_1 j_2 m_1 m_2\rangle\}$ basis.

The square matrix is obviously <u>not</u> diagonal (because an op. like S_{1+} connects $|-+\rangle$ to $|++\rangle$ for example).

The <u>unitary matrix</u> that diagonalizes this matrix, <u>carries</u> the $|m_1 m_2\rangle$ base kets into the $|S m_s\rangle$ base kets. ($\{S, m_s\} = \{|S_1 S_2 S\, m_s\rangle\}$). The elements of this unitary matrix are precisely the Clebsch-Gordan coeffs. for this prob.

3.8 Schwinger's Oscillator Model of Angular Momentum

Angular Momentum and Uncoupled Oscillators

There exists a very interesting connection between the algebra of ang. mom. and the algebra of two independent (uncoupled) oscillators.

Consider two simple harmonic osc., which we call plus type and minus type.

$$N_+ \equiv a_+^\dagger a_+ \qquad N_- \equiv a_-^\dagger a_- \qquad (1)$$

$$[a_+, a_+^\dagger] = I \qquad [a_-, a_-^\dagger] = I$$

$$[N_+, a_+] = -a_+ \qquad [N_-, a_-] = -a_- \qquad (2)$$

$$[N_+, a_+^\dagger] = a_+^\dagger \qquad [N_-, a_-^\dagger] = a_-^\dagger$$

However, $[a_+, a_-^\dagger] = [a_-, a_+^\dagger] = 0$ (uncoupled) (3)

$\rightarrow [N_+, N_-] = 0$ $\rightarrow$ So we can build up simultaneous eigenkets of N_- and N_+

$$N_+|n_+, n_-\rangle = n_+|n_+, n_-\rangle \quad , \quad N_-|n_+, n_-\rangle = n_-|n_+, n_-\rangle \qquad (4)$$

Also, $a_+^\dagger|n_+, n_-\rangle = \sqrt{n_+ + 1}\,|n_+ + 1, n_-\rangle$, $a_-^\dagger|n_+, n_-\rangle = \sqrt{n_- + 1}\,|n_+, n_- + 1\rangle$

$$a_+|n_+, n_-\rangle = \sqrt{n_+}\,|n_+ - 1, n_-\rangle \quad , \quad a_-|n_+, n_-\rangle = \sqrt{n_-}\,|n_+, n_- - 1\rangle \qquad (5)$$

$$a_+|0,0\rangle = 0 \qquad a_-|0,0\rangle = 0 \qquad (6)$$

(7)

$$|n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}}|0,0\rangle \qquad (8)$$

Next we define; $J_+ \equiv \hbar a_+^\dagger a_- \qquad J_- \equiv \hbar a_-^\dagger a_+ \qquad (9)$

and $J_z \equiv (\frac{\hbar}{2})(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{\hbar}{2}(N_+ - N_-) \qquad (10)$

We can prove that $\quad [J_z, J_\pm] = \pm\hbar J_\pm, \quad [J_+, J_-] = 2\hbar J_z$

For example; $\hbar^2[a_+^\dagger a_-, a_-^\dagger a_+] = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ - \hbar^2 a_-^\dagger a_+ a_+^\dagger a_-$

$= \hbar^2 a_+^\dagger(a_-^\dagger a_- + 1)a_+ - \hbar^2 a_-^\dagger(a_+^\dagger a_+ + 1)a_- = \hbar^2(a_+^\dagger a_+ - a_-^\dagger a_-) = 2\hbar J_z$

Defining the total N to be $\quad N \equiv N_+ + N_- = a_+^\dagger a_+ + a_-^\dagger a_- \qquad (11)$

We can also prove. $J^2 \equiv J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+) = \frac{\hbar^2}{2} N(\frac{N}{2} + 1)$

(12)

What are the physical interpretations of all this?

We associate spin up $(m = \frac{1}{2})$ with one Q. unit of the plus-type osc., and
" down $(m = \frac{-1}{2})$ " " " " " " minus " " "

→ n_+: number of spin up $\qquad n_-$: number of spin down

J_+: destroys one unit of spin down with z-component $-\frac{\hbar}{2}$ and
creates " " " " up " " " $+\frac{\hbar}{2}$

→ The z-component is therefore increased by $\hbar$.

J_- has the <u>reverse</u> action.

$$J_+ |n_+, n_-\rangle = \hbar a_+^\dagger a_- |n_+, n_-\rangle = \sqrt{n_-(n_+ + 1)}\,\hbar |n_+ + 1, n_- - 1\rangle \qquad (13)$$

$$J_- |n_+, n_-\rangle = \hbar a_-^\dagger a_+ |n_+, n_-\rangle = \sqrt{n_+(n_- + 1)}\,\hbar |n_+ - 1, n_- + 1\rangle \qquad (14)$$

$$J_z |n_+, n_-\rangle = \frac{\hbar}{2}(N_+ - N_-)|n_+, n_-\rangle = \underbrace{\frac{1}{2}(n_+ - n_-)\hbar}_{\text{z-component of the total ang. mom.}} |n_+, n_-\rangle \qquad (15)$$

Remark: $\frac{1}{2}(n_+ - n_-)\hbar = n_+\left(\frac{\hbar}{2}\right) + n_-\left(-\frac{\hbar}{2}\right)$

Note that in all these operations (i.e. (13), (14) (15)) the sum $n_+ + n_-$ the total number of spin $\frac{1}{2}$ particles remains <u>unchanged</u>.

Now if we substitute, $n_+ \to j+m$, $n_- \to j-m$ (16)

in (13) (14) and (15)

$$\rightarrow \begin{aligned} \sqrt{n_-(n_+ + 1)} &\to \sqrt{(j-m)(j+m+1)} \\ \sqrt{n_+(n_- + 1)} &\to \sqrt{(j+m)(j-m+1)} \end{aligned} \qquad (17)$$

$$\rightarrow \begin{aligned} J_+ |j+m, j-m\rangle &= \sqrt{(j-m)(j+m+1)}\, |j+m+1, j-m-1\rangle \\ J_- | \quad , \quad \rangle &= \sqrt{(j+m)(j-m+1)}\, |j+m-1, j-m+1\rangle \end{aligned} \qquad (18)$$

Also; the eigenvalue of $\vec{J}^2$;

$$\frac{\hbar^2}{2}(n_+ + n_-)\left[\frac{(n_+ + n_-)}{2} + 1\right] \to \hbar^2 j(j+1) \qquad (19)$$

All these may not be too surprizing, because we have already proved that $J_\pm$ and J^2 ops., we constructed out of the osc. ops. satisfy the usual ang. mom. commutation relations.

It is now natural to use.

$$(16) \to \quad j \equiv \frac{(n_+ + n_-)}{2}, \quad m \equiv \frac{n_+ - n_-}{2} \qquad (20)$$

in place of n_+ and n_- to characterize the simultaneous eigenkets of J^2 and J_z.

Acc. to (13) $\quad n_+ \to n_+ + 1,\ n_- \to n_- - 1 \to j$: unchanged

but $\quad m \longrightarrow m+1$

Likewise acc. to (14), j: unchanged, but $m \to m-1$

$$\text{Now } (8)(20) \longrightarrow |j\, m\rangle = \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\, |0\, 0\rangle \qquad (21)$$

As a special case: let $m = j$

$$(8) \to |j\, j\rangle = \frac{(a_+^\dagger)^{2j}}{\sqrt{(2j)!}}\, |0\rangle$$

We can imagine this state to be built up of 2j spin $\frac{1}{2}$ particles with their spins all pointing in the positive z-dir

In general, we note that a complicated object of high j can be visualized as being made up of primitive,

$$\text{spin } \tfrac{1}{2} \text{ particles} \begin{cases} j+m \text{ of them spin up} \\ j-m \text{ " " " down} \end{cases}$$

This picture is extremely convenient even though we obviously cannot always regard an object of ang. mom. j literally as a composite system of spin $\frac{1}{2}$ particles (like bosons).

All we are saying is that;

> As far as transformation properties under rots. are concerned, we can visualize any object of ang. mom. j as a composite system of $2j$ spin $\frac{1}{2}$ particles formed in the manner of (21).

Ex. Two spin $\frac{1}{2}$ particles can be coupled to $S=0$ as well as $S=1$.

In Schwinger's Osc. Model:

$$n_+ \to j+m \quad ; \quad n_+ \to 0+0 \quad \to n_+=0$$
$$n_- \to j-m \quad ; \quad n_- \to 0-0 \quad \to n_-=0$$

$$(8) \to |0,0\rangle = \frac{1}{1}|0\rangle$$

$n_+ \to j+m$: $n_+ \to 1+0 \quad \to n_+ = 1$

$n_- \to j-m$: $n_- \to 1-0 \quad \to n_- = 1$

$$(8) \to \quad |1\,0\rangle = \frac{a_+^+ a_-^+}{\sqrt{1}\sqrt{1}}\,|0,0\rangle = |1,0\rangle$$

$$\frac{n_+ + n_-}{2} \qquad \frac{n_+ - n_-}{2}$$

$n_+ \longrightarrow j+m$: $n_+ \to 1+1 \quad \longrightarrow n_+ = 2$

$n_- \longrightarrow j-m$: $n_- \longrightarrow 1-1 \quad \longrightarrow n_- = 0$

$$(8) \to \quad |1\,1\rangle = \frac{(a_+^+)^2 (a_-^+)^0}{\sqrt{2!}\,\sqrt{0!}}\,|0,0\rangle = |1\,1\rangle$$

. . . . .

. . . . .

In Schwinger's Osc. scheme, however, we obtain only states with ang. mom. j when we start with $2j$ spin $\frac{1}{2}$ particles. In the language of permutation symmetry (Chap. 6), only totally symmetrical states are constructed by this method.

The primitive spin $\frac{1}{2}$ particles appearing here are actually bosons! This method is quite adequate if our purpose is to examine the properties under rots. of states characterized by j and m without asking how such states are built up initially.

Ex.

Isospin formalism:

$J_+ \longrightarrow T_+$: annihilates one neutron and creates one proton

$J_z \longrightarrow T_z$: its eigenvalue counts the difference between neutrons and protons.

Tensor Operators

Vector Operator

The operators corresponding to various physical quantities are characterized by their behavior under rot., as scalars, vectors and tensors.

$$V_i = \sum_j R_{ij} V_j \qquad \text{Def. of vector in cl.M.} \qquad (1)$$

It is reasonable to demand that;

A vector op. is an op. that expectation values of its components transform like the components of a cl. vector under rot.

$$\begin{pmatrix} \psi^{+\prime} V_x \psi' \\ \psi'^{+} V_y \psi' \\ \psi'^{+} V_z \psi' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} \psi^{+} V_x \psi \\ \psi^{+} V_y \psi \\ \psi^{+} V_z \psi \end{pmatrix} \qquad (2)$$

Indeed $|\alpha\rangle \xrightarrow{\text{rot}} D(R)|\alpha\rangle = |\alpha\rangle'$

$${}'\langle\alpha| V_i |\alpha\rangle' = \langle\alpha| D^{+}(R) V_i D(R) |\alpha\rangle = \sum_j R_{ij} \langle\alpha| V_j |\alpha\rangle$$

or $\langle V_i\rangle' = \sum_j R_{ij} \langle V_j\rangle \qquad \forall |\alpha\rangle$ (arbitrary) (3)

$$\longrightarrow \quad D^{+}(R)\, V_i\, D(R) = \sum_j R_{ij} V_j \qquad (4)$$

Fundamental criterion for V being a vector op.

Now consider an infinitesimal rot.;

$$R_{ij} = \delta_{ij} + \epsilon_{ij} \qquad \sum_j R_{ij} R_{kj} = \delta_{ik} \quad \text{orthogonality cond.} \quad (5)$$

$$\to \sum_j (\delta_{ij} + \epsilon_{ij})(\delta_{kj} + \epsilon_{kj}) = \delta_{ik} \quad \to \underset{(6)}{\epsilon_{ik} + \epsilon_{ki} = 0} \quad (\text{to first order in } \epsilon)$$

$$\epsilon_{ij} = \begin{pmatrix} 0 & \epsilon_{12} & \epsilon_{13} \\ -\epsilon_{12} & 0 & \epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & 0 \end{pmatrix} , \quad \text{setting} \begin{cases} \epsilon_{12} = -\epsilon_z \\ \epsilon_{23} = -\epsilon_x \\ \epsilon_{31} = -\epsilon_y \end{cases} \to R = \begin{pmatrix} 1 & -\epsilon_z & \epsilon_y \\ \epsilon_z & 1 & -\epsilon_x \\ -\epsilon_y & \epsilon_x & 1 \end{pmatrix} \quad (7)$$

where $R \equiv R(\hat{n}, |\epsilon|) \qquad \bar{\epsilon} = (\epsilon_x, \epsilon_y, \epsilon_z) \quad , \hat{n} \parallel \bar{\epsilon}$

Also $D(R) = I - \frac{i}{\hbar} \epsilon J \cdot \hat{n} = I - \frac{i}{\hbar} \bar{\epsilon} \cdot J$

$$\to \left(I + \frac{i}{\hbar} \epsilon J \cdot \hat{n}\right) V_i \left(I - \frac{i}{\hbar} \epsilon J \cdot \hat{n}\right) = (\delta_{ij} + \epsilon_{ij}) V_j$$

$$\cancel{V_i} - \frac{i}{\hbar} \epsilon n_k V_i J_k + \frac{i}{\hbar} \epsilon n_k J_k V_i + O(\epsilon^2) = (\delta_{ij} - \epsilon\, \epsilon_{ijk} n_k) V_j$$
$$= \cancel{V_i} - \epsilon\, \epsilon_{ijk} n_k V_j$$

$$\to \frac{i}{\hbar} \cancel{\epsilon} \cancel{n_k} [J_k, V_i] = - \cancel{\epsilon}\, \epsilon_{ijk} V_j \cancel{n_k}$$

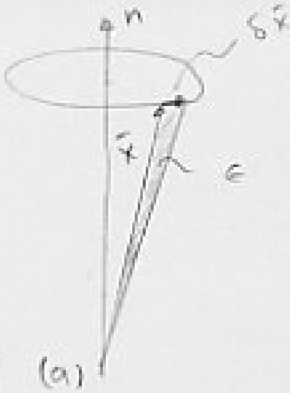
$$\to [J_k, V_i] = i\hbar\, \epsilon_{ijk} V_j \quad \to [V_i, J_j] = i\hbar \epsilon_{ijk} V_k \quad (8)$$

This relation is equivalent to $D^\dagger(R) V_i D(R) = \sum_j R_{ij} V_j$ for <u>infinitesimal</u> rot. It is <u>sufficient</u> to <u>ensure</u> the cond. (4) is satisfied for <u>finite</u> rots. Indeed any <u>finite</u> rot. can be built up by a <u>large</u> number of the infinitesimal rots.

$$D^\dagger(R) V_i D(R) = \sum_j R_{ij} V_j , \quad D^\dagger(S) V_i D(S) = \sum_j S_{ij} V_j$$
$$D^\dagger(SR) V_i D(SR) = D^\dagger(R) D^\dagger(S) V_i D(S) D(R) = D^\dagger(R) \sum_j S_{ij} V_j D(R)$$
$$= \sum_j \sum_k S_{ij} R_{jk} V_k = \sum_k (SR)_{ik} V_k$$

Remark:

$R_{ij} = \delta_{ij} + \epsilon_{ij}$ $\qquad \epsilon_{ij}$: infinitesimal



$$\begin{cases} x'_i = R_{ij} x_j \approx x_i + \epsilon_{ij} x_j \\ x'_i = x_i + \delta x_i \quad \text{for an infinitesimal rot.} \end{cases} \rightarrow \delta x_i = \epsilon_{ij} x_j \quad (9)$$

Since, $\delta \bar{x} = \delta \bar{\omega} \times \bar{x}$ $\qquad \delta x_i = \epsilon_{ijk} \omega_j x_k = \epsilon_{ijk} (\epsilon \hat{n}_j) x_k$

$$= -\epsilon_{ijk} \epsilon n_k x_j \quad (10)$$

$(9)(10) \rightarrow \quad \epsilon_{ij} = -\epsilon \, \epsilon_{ijk} n_k$

For <u>finite</u> rot. we apply;

$$e^{iB\lambda} A e^{-iB\lambda} = A + i\lambda [B, A] + \cdots + \left(\frac{i^n \lambda^n}{n!}\right) [B, [B, \cdots [B, A]] \cdots] + \quad (B = B^\dagger, \lambda = \lambda^*)$$

to $\quad e^{\frac{i}{\hbar} J_j \varphi} V_i e^{-\frac{i}{\hbar} J_j \varphi}$

We simply need to calculate $[J_j, [J_j, [\cdots [J_j, V_i]] \cdots]$

Multiple commutators keep on giving back to us V_i or V_k $(k \neq i, j)$

Now look that J_i's <u>are not</u> <u>arbitrary</u> and they must satisfy;

$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$ $\qquad$ (from def. of ang mom. op.)

Compare with (8) $\longrightarrow$ J is a vector op.

Whether or not a given op. V constitutes a vector op.

depends on $\begin{cases} 1- \text{Def. of the Physical syst.} \\ 2- \text{The structure of its ang. mom } J \end{cases}$

Ex. x, y, z (coord. ops.) provide a complete description of the dynamical system (a particle without spin).

In this case $\bar{J} = \bar{L} = \bar{r} \times \bar{P}$

The quantities $\bar{r}, \bar{P}, \bar{L}$ are all vector ops.

Because; $[L_i, r_j] = i\hbar \epsilon_{ijk} r_k$

$[L_i, P_j] = i\hbar \epsilon_{ijk} P_k$

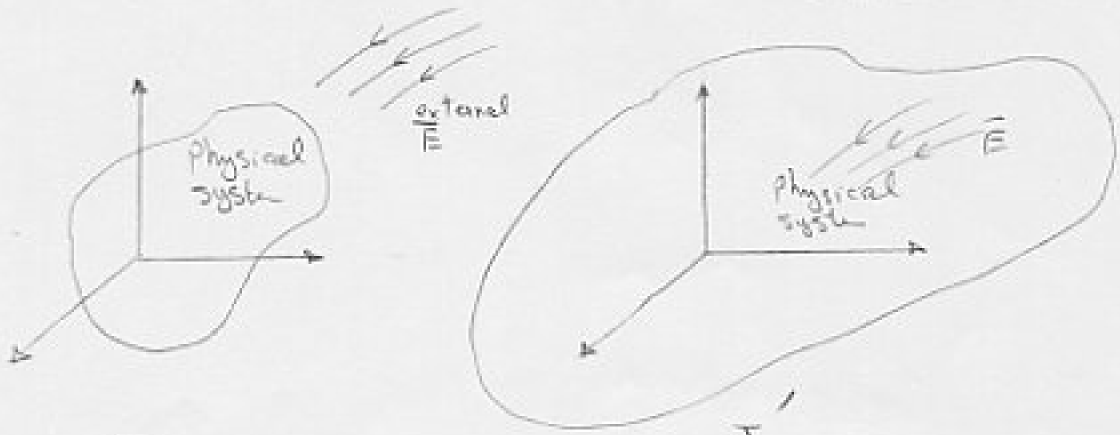
$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$

Ex. On the other hand an external electric field $\bar{E}$ acting on the system does not in general make up a vector op. with respect to this system, even though E is of course a vector. Cond. (4) does not satisfy for such $\bar{E}$. Because is external to the system and not subject to rot. together with it.

But if the system is enlarged so as to include the sources of the electric field in the dynamical description

$\longrightarrow$ $\bar{E}$ would become a proper vector op.

$\longrightarrow$ This would result in a much more complicated J.



J $\qquad\qquad\qquad\qquad$ J'

Ex.

Consider a particle with spin:

If $J = L$ (our physical sys.)

$\rightarrow [J_k, S_i] = 0 \neq i\hbar\epsilon_{ijk} S_k \quad , \quad D^{\dagger}(R)\, S_i\, D(R) \neq \sum_j R_{ij} S_j$

But if $J = L + S$ (enlarged space)

$\rightarrow [J_k, S_i] = i\hbar\epsilon_{ijk} S_j \quad , \quad D^{\dagger}(R)\, S_i\, D(R) = \sum_j R_{ij} S_j$

i.e. Spin wave func. must be rotated with the space wave func.

Cartesian Tensors Versus Irreducible Tensors

Def. - $\quad T_{ijk\ldots} = \sum_{i'}\sum_{j'}\sum_{k'} \cdots R_{ii'} R_{jj'} R_{kk'} \cdots T_{i'j'k'\ldots} \qquad (1)$

Classical Cartesian tensor (not Tensor op.)

Because; for example: $\quad \langle \Psi' | T_{ij} | \Psi' \rangle = \sum_{k,l} R_{ik} R_{jl} \langle \Psi | T_{kl} | \Psi \rangle$

$\to \cancel{D^{\dagger}(R)\, T_{ij}\, D(R) = \sum_{kl} R_{ik} R_{jl} T_{kl}} \cdots$

rank = the number of indices $\quad R$; 3×3 orthogonal matrix

<u>Ex.</u> Dyadic , rank = 2

$$T_{ij} \equiv U_i V_j \qquad (2) \qquad U, V;\ \text{two vectors}$$

The trouble with Cartesian tensor like (2) is that, it is <u>reducible</u>, that is, it can be <u>decomposed</u> into objects that transforms <u>differently</u> under rot.

For a dyadic;

$$T_{ij} = U_i V_j = \underbrace{\frac{U\cdot V}{3}\delta_{ij}}_{\text{scalar invariant under rot.}} + \underbrace{\frac{(U_i V_j - U_j V_i)}{2}}_{=\epsilon_{ijk}(U\times V)_k \text{ antisymmetric tensor with 3-indep components}} + \underbrace{\left(\frac{U_i V_j + U_j V_i}{2} - \frac{U\cdot V}{3}\delta_{ij}\right)}_{\text{symmetric traceless tensor with (6-1) indep. components, traceless}} \qquad (3)$$

$\to 3\times 3 = 1 + 3 + 5$

$\left\{\begin{matrix}1\\3\\5\end{matrix}\right.$ are the multiplicities of objects with ang. mom. $\left\{\begin{matrix}l=0\\l=1\\l=2\end{matrix}\right.$

This suggests that a dyadic has been decomposed into tensors that can transforms like spherical harmonics.

A point: An example of spherical tensor;

$Y_\ell^m(\theta,\varphi) \equiv Y(\hat{n})$ spherical harmonic

Replace; $\hat{n} \xrightarrow{by} \bar{V}$ (vector) $\longrightarrow T_k^q = Y_{\ell=k}^{m=q}(\bar{V})$

This is a spherical tensor.

Specifically; in the $k=1$ case; we take spherical harmonics with $\ell=1$ and replace

$\frac{z}{r} = (\hat{n})_z$ by V_z, $\frac{x \pm iy}{r} = (\hat{n})_{x\pm iy}$ by $V_x \pm iV_y$

$\rightarrow Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \rightarrow T_1^0 = \sqrt{\frac{3}{4\pi}} V_z$

$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{4\pi}} \frac{x \pm iy}{\sqrt{2}r} \rightarrow T_1^{\pm 1} = \sqrt{\frac{3}{4\pi}} \left(\mp \frac{V_x \pm iV_y}{\sqrt{2}} \right)$

Also, for $k=2$

$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \frac{(x \pm iy)^2}{r^2} \longrightarrow T_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} (V_x \pm iV_y)^2$

(4)

A General def. for the Tensor Op.:

First look:

$$D^{+}(R)\, V_i\, D(R) = \sum_j R_{ij} V_j \quad (5) \rightarrow D(R)\left(D^{+}(R) V_i D(R)\right) D^{+}(R) =$$

$$= \sum_j D(R) R_{ij} V_j D^{+}(R)$$

$$\rightarrow V_i = \sum_i D(R) R_{ij} V_j D^{+}(R)$$

$$\rightarrow \sum_i V_i R_{ik} = \sum_i \sum_j R_{ij} R_{ik} D(R) V_j D^{+}(R)$$

$$\rightarrow \sum_i V_i R_{ik} = \sum_j \delta_{jk} D(R) V_j D^{+}(R)$$

$$\rightarrow D(R)\, V_k\, D^{+}(R) = \sum_i V_i R_{ik} \quad (5')$$

$$V_k' = D(R)\, V_k\, D^{+}(R) \quad \text{rotational transform of } V_k \quad (5'')$$

Obviously $\langle V_k' \rangle' = \langle V_k \rangle$ (6)

→ Def: A vector op is thus a set of three operators V_x, V_y, V_z, whose rotational transforms V_x', V_y', V_z' are certain linear funcs of V_x, V_y, V_z.

Generalization to Tensors:

A tensor op. is a set of n ops. $T_1, T_2 \cdots T_n$, such that their rotational transforms are linear funcs. of the n ops.:

$$T_k' = D(R)\, T_k\, D^{+}(R) = \sum_i T_i\, D_{ik}(R) \quad \text{(reducible)} \quad (7)$$

The coeffs. $D_{ik}(R)$ depend on the rot. and are obviously representations of the rot. group.

The irreducible spherical tensor op. of rank k is a set of $2k+1$ ops. T_k^q which satisfy the tr. equ.:

$$D(R)\, T_k^q\, D^+(R) = \sum_{q'=-k}^{k} T_k^{q'}\, D^{(k)}_{q'q}(R) \quad (8)$$

Thus T_k^q under rot. it transforms like $|k,q\rangle$, because;

$$\begin{aligned} D(R)|k,q\rangle &= \sum_{q'=-k}^{k} |k,q'\rangle\langle k,q'|D(R)|kq\rangle \\ &= \sum_{q'=-k}^{k} |kq'\rangle\, D^{(k)}_{q'q}(R) \end{aligned} \quad (8')$$

Remark: Note that we obtained (5') from (5). The equ. (8) is counterpart of (5'). This form of (8) is considered, because it is comparable with (8'). It equivalent form is given in (P 373).

Remark: $T_k^q \in \{ T_k^q \;\; q=-k \ldots q=k \}$

$D(R)\, T_k^q\, D^+(R) \in \{ \quad " \quad \}$

$\rightarrow T_k^q$: irreducible tensor op.

A More Convenient Def.
of a Spherical Tensor:

$$(8) \rightarrow \left(I - \frac{i}{\hbar} J \cdot \hat{n}\,\epsilon\right) T_k^q \left(I + \frac{i}{\hbar} J \cdot n\,\epsilon\right) = \sum_{q'=-k}^{k} T_k^{q'} \langle k q' | \left(I - \frac{i}{\hbar} J \cdot n\,\epsilon\right) | k q \rangle$$

$$\Rightarrow \left[J \cdot n , T_k^q \right] = \sum_{q'=-k}^{k} T_k^{q'} \langle k q' | J \cdot n | k q \rangle$$

$$n_i J_i = n_- J_+ + n_+ J_- + n_0 J_0$$

$$J_\pm = J_x \pm i J_y \qquad J_0 = J_z$$

$$n_\pm = n_x \pm i n_y \qquad n_0 = n_z$$

$$n_- \left[J_+ , T_k^q \right] = n_- \sum_{q'} T_k^{q'} \langle k q' | J_+ | k q \rangle$$

$$n_+ \left[J_- , T_k^q \right] = n_+ \sum_{q'} T_k^{q'} \langle k q' | J_- | k q \rangle$$

$$n_0 \left[J_0 , T_k^q \right] = n_0 \sum_{q'} T_k^{q'} \langle k q' | J_0 | k q \rangle$$

$$\rightarrow \quad \left[J_0 , T_k^q \right] = \hbar q\, T_k^q$$

$$\left[J_\pm , T_k^q \right] = \hbar \sqrt{(k \mp q)(k \pm q + 1)}\; T_k^{q \pm 1}$$

These relations serve to test whether a set of $(2k+1)$ ops. T_k^q constitutes an irreducible spherical tensor op. with respect to the sys. whose rots are generated by J.

Product of Tensors:

Consider.

$$T_0^0 = -\frac{U\cdot V}{3} = \frac{U_+V_- + U_-V_+ - U_0V_0}{3}$$

$$T_1^q = \frac{(U\times V)_q}{i\sqrt{2}} \qquad\qquad U_{\pm 1} = \frac{\mp(U_x \pm iU_y)}{\sqrt{2}}$$

$$T_2^{\pm 2} = U_\pm V_\pm \qquad\qquad U_0 = U_z$$

$$T_2^{\pm 1} = \frac{U_\pm V_0 + U_0 V_\pm}{\sqrt{2}}$$

$$T_2^0 = \frac{U_+V_- + 2U_0V_0 + U_-V_+}{\sqrt{6}}$$

These are examples of tensor products.

A more systematic way of forming tensor products goes as follows:

Theorem: Let T_{k1}^{q1} and T_{k2}^{q2} be irreducible spherical tensors of rank $k1$ and $k2$, then;

$$T_k^q = \sum_{q1}\sum_{q2} \langle k1, q1, k2, q2 | k1\, k2; k\, q\rangle\, T_{k1}^{q1}\, T_{k2}^{q2} \qquad (1)$$

is spherical (irreducible) tensor of rank k.

Proof: We must show that under rot. T_k^q must transform acc.

$$D^+(R)\, T_k^q\, D(R) = \sum_{q'=-k}^{k} D_{qq'}^{(k)\,*}(R)\, T_k^{q'} \qquad (2)$$

or equivalently $D(R)\, T_k^q\, D^+(R) = \sum_{q'=-k}^{k} D_{q'q}^{(k)}(R)\, T_k^{q'}$ (3)

(1) $\to$ $D^{\dagger}(R)\, T_k^{q}\, D(R) = \sum_{q_1}\sum_{q_2} \langle k_1 q_1, k_2 q_2 | k_1 k_2 ; k q\rangle$

$\cdot D^{\dagger}(R)\, T_{k_1}^{q_1}\, D(R)\, D^{\dagger}(R)\, T_{k_2}^{q_2}\, D(R)$

(2) $\to$

$$= \sum_{q_1}\sum_{q_2}\sum_{q_1'}\sum_{q_2'} \langle k_1 q_1, k_2 q_2 | k_1 k_2 ; k, q\rangle\, T_{k_1}^{q_1'}\, D^{(k_1)*}_{q_1 q_1'}(R)\, T_{k_2}^{q_2'}\, D^{*(k_2)}_{q_2 q_2'}(R)$$

$$= \quad '' \quad '' \quad '' \quad D^{(k_1)}_{q_1' q_1}(R^{-1}) \quad '' \quad D^{(k_2)}_{q_2' q_2}(R^{-1})$$

(5) $\to$

$$= \sum_{k''}\sum_{q_1}\sum_{q_2}\sum_{q_1'}\sum_{q_2'}\sum_{q''}\sum_{q'} \langle k_1 q_1, k_2 q_2 | k_1 k_2 ; k, q\rangle \langle k_1 q_1', k_2 q_2' | k_1 k_2 ; k'' q'\rangle$$

$$\cdot \langle k_1 q_1, k_2 q_2 | k_1 k_2 ; k'' q''\rangle\, D^{(k'')}_{q' q''}(R^{-1})\, T_{k_1}^{q_1'}\, T_{k_2}^{q_2'}$$ (4)

where we have used;

$$D^{(j_1)}_{m_1 m_1'}(R)\, D^{(j_2)}_{m_2 m_2'}(R) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m}\sum_{m'} \langle j_1 m_1, j_2 m_2 | j_1 j_2 ; j m\rangle$$

$$\cdot \langle j_1 m_1', j_2 m_2' | j_1 j_2 ; j m'\rangle\, D^{(j)}_{m m'}(R)$$ (5)

Using the orthogonality of the Clebsch-Gordan coeffic.;

$$\sum_{m_1}\sum_{m_2} \langle j_1 m_1, j_2 m_2 | j_1 j_2 ; j m\rangle \langle j_1 m_1, j_2 m_2 | j_1 j_2 ; j' m'\rangle = \delta_{jj'}\, \delta_{mm'}$$ (6)

$$D^{\dagger}(R)\, T_k^{q}\, D(R) = \sum_{k''}\sum_{q_1'}\sum_{q_2'}\sum_{q''}\sum_{q'} \delta_{kk''}\, \delta_{qq''} \langle k_1 q_1', k_2 q_2' | k_1 k_2 ; k'' q'\rangle\, D^{(k'')}_{q' q''}(R^{-1})$$

$$\cdot T_{k_1}^{q_1'}\, T_{k_2}^{q_2'}$$

$$= \sum_{q'} \left(\sum_{q_1'}\sum_{q_2'} \langle k_1 q_1', k_2 q_2' | k_1 k_2 ; k q'\rangle\, T_{k_1}^{q_1'}\, T_{k_2}^{q_2'} \right) \cdot D^{k}_{q' q}(R^{-1})$$ (7)

(1) $\to$

$$= \sum_{q'} T_k^{q'}\, D^{(k)}_{q' q}(R^{-1}) = \sum_{q'} D^{*(k)}_{q q'}(R)\, T_k^{q'}$$

A point; look th $T_k^q|jm\rangle$ under the rot.;

$$D(R)\left(T_k^q|jm\rangle\right) = D(R)\,T_k^q\left(D^\dagger(R)D(R)\right)|jm\rangle$$

$$= D(R)\,T_k^q\,D^\dagger(R)\sum_{m'}|jm'\rangle\langle m'j|D(R)|jm\rangle$$

$$= \sum_{m'}\underbrace{D(R)\,T_k^q\,D^\dagger(R)}_{\sum_{q'} T_k^{q'} D^{(k)}_{q'q}(R)}|jm'\rangle\, D^{(j)}_{m'm}(R)$$

$$= \sum_{m'q'} D^{(j)}_{m'm}(R)\, D^{(k)}_{q'q}(R)\; T_k^{q'}|jm'\rangle$$

Lesson: $T_k^q|jm\rangle$ transforms like $|kq\rangle\otimes|jm\rangle$

because;

$$D(R)|kq\rangle|jm\rangle = D^k(R)|kq\rangle\, D^j(R)|jm\rangle$$

$$= \sum_{q'}\sum_{m'}|kq'\rangle\langle kq'|D^k(R)|kq\rangle\,|jm'\rangle\langle jm'|D^j(R)|jm\rangle$$

$$= \sum_{q'}\sum_{m'}|kq'\rangle|jm'\rangle\, D^k_{q'q}(R)\, D^j_{m'm}(R)$$

Remark: $D(R) = e^{-\frac{i}{\hbar}\underbrace{(k+j)}_{J}\cdot\hat{n}\,\varphi} = D_1(R)\otimes D_2(R)$

Remark: By th orthonormality of th Clebsch-G. coeffs., we have also

$$(1)\rightarrow\quad T_{k_1}^{q_1}T_{k_2}^{q_2} = \sum_{k=|k_1-k_2|}^{k_1+k_2}\sum_q T_k^q\,\langle kq|k_1q_1,k_2q_2\rangle$$

$$= \sum_k\sum_q T_k^q\,\langle k_1q_1,k_2q_2|kq\rangle$$

Matrix Elements of Tensor Operators: The Wigner-Eckart Theorem

Some properties of the matrix elements that follow from kinematic or geometric considerations, are as below;

i) m-selection rule

$$\langle \alpha', j'm' | T_k^q | \alpha, jm\rangle = 0 \qquad \text{unless } m' = q + m$$

Using $[J_z, T_k^q] = \hbar q\, T_k^q$

$$\to \langle \alpha', j'm' | \left([J_z, T_k^q] - \hbar q\, T_k^q \right) | \alpha, jm\rangle = [(m'-m)\hbar - \hbar q]$$

$$\cdot \langle \alpha', j'm' | T_k^q | \alpha, jm\rangle = 0$$

$$\to \langle \alpha', j'm' | T_k^q | \alpha, jm\rangle = 0 \quad \text{unless } m' = q + m$$

Alternatively; consider the tr. property of $T_k^q |\alpha, jm\rangle$ under rot.,

$$D(R)\, T_k^q |\alpha, jm\rangle = D(R)\, T_k^q D^\dagger(R)\, D(R) |\alpha, jm\rangle$$

Now consider a rot. about z-axis

$$D(\hat{z},\varphi)\, T_k^q |\alpha, jm\rangle = \sum_{q'=-k}^{k} D^{(k)}_{q'q}(\hat{z},\varphi)\, T_k^{q'}\, D(\hat{z},\varphi) |\alpha, jm\rangle$$

$$= \sum_{q'} \delta_{qq'}\, e^{-iq\varphi}\, T_k^{q'}\, e^{-im\varphi} |\alpha, jm\rangle = e^{-i(q+m)\varphi}\, T_k^q |\alpha, jm\rangle$$

$$\to \langle \alpha, jm' | D(\hat{z},\varphi)\, T_k^q |\alpha, jm\rangle = e^{-i(q+m)\varphi} \langle \alpha, jm' | T_k^q | \alpha jm\rangle \qquad (1)$$

on the other hand;

$$\langle \alpha, jm' | D(\hat{z},\varphi)\, T_k^q |\alpha, jm\rangle = e^{-im'\varphi} \langle \alpha, jm' | T_k^q | \alpha, jm\rangle \qquad (2)$$

$$(1)(2) \rightarrow e^{-i(q+m)\varphi} \langle \alpha j m' | T_k^q | \alpha j m \rangle = e^{-i m' \varphi} \langle \alpha j m' | T_k^q | \alpha j m \rangle$$

$\rightarrow$ For $\langle \alpha j m' | T_k^q | \alpha j m' \rangle \neq 0$

$\rightarrow m' = m+q$

Remark:

$\langle \alpha j m' | D(\hat{z},\varphi) \longleftrightarrow D^{+}(\hat{z},\varphi) | \alpha j m \rangle$

ii) Wigner-Eckart Theorem:

The matrix elements of tensor ops. with respect ang. mom. eigenstates satisfy

$$\langle \alpha', j' m' | T_k^q | \alpha, j m \rangle = \langle j m, k q | j k ; j' m' \rangle \frac{\langle \alpha' j' \| T_k \| \alpha j \rangle}{\sqrt{2j+1}}$$

depending on geometry (i.e. the orientation of the system w.r.t. z-axis)

There is no reference whatsoever to the particular nature of the tensor op.

The second factor does depend on the dynamics, for instance, α may stand for the radial Q. number and its evaluation may involve, for example, evaluation of radial integrals.

On the other hand it is indep. of mag. Q. numbers, m, m' and q, which satisfy the orientation of the physical system.

Selection rules:

i) $m' = m+q$ (non vanishing Clebsch-Gordan coeffs.)

ii $|j-k| \leq j' \leq j+k$ 〃

Proof:

$$\langle \alpha' j' m' | [J_\pm , T_k^q] | \alpha j m\rangle = \hbar \sqrt{(k\mp q)(k\pm q+1)}\ \langle \alpha' j' m' | T_k^{q\pm 1} | \alpha j m\rangle$$

on the other hand

$$\langle \alpha' j' m' | (J_\pm T_k^q - T_k^q J_\pm) | \alpha j m\rangle =$$

$$= \sqrt{(j'\pm m')(j'\mp m'+1)}\ \langle \alpha' j' m'\mp 1 | T_k^q | \alpha j m\rangle -$$

$$\sqrt{(j\mp m)(j\pm m+1)}\ \langle \alpha' j' m' | T_k^q | \alpha j\, m\pm 1\rangle$$

$$\sqrt{(\underset{j}{j'}\pm \underset{m}{m'})(\underset{j_1}{j}\mp \underset{m_1}{m'}+1)}\ \langle \alpha' j' m'\mp 1 | T_k^q | \alpha j m\rangle =$$

$$\sqrt{(\underset{j_1}{j}\mp \underset{m_1}{m})(\underset{j_1}{j}\pm \underset{m_1}{m}+1)}\ \langle \alpha' j' m' | T_k^q | \alpha j\, m\pm 1\rangle + \sqrt{(\underset{j_2}{k}\mp \underset{m_2}{q})(\underset{j_2}{k}\pm \underset{m_2}{q}+1)}\ \langle \alpha' j' m' | T_k^{q\pm 1} | \alpha j m\rangle$$

Compare this with the <u>recursion</u> relation for the Clebsch-Gordan coeff.

$$\sqrt{(j\mp m)(j\pm m+1)}\ \langle j_1 m_1, j_2 m_2 | j_1 j_2, j, m\pm 1\rangle =$$

$$\sqrt{(j_1\mp m_1+1)(j_1\pm m_1)}\ \langle j_1\, m_1\mp 1, j_2 m_2 | j_1 j_2; j m\rangle$$

$$+ \sqrt{(j_2\mp m_2+1)(j_2\pm m_2)}\ \langle j_1 m_1, j_2\, m_2\mp 1 | j_1 j_2; j m\rangle$$

Note the striking similarity if we substitute

$j' \longrightarrow j$ $\qquad$ $m \longrightarrow m_1$

$m' \longrightarrow m$ $\qquad$ $k \longrightarrow j_2$

$j \longrightarrow j_1$ $\qquad$ $q \longrightarrow m_2$

Both recursion relations are of the form $\sum_j a_{ij} x_j = 0$ (i.e. first order linear homogeneous equs.) with the <u>same</u> coeffs. <u>a_{ij}</u>.

Whenever we have $\sum_j a_{ij} x_j = 0 \quad \sum_j a_{ij} y_j = 0$

we <u>cannot</u> solve for the x_j (or y_j) <u>individually</u> but we can solve for the ratios, so

$$\frac{x_j}{x_k} = \frac{y_j}{y_k} \quad \text{or} \quad x_j = c\, y_j \qquad \text{(the sols. must be proportional)}$$

where c: universal proportionality factor. Noting that, $\langle j_1 m_1, j_2 m_2 \pm 1 | j_1 j_2 ; j m \rangle$ in the Clebsch-Gordan recursion relation corresponds $\langle \alpha' j' m' | T_k^{q\pm 1} | \alpha j m \rangle$ we see that

$$\langle \alpha' j' m' | T_k^{q\pm1} | \alpha j m \rangle = \alpha (\text{indep. of } m, q, m') \langle j m, k\, q\pm 1 | j k ; j' m' \rangle$$

$$\rightarrow \langle \alpha' j' m' | T_k^{q} | \alpha j m \rangle = \alpha (\quad " \quad) \langle j m, k q | j k ; j' m' \rangle$$

$$\alpha \equiv \frac{\langle \alpha' j' \| T_k^q \| \alpha j \rangle}{\sqrt{2j+1}} \qquad \text{reduced matrix element}$$

which proves the theorem.

From the selection rules, $\begin{cases} q = m' - m \\ |j - j'| \leq k \leq j + j' \end{cases}$

we infer that:

Remark: α is indep. of m, q, m' because in $x_j = c\, y_j$, c is the same for all values of j. ($\rightarrow$ different values of magnetic q. number)

i) The scalar operator ($k=0$) has non-vanishing matrix elements only if: $\begin{cases} m = m' \\ j = j' \end{cases}$

ii) The selection rules for a vector op. are:

$$\Delta m = m' - m = 0, \pm 1$$
$$\Delta j = j' - j = 0, \pm 1 \qquad (j = j' = 0 \text{ excluded})$$
because, $k=1$

Combination of Operator and Eigenstate:

Analog of equs; $\begin{cases} |jm\rangle = \sum_{m_1}\sum_{m_2} |j_1 m_1\rangle |j_2 m_2\rangle \langle j_1 m_1, j_2 m_2 | jm\rangle \\ \text{and} \\ T_k^q = \sum_{q_1}\sum_{q_2} T_{k_1}^{q_1} T_{k_2}^{q_2} \langle k_1 q_1, k_2 q_2 | kq\rangle \end{cases}$

we have;
$$|jm\rangle = \sum_{q_1}\sum_{m_2} T_{k_1}^{q_1} |j_2 m_2\rangle \langle k_1 q_1, j_2 m_2 | jm\rangle \quad (1)$$

α-quant numbers which in general cause radial dependence have been omitted.

$|jm\rangle$ established in this way are ang. mom eigenstates because they satisfy equs. like

$$J_z |jm\rangle = m\hbar |jm\rangle, \quad J_\pm |jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\, |j, m\pm 1\rangle \ldots$$

But in general $\langle jm|jm\rangle \neq 1$

$$(1) \rightarrow N = \langle jm|jm\rangle = \sum_{q_1}\sum_{m_2} \langle jm| T_{k_1}^{q_1} |j_2 m_2\rangle \langle k_1 q_1, j_2 m_2 | jm\rangle \quad (2)$$

$\rightarrow N$: indep of q_1, m_2

Now $\langle jm| J_+^\dagger J_+ |jm\rangle = \langle jm| (\vec{J}^2 - J_z^2 - \hbar J_z) |jm\rangle$

$$\rightarrow |\varsigma_+(jm)|^2 \langle j, m+1 | j, m+1\rangle = [j(j+1) - m(m+1)] \langle jm|jm\rangle$$

$\rightarrow \underbrace{[j(j+1)-m(m+1)]}_{|c_+(j,m)|^2 ?} \langle j\, m+1 | j\, m+1\rangle = [j(j+1)-m(m+1)] \langle jm | jm\rangle$

$\rightarrow \langle j, m+1 | j, m+1 \rangle = \langle jm | jm \rangle$

$\rightarrow N(j, m+1) = N(j,m) \longrightarrow N$: indep of m

But from (2); $\quad N = N(j, k_1, j_2, T_{k_1})$

Now by the use of orthonormality of the Clebsch-G. coeffs.;

$$T_{k_1}^{q_1} |j_2 m_2\rangle = \sum_j \sum_m |jm\rangle\langle jm | q_1 m_2\rangle = \sum_j \sum_m |jm\rangle\langle q_1 m_2 | jm\rangle$$

Matrix Elements of R_1^q:

$$R_1^q = \sqrt{\frac{4\pi}{3}}\; r\, Y_1^q(\theta,\varphi)$$

$$\langle \alpha' j' m' | R_1^q | \alpha j m\rangle = ?$$

Let us ignore the spin,

$$\langle n'l'm' | R_1^q | nlm\rangle = \int r^2 dr\, d\Omega\; \Psi^*_{n'l'm'}(r,\theta,\varphi)\, R_1^q\, \Psi_{nlm}(r,\theta,\varphi)$$

$$= \left[\int r^2 dr\, R^*_{n'l'}(r)\; r\, R_{nl}(r)\right]\left[\sqrt{\frac{4\pi}{3}}\int d\Omega\, Y^*_{l'm'}(\theta,\varphi)\, Y_1^q(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\right]$$

$$= \langle n'l' | r | nl\rangle \sqrt{\frac{4\pi}{3}} \langle l'm' | Y_1^q(\theta,\varphi) | lm\rangle$$

Now $\quad Y_1^q |lm\rangle = \sum_{l''m''} |l''m''\rangle\langle qm | l''m''\rangle \quad ?$

$$\rightarrow \langle l'm' | Y_1^q | lm\rangle = \sum_{l''m''} \langle l'm' | l''m''\rangle\langle qm | l''m''\rangle$$

$$\rightarrow \langle l'm'|Y_1^q|lm\rangle = \langle l'm'|l'm'\rangle \langle 1q, lm|l'm'\rangle$$

(Because $\langle l'm'|l''m''\rangle = 0$ unless $l' = l''$, $m' = m''$)

$$\langle l'm'|Y_1^q|lm\rangle = N(l', l, k=1, Y_1)\, \langle 1q, lm|l'm'\rangle$$

$$= \frac{\langle l'\|Y_1\|l\rangle}{\sqrt{2l+1}} \langle 1q, lm|l'm'\rangle$$

$$\rightarrow \langle n'l'm'|R_1^q|nlm\rangle = \frac{\langle n'l'\|R_1\|nl\rangle}{\sqrt{2l+1}} \langle 1q, lm|l'm'\rangle$$

Remark:

If T_k^q is an irreducible tensor op. $\longrightarrow$ $T_k^{q\dagger}$ <u>is not</u>.

Proof.

$$D(R)\, T_k^q\, D^\dagger(R) = \sum_{q'} D^k_{q'q}\, T_k^{q'}$$

$$\left[D(R)\, T_k^q\, D^\dagger(R)\right]^\dagger = \sum_{q'} D^{k*}_{q'q}\, T_k^{q'\dagger}$$

$$\rightarrow D(R)\, T_k^{q\dagger}\, D^\dagger(R) = \sum_{q'} \underset{\sim\sim}{D^{k*}_{q'q}}\, T_k^{q'\dagger}$$

Ex.! $T_1^1 = -\frac{A_x + iA_y}{\sqrt{2}}$ $\quad T_1^0 = A_z$ $\quad T_1^{-1} = \frac{A_x - iA_y}{\sqrt{2}}$

$$T_1^{1\dagger} = -T_1^{-1} \qquad T_1^{0\dagger} = T_1^0 \qquad T_1^{-1\dagger} = -T_1^1$$

For example $\displaystyle T_1^{1\dagger} = -T_1^{-1} = -\sum_{q'} D^1_{q'q}\, T_1^{q'} \neq \sum_{q'} D^1_{q'q}\, T_1^{q'\dagger}$

Wigner-Eckart Theorem:

Alternative approach:

$$\langle j_2 m_2 | T_k^q | j_1 m_1 \rangle = \langle j_2 m_2 | D^\dagger(R) D(R)\, T_k^q\, D^\dagger(R) D(R) | j_1 m_1 \rangle$$

$$= \sum_{q'} \sum_{m_1'} \sum_{m_2'} \langle j_2 m_2 | D^\dagger(R) | j_2 m_2' \rangle \langle j_2 m_2' | T_k^{q'} | j_1 m_1' \rangle \langle j_1 m_1' | D(R) | j_1 m_1 \rangle \cdot D^{k}_{q'q}(R)$$

$$= \sum_{q'} \sum_{m_1'} \sum_{m_2'} \langle j_2 m_2' | T_k^{q'} | j_1 m_1' \rangle\, D^{j_2 *}_{m_2' m_2}(R) \cdot D^{j}_{m_1' m_1}(R) \cdot D^{k}_{q'q}(R)$$

Now we integrate both sides over Euler angles, and use the eqn.:

$$\int dR\, D^{j_1}_{m_1 m_1'}(R)\, D^{j_2}_{m_2 m_2'}(R)\, D^{j_3 *}_{m_3 m_3'}(R) = \frac{\langle j_1 m_1, j_2 m_2 | j_3 m_3 \rangle \langle j_1 m_1', j_2 m_2' | j_3 m_3' \rangle}{2 j_3 + 1}$$

Since $\int dR = \int_0^{4\pi} \frac{d\alpha}{4\pi} \int_0^{\pi} \frac{1}{2} \sin\beta\, d\beta \int_0^{4\pi} \frac{d\gamma}{4\pi} = 1$

normalization factors

Remark:

The parameter manifold of $O(3)$ in terms of Euler angles are $0 \le \alpha \le 2\pi$, $0 \le \beta \le \pi$, $0 \le \gamma \le 2\pi$ (1)

Because $\alpha = 0$ and $\alpha = 2\pi$ are the same point in the space of group elements of $O(3)$, the double-valued representations are discontinuous at those points. Hence (1) can not be the manifold for the universal covering group, because all its representations $D^{(j)}(a,b)$ are everywhere continuous and single valued.

Instead of (1) the $\underline{\text{correct}}$ manifold for the covering group $SU(2)$ is; $0 \le \alpha \le 4\pi$, $0 \le \beta \le \pi$, $0 \le \gamma \le 4\pi$ (2)

Remark: The manifold is actually covered completely if one $\underline{\text{only}}$ doubles the range of either α or γ. We use the more symmetric ranges in (2) for later convenience.

Thus; $\langle j_2 m_2 | T_k^q | j_1 m_1 \rangle (1) = \dfrac{\langle j_1 m_1 kq | j_2 m_2 \rangle}{2 j_2 + 1} \sum_{q' m_1' m_2'} \langle j_2 m_2' | T_k^{q'} | j_1 m_1' \rangle \cdot \langle j_1 m_1' k q' | j_2 m_2' \rangle$

$$\underbrace{\sum_{q' m_1 m_2'} \frac{1}{\sqrt{2j_2+1}} \langle j_2 m_2' | T_k^q | j_1 m_1' \rangle \langle j_1 m_1' kq | j_2 m_2' \rangle}_{\text{indep. of } q', m_1', m_2'} = \langle j_2 \| T_k \| j_1 \rangle$$

i.e.: indep. of any projection Q. number

$$\to \langle \alpha' j' m' | T_k^q | \alpha j m \rangle = \frac{\langle \alpha' j' \| T_k \| \alpha j \rangle}{\sqrt{2j'+1}} \langle j m kq | jk, j'm' \rangle$$

Wigner-Eckart Theorem;

Altenative approach;

$$T_{k_1}^{q_1} | j_2 m_2 \rangle = \sum_j \sum_m | jm \rangle \langle jm | q_1 m_2 \rangle = \sum_j \sum_m | jm \rangle \langle q_1 m_2 | jm \rangle$$

(by orthonormality of the Clebsch-G. coeffs.)

$$\langle j' m' | T_{k_1}^{q_1} | j_2 m_2 \rangle = \sum_j \sum_m \langle j' m' | jm \rangle \langle q_1 m_2 | jm \rangle$$

But $\langle j' m' | j m \rangle = \delta_{jj'} \delta_{mm'} N(j', k_1, j_2, T_{k_1})$

$$\to \langle j' m' | T_{k_1}^{q_1} | j m \rangle = N(j', k_1, j_2, T_{k_1}) \langle q_1 m_2 | j' m' \rangle$$

$$\to \langle j' m' | T_k^q | j m \rangle = \frac{\langle j' \| T_k \| j \rangle}{\sqrt{2j'+1}} \langle q m | j' m' \rangle$$

The matrix elements of Vector ops.:

$$J_1^1 = -\frac{J_x + iJ_y}{\sqrt{2}} \qquad J_1^0 = J_z \qquad J_1^{-1} = \frac{J_x - iJ_y}{\sqrt{2}}$$

By Wigner-Eckart theorem:

$$\langle \alpha' j j | J_1^0 | \alpha j j \rangle = \frac{\langle \alpha' j \| J_{(1)} \| \alpha j \rangle}{\sqrt{2j+1}} \langle j j | 1 0, j j \rangle$$

where we have chosen $j' = j$, $m' = j$ $k = 1$, $q = 0$

$$\langle \alpha' j j | \alpha j j \rangle \hbar j = \frac{\langle \alpha' j \| J_{(1)} \| \alpha j \rangle}{\sqrt{2j+1}} \sqrt{\frac{j}{j+1}}$$

$$\delta_{\alpha\alpha'} \hbar j = \langle \alpha' j \| J_{(1)} \| \alpha j \rangle \sqrt{\frac{j}{j+1}} \frac{1}{\sqrt{2j+1}}$$

$$\rightarrow \langle \alpha' j' \| J_{(1)} \| \alpha j \rangle = \sqrt{j(j+1)}\sqrt{(2j+1)}\, \hbar\, \delta_{jj'} \delta_{\alpha\alpha'} \quad (1)$$

Now;

$$J \cdot V = J_x V_x + J_y V_y + J_z V_z = \left(\frac{J_1^1 - J_1^{-1}}{\sqrt{2}}\right)\left(\frac{V_1^1 - V_1^{-1}}{\sqrt{2}}\right) + \left(\frac{J_1^1 + J_1^{-1}}{i\sqrt{2}}\right)\left(\frac{V_1^1 + V_1^{-1}}{i\sqrt{2}}\right) + J_1^0 V_1^0$$

$$= -\left(J_1^{-1} V_1^1 + J_1^1 V_1^{-1}\right) + J_1^0 V_1^0$$

$$= \sum_{q=-1}^{1} (-)^q J_1^{-q} V_1^q \qquad V: \text{Vector op.}$$

The projection theorem:

$$\langle \alpha' j m' | V_1^q | \alpha j m \rangle = \frac{\langle \alpha' j m | J \cdot V | \alpha j m \rangle}{\hbar^2 j(j+1)} \langle j m' | J_1^q | j m \rangle$$

Proof:

$$\langle \alpha' j m | J \cdot V | \alpha j m \rangle = \sum_q \langle \alpha' j m | (-)^q J_1^{-q} V_1^q | \alpha j m \rangle$$

$$= \sum_{\alpha''} \sum_{j''} \sum_{m''} \sum_q (-)^q \langle \alpha' j m | J_1^{-q} | \alpha'' j'' m'' \rangle \langle \alpha'' j'' m'' | V_1^q | \alpha j m \rangle$$

$$\langle \alpha' j m | J \cdot V | \alpha j m \rangle = \sum_{m''} \sum_{q} (-)^q \langle j m | J_1^{-q} | j m'' \rangle \langle j m, 1 q | j m'' \rangle \cdot \frac{\langle \alpha' j \| V_1 \| \alpha j \rangle}{\sqrt{2j+1}} = C_{jm} \langle \alpha' j \| V_1 \| \alpha j \rangle \quad (1)$$

C_{jm}: indep. of α, α' and V ($K=1$ for J and V).

Therefor we may evaluate it by substituting J for V

$$\langle \alpha, j, m | J \cdot J | \alpha, j, m \rangle = C_{jm} \langle \alpha j \| J_1 \| \alpha j \rangle$$

$\hbar^2 j(j+1) = C_{jm} \sqrt{j(j+1)} \sqrt{2j+1}\, \hbar$ (by Equ 1, P385) $\rightarrow C_{jm} = \sqrt{\frac{j(j+1)}{2j+1}}\, \hbar$

Remark: Since $J \cdot V$ is a scalar op. $\longrightarrow C_{jm}$ must be indep. of m; $\quad C_{jm} \longrightarrow C_j$

$$\langle \alpha, j m | J^2 | \alpha, j, m \rangle = C_j \langle \alpha j \| J_1 \| \alpha j \rangle \quad (2) \qquad \left(C_j = \sqrt{\frac{j(j+1)}{2j+1}}\, \hbar \right)$$

Then
$$\begin{cases} \langle \alpha' j m | J \cdot V | \alpha j m \rangle = C_j \langle \alpha' j \| V_1 \| \alpha j \rangle \\ \langle \alpha j m | J \cdot J | \alpha j m \rangle = C_j \langle \alpha j \| J_1 \| \alpha j \rangle \end{cases}$$

Remark: $J \cdot J$ and $J \cdot V$ are scalar ops., then we have nonvanishing matrix elements for $m = m'$, $j = j'$

$\longrightarrow$

$$\frac{\langle \alpha' j m | J \cdot V | \alpha j m \rangle}{\langle \alpha j m | J \cdot J | \alpha j m \rangle} = \frac{\langle \alpha' j \| V_1 \| \alpha j \rangle}{\langle \alpha j \| J_1 \| \alpha j \rangle} \quad (3)$$

By Wigner-Eckart theorem:

$$\langle \alpha' j m' | V_1^q | \alpha j m \rangle = \langle 1q, jm | jm' \rangle \frac{\langle \alpha' j \| V_{(1)} \| \alpha j \rangle}{\sqrt{2j+1}}$$

$$\langle \alpha j m' | J_1^q | \alpha j m \rangle = \langle 1q, jm | jm' \rangle \frac{\langle \alpha j \| J_{(1)} \| \alpha j \rangle}{\sqrt{2j+1}}$$

$$\rightarrow \langle \alpha' j m' | V_1^q | \alpha j m \rangle = \frac{\langle \alpha' j \| V_{(1)} \| \alpha j \rangle}{\langle \alpha j \| J_{(1)} \| \alpha j \rangle} \langle \alpha j m' | J_1^q | \alpha j m \rangle \qquad (4)$$

(3) in (4) $\rightarrow \langle \alpha' j m' | V_1^q | \alpha j m \rangle = \dfrac{\langle \alpha' j m | J \cdot V | \alpha j m \rangle}{\langle \alpha j m | J^2 | \alpha j m \rangle} \langle \alpha j m' | J_1^q | \alpha j m \rangle$

$$\rightarrow \langle \alpha' j m' | V_1^q | \alpha j m \rangle = \frac{\langle \alpha' j m | J \cdot V | \alpha j m \rangle}{\hbar^2 j(j+1)} \langle j m' | J_1^q | j m \rangle$$