Dr. M.H. Shahnas

Quantum Mechanics-I

Useful Books : Quantum Mechanics
Eugen Merzbacher
Modern Quantum Mechanics
J.J. Sakurai

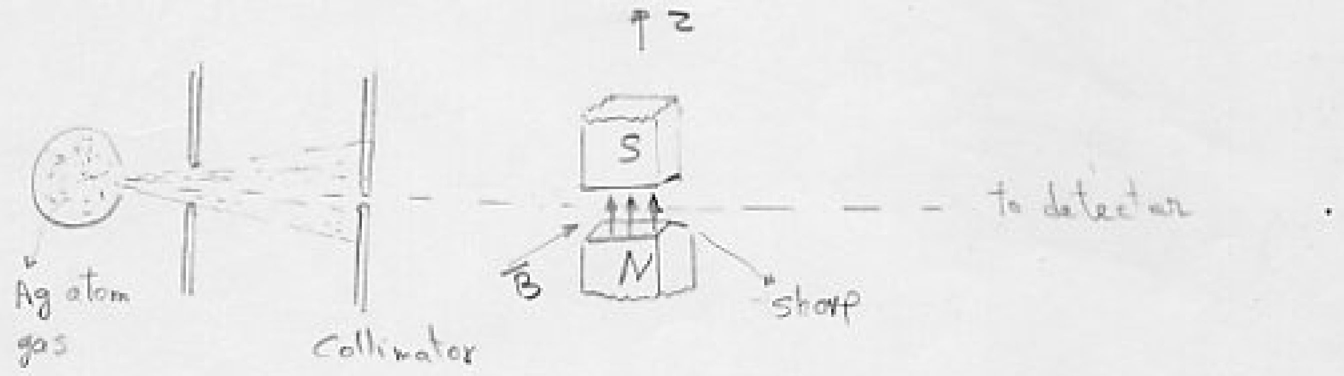
Chapter 1
Fundamental Concepts

1-1 The Stern-Gerlach Experiment

1921 by O. Stern

1922 " " " and W. Gerlach (the experiment)

In a certain sences a <u>two-state</u> system of the Stern-Gerlach type is the <u>least</u> classical, <u>most</u> quantum-mechanical system.



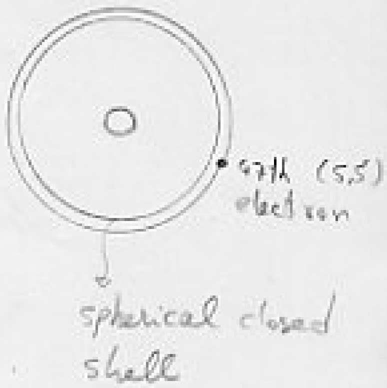
$\bar{B}$: inhomogeneous mag. field.

nucleus $\sim 2\times10^5$ electron mass

mag. moment $\sim \frac{1}{m}$

$\rightarrow \mu_{nucleus} \rightarrow$ small

$\mu_{atom} \approx \mu_{intrinsic}$ spin mag. moment of 47th electron



$\mu = \frac{e}{m_e c} S$ $\quad (e<0)$

$U = -\mu \cdot B$ $\quad$ potential $\quad F = -\nabla U$

$F_z = \frac{-\partial}{\partial z}(-\mu \cdot B) \simeq \mu_z \frac{\partial B_z}{\partial z}$ $\quad (B_x \approx B_y \simeq 0)$

Since the atom is heavy → classical trajectory can be applied.

Acc. to the Fig.: $\quad \mu_z > 0 \; (S_z < 0) \rightarrow F_z \downarrow$

$\mu_z < 0 \; (S_z > 0) \rightarrow F_z \uparrow$

→ Stern-Gerlach experiment <u>measures</u> μ_z (S_z).

If the electron is like a classical spinning object,

→ $-|\mu| \leq \mu_z \leq |\mu|$ (continuous)

→  classical expectation

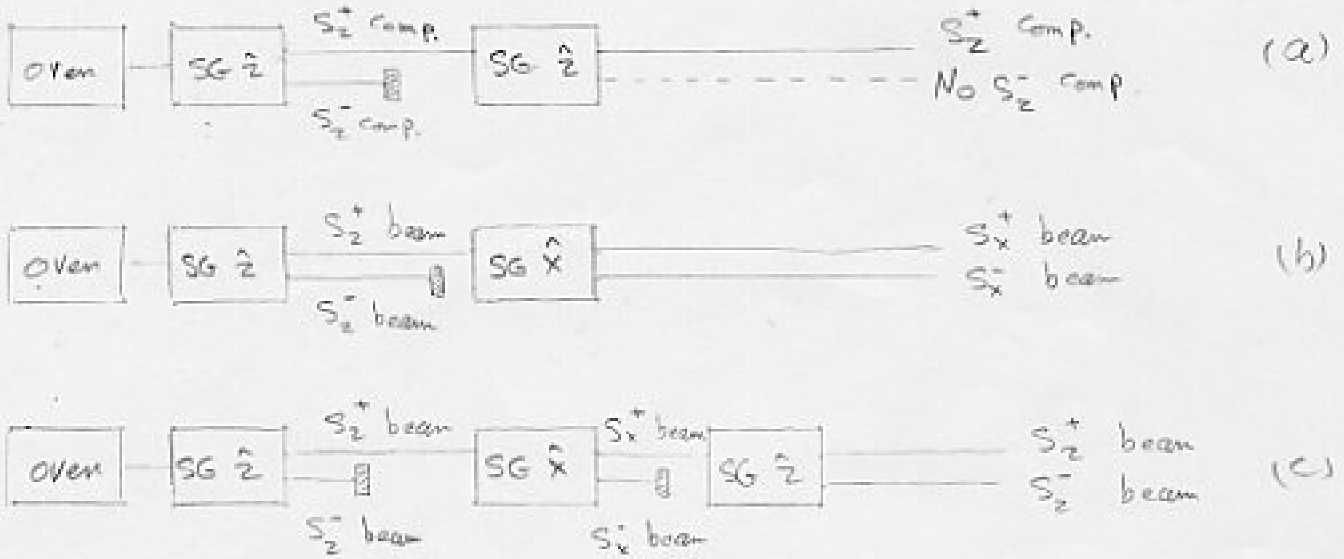
 quantum-mechanical expectation (observed)

→ $\bar{S}$ has two possible values along z-dir. $\begin{cases} S_z^+ \\ S_z^- \end{cases}$

$S_z = \hbar/2$ $\quad \hbar = 1.0546 \times 10^{-27}$ erg·s $= 6.5822 \times 10^{-16}$ eV·s

Of course there is nothing scared about the up-down dir. or the z-axis.

Sequential Stern-Gerlach Experiment:



$\rightarrow$ In quantum mechanics we cannot determine both S_x and S_z simultaneously.

More precisely;

selection of the S_x^+ by SG $\hat{x}$ completely destroys any previous information of S_z.

Analogy with Polarization of Light:
(classical situation)

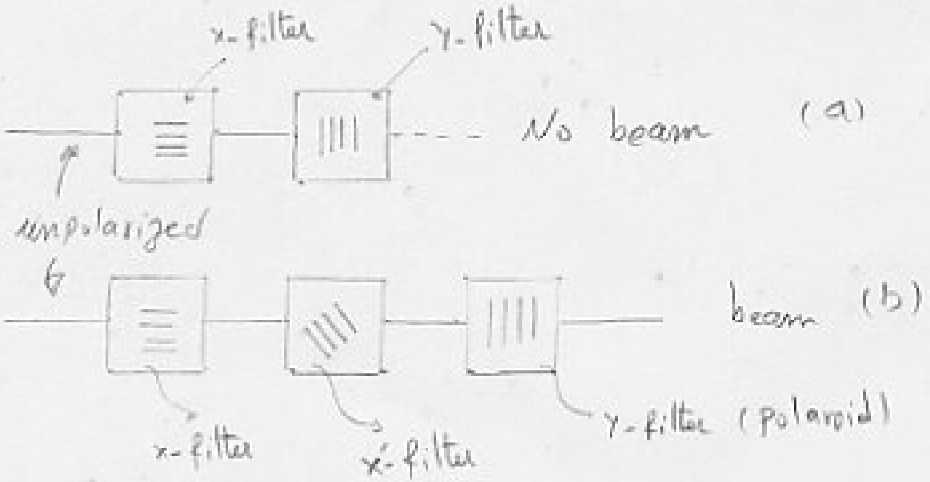
Consider a <u>linearly</u> (or plane polarized) light wave propagating in <u>z-dir.</u>

$E = E_0 \hat{x} \cos(kz - \omega t)$ x-polarized light (electric field)

or

$E = E_0 \hat{y} \cos(kz - \omega t)$ y-polarized light (")

(monochromatic)

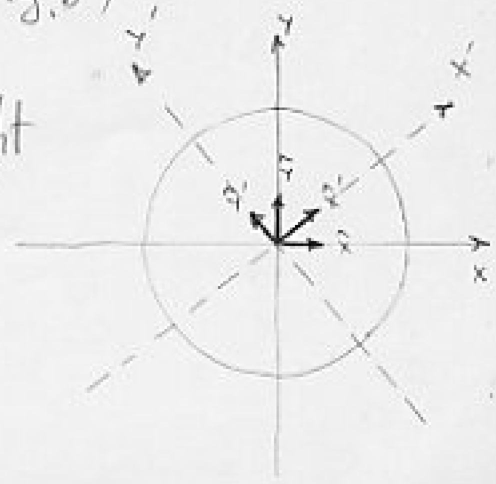


destroys any previous information

Correspondence with the SG experiment (Fig. b),

$S_z^{\pm}$ atoms $\longleftrightarrow$ x-, y polarized light

$S_x^{\pm}$ " $\longleftrightarrow$ x'-, y' " "



Acc. to classical electrodynamics :

$$\begin{cases} E_0 \hat{x}' \cos(kz-\omega t) = E_0 \left[\frac{1}{\sqrt{2}} \hat{x} \cos(kz-\omega t) + \frac{1}{\sqrt{2}} \hat{y} \cos(kz-\omega t) \right] \\ E_0 \hat{y}' \cos(kz-\omega t) = E_0 \left[-\frac{1}{\sqrt{2}} \hat{x} \cos(k-\omega t) + \frac{1}{\sqrt{2}} \hat{y} \cos(kz-\omega t) \right] \end{cases} \tag{1}$$

In Fig. b (P4);

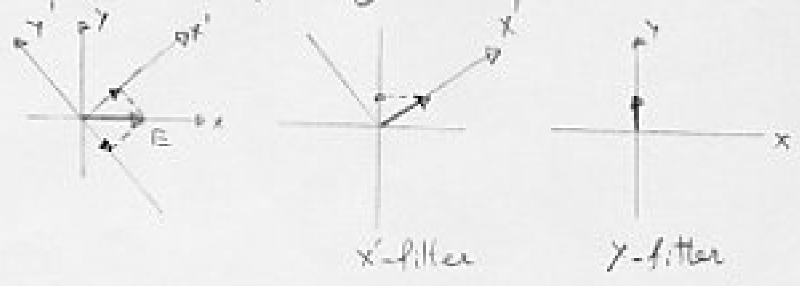
The beam comming out from <u>first</u> polaroid is $\hat{x}$-polarized beam,

$\longrightarrow$ which can be regarded as $E_0 \hat{x} \cos(kz-\omega t) = f(k,z,\omega,t)\,\hat{x}' + g(k,z,\omega t)\,\hat{y}'$

The <u>second</u> polaroid selects the x'-polarized beam;

$\rightarrow \quad E_0' \hat{x}' \cos(kz-\omega t) = e(k,z,\omega,t)\,\hat{x} + h(k,z,\omega t)\,\hat{y}$

The <u>third</u> polaroid selects the y-polarized component.



Applying Correspondence ;

SG, Fig c (P3) $\longrightarrow$ Polarization, Fig b (P4)

suggests $\longrightarrow$ spin state may be represented by some kind of <u>vector</u> in a <u>new</u> kind of two-dimentional vector space (abstract vector space) <u>not</u> to be confused with the usual two-dim. (xy) space.

Correspondence; $\rightarrow$ $\hat{x}, \hat{y} \longleftrightarrow$ $|S_z +\rangle$
$|S_z -\rangle$

In analogy, we may conjecture;

$$\begin{cases} |S_x;+\rangle \overset{?}{=} \frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}}|S_z;-\rangle \\ |S_x;-\rangle \overset{?}{=} -\frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}}|S_z;-\rangle \end{cases} \quad (2)$$

in analogy with (1).

What about $S_y^{\pm}$ states;

Consider a <u>right</u> circularly polarized light

$$E = E_0\left[\frac{1}{\sqrt{2}}\hat{x}\cos(kz-\omega t) + \frac{1}{\sqrt{2}}\hat{y}\cos(kz-\omega t+\frac{\pi}{2})\right]$$

More generally;

$$\mathcal{E} = E_0\left[\frac{1}{\sqrt{2}}\hat{x}e^{i(kz-\omega t)} + \frac{i}{\sqrt{2}}\hat{y}e^{i(kz-\omega t)}\right] \quad \text{(right)}$$

$$E = \mathrm{Re}(\mathcal{E})$$

Correspondence;

S_y^+ atom $\longleftrightarrow$ right circularly polarized beam

S_y^- " $\longleftrightarrow$ left " " "

$$|S_y;\pm\rangle \stackrel{?}{=} \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{i}{\sqrt{2}}|S_z;-\rangle$$

$\longrightarrow$ Two-dim. vector space must be complex.

Otherwise we cannot express $|S_y;\pm\rangle$ in terms of $|S_z;+\rangle$ and $|S_z;-\rangle$ (remember we have used available possibilities of real vector space in expressing $|S_x;\pm\rangle$).

Result: $(x,y,z) \longrightarrow (x,y,z,\sigma)$ dynamic variables

$\sigma = \pm 1$ spin dynamic variable

$\sigma = \pm \longrightarrow \mu_B = \mp \frac{\hbar}{2}\frac{e}{mc} \longrightarrow S_z = \pm\frac{\hbar}{2}$

Ex. $|\psi(x,y,z)|^2 dx\,dy\,dz \longrightarrow |\psi(x,y,z,\sigma)|^2 dx\,dy\,dz$

1.2 Kets, Bras and Operators

Ket space;

We consider a complex vector space whose dimensionality is specified by the nature of a physical system under consideration.

In SG experiment;

the only quantum mechanical deg. of freedom is spin of atom

$\longrightarrow$ two different paths $\xrightarrow[\text{related to}]{}$ S_z^+, S_z^- $\longrightarrow$ dimentionatity=2

Ex. – For Position (coordinate) or momentum, (continuous spectra),

dimensionality = nondenumerably infinite.

$\longrightarrow$ the vector space is a Hilbert space

In quantum mechanics a physical state, for example an atom with a definite spin orientation is represented by a <u>state vector</u> in a complex vector space.

Following Dirac, we call such a vector a <u>Ket</u>.

$|\alpha\rangle$ (Ket) contain all information about the physical state.

Addition; $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$ another ket

$c|\alpha\rangle = |\alpha\rangle c$ 〃 〃

If $c = 0$ $\rightarrow$ resulting Ket is <u>null</u> Ket.

One of Physics postulates;

$|\alpha\rangle$ and $c|\alpha\rangle$ (with $c \neq 0$) represent the same physical state.

In other words only the direction in vector space is of significance.

An observable, such as momentum and spin components, can be represented by an operator such as A in the vector space in question.

$$A \cdot (|\alpha\rangle) = A|\alpha\rangle \qquad \text{another ket}$$

In general $A|\alpha\rangle \neq c|\alpha\rangle$

However there are particular kets of importance, known as eigenkets of A, denoted by

$$|a'\rangle,\ |a''\rangle,\ |a'''\rangle, \cdots$$

with the property

$$A|a'\rangle = a'|a'\rangle,\quad A|a''\rangle = a''|a''\rangle, \cdots$$

The set $\{a'\} \equiv \{a', a'', a''', \cdots\}$ are eigenvalues.

The physical state corresponding to an eigenket is called an eigenstate.

Ex. $S_z|S_z;\pm\rangle = \pm\frac{\hbar}{2}|S_z;\pm\rangle$

$\searrow |\pm\hbar/2\rangle$

or $S_x|S_x;\pm\rangle = \pm\frac{\hbar}{2}|S_x;\pm\rangle$

In N-dimensional vector space;

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

$|\alpha\rangle$: arbitrary ket

$|a'\rangle$: eigenket of A

$c_{a'}$: complex coeff.

Ex: In two-dim. case

$$|\alpha\rangle = c_+|S_z;+\rangle + c_-|S_z;-\rangle$$

Bra Space and Inner Products

Concerned vector space : Ket space

bra space : a vector space dual to ket space

Postulate: $\forall\ |\alpha\rangle\ \exists\ \langle a'|$ in dual or bra space

The bra space is spanned by eigenbras $\{\langle a'|\}$

$$|\alpha\rangle \xleftrightarrow{\text{dual Correspondence}} \langle\alpha|$$

$$|a'\rangle, |a''\rangle, \dots \xleftrightarrow{DC} \langle a'|, \langle a''|, \dots$$

$$|\alpha\rangle + |\beta\rangle \xleftrightarrow{DC} \langle\alpha| + \langle\beta|$$

Roughly speaking: bra space : Some kind of mirror image of Ket space.

Postulate :

The dual of $c|\alpha\rangle$ is $c^*\langle\alpha|$

More generally: $c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \xleftrightarrow{DC} c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|$

Def. - The inner product:

$$\langle\beta|\alpha\rangle = (\underbrace{\langle\beta|}_{\text{bra}}) \cdot (\underbrace{|\alpha\rangle}_{\text{Ket}})$$ in general complex number

Postulate: $\langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^*$

Notice the difference between $a \cdot b$ and $\langle \alpha | \beta \rangle$;

$a \cdot b = b \cdot a$ (real vector space)

Conclusion: $\langle \alpha | \alpha \rangle$: real number

Postulate: $\langle \alpha | \alpha \rangle \geqslant 0$ $\begin{cases} \langle \alpha | \alpha \rangle > 0 & \text{if } |\alpha\rangle \neq \underset{\text{null}}{|0\rangle} \\ \langle \alpha | \alpha \rangle = 0 & \text{"} \ |\alpha\rangle = |0\rangle \end{cases}$

Positive definite metric

This postulate is <u>essential</u> for the probabilistic interpretation of quantum mechanics

<u>Orthogonality:</u>

$|\alpha\rangle$ and $|\beta\rangle$ are orthogonal if; $\langle \alpha | \beta \rangle = 0$

$\longrightarrow \langle \beta | \alpha \rangle = 0$ $\quad (|\alpha\rangle \neq |\beta\rangle)$

<u>Normalization:</u>

$$|\bar{\alpha}\rangle = \frac{1}{\sqrt{\langle \alpha | \alpha \rangle}} |\alpha\rangle \qquad (|\alpha\rangle \neq |0\rangle)$$

~~(for discrete spectra)~~

$\langle \bar{\alpha} | \bar{\alpha} \rangle = 1$ $\qquad \sqrt{\langle \alpha | \alpha \rangle}$: Norm of $|\alpha\rangle$

Operators:

Notation:

$X, Y, \ldots$ General class of op. (acting on kets)

$A, B, \ldots$ observables (" ")

$X \cdot (|\alpha\rangle) = X|\alpha\rangle$

Equality: Operators X and Y are said to be equal;

$X = Y$

if $X|\alpha\rangle = Y|\alpha\rangle \quad \forall |\alpha\rangle$

Def.: X is said to be the null op.

if $X|\alpha\rangle = 0 \quad \forall |\alpha\rangle$

Addition: $X + Y = Y + X$ commutative

$X + (Y + Z) = (X + Y) + Z$ associative

Def. - Operator X is said to be linear,

if $X(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$

Note: Time-reversal op. is excepted.

$$(\langle\alpha|)\cdot X = \langle\alpha|X$$

$$X|\alpha\rangle \overset{DC}{\not\longleftrightarrow} \langle\alpha|X \qquad \text{in general}$$

But $X|\alpha\rangle \overset{DC}{\longleftrightarrow} \langle\alpha|X^+$

X^+: Hermitian adjoint of X

Def. – An op. X is said to be Hermitian if

$$X = X^+$$

Def. – Hermitian adjoint op. is defined by the requirement:

$$\int f^* F^+ g \, d\tau = \int (Ff)^* g \, d\tau \qquad \text{or} \qquad \langle Ax, y\rangle = \langle x, A^+ y\rangle$$

Multiplication

$$XY \neq YX$$ noncommutative in general

However;

$$X(YZ) = (XY)Z = XYZ$$ associative

Also, $X(Y|\alpha\rangle) = (XY)|\alpha\rangle = XY|\alpha\rangle$

$$(\langle\beta|X)Y = \langle\beta|(XY) = \langle\beta|XY$$

Notice that $(XY)^{\dagger} = Y^{\dagger}X^{\dagger}$

because $XY|\alpha\rangle = X(Y|\alpha\rangle) \xleftrightarrow{DC} (\langle\alpha|Y^{\dagger})X^{\dagger} = \langle\alpha|Y^{\dagger}X^{\dagger}$

onth other hand

$$XY|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|(XY)^{\dagger}$$

Or;

$$\langle CDX, Y\rangle = \langle DX, C^{\dagger}Y\rangle = \langle X\ D^{\dagger}C^{\dagger}Y\rangle$$

onth other hand;

$$\langle CDX, Y\rangle = \langle X, (CD)^{\dagger}Y\rangle$$

Outer product;

$$(|\beta\rangle)\cdot(\langle\alpha|) = |\beta\rangle\langle\alpha| \qquad \text{an op.}$$

Illegal products;

$$|\alpha\rangle X \quad , \quad X\langle\alpha|$$

$|\alpha\rangle|\beta\rangle$, $\langle\alpha|\langle\beta|$ if they belong to the same vector space or the same particle

But; the product is <u>legal</u> if

1- They belong to different vector spaces

<u>Ex.</u> $\underset{\text{spin}}{|\alpha\rangle}\,\underset{\text{orbital angular momentum}}{|\beta\rangle}$ (more appropriately $|\alpha\rangle \otimes |\beta\rangle$)

2- They belong to different particles;

<u>Ex</u> $\underset{\text{spin of particle 1}}{|\alpha\rangle}\,\underset{\text{spin of particle 2}}{|\beta\rangle}$

The Associative Axiom;

Associative property is <u>postulated</u> to hold quite generally among <u>kets</u>, <u>bras</u> and <u>operators</u> (in legal multiplications).

Dirac calls this postulate, the <u>associative axiom of multiplication</u> (AAM).

<u>Ex.</u>

$$(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle \overset{\text{AAM}}{=\!=\!=} |\beta\rangle\cdot(\underbrace{\langle\alpha|\gamma\rangle}_{\text{c-number}}) = c|\beta\rangle$$

$\xrightarrow{\text{Conclusion}}$ $|\beta\rangle\langle\alpha|$ is an op.

Notation;

$$(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle = \langle\alpha|\gamma\rangle|\beta\rangle$$

The op. $|\beta\rangle\langle\alpha|$ rotates $|\gamma\rangle$ to into the dir of $|\beta\rangle$

It is easy to see that if

$$X = |\beta\rangle\langle\alpha| \longrightarrow X^{\dagger} = |\alpha\rangle\langle\beta|$$

Because $(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = \langle\alpha|\gamma\rangle|\beta\rangle = c|\beta\rangle$ $\qquad c|\beta\rangle \overset{DC}{\longleftrightarrow} c^{*}\langle\beta|$

$\rightarrow c^{*}\langle\beta| = \underbrace{\langle\gamma|\alpha\rangle\langle\beta|}_{} = \langle\gamma|X^{\dagger}$

Ex. $\quad \underbrace{(\langle\beta|)}_{\text{bra}} \cdot \underbrace{(X|\alpha\rangle)}_{\text{ket}} \;\underline{\underline{\text{AAM}}}\; \underbrace{(\langle\beta|X)}_{\text{bra}} \cdot \underbrace{(|\alpha\rangle)}_{\text{ket}}$

Since they are equal, we might use more compact notation,

$$= \langle\beta|X|\alpha\rangle$$

Since $\quad X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^{\dagger}$

$$\rightarrow \langle\beta|X|\alpha\rangle = \langle\beta|\,(X|\alpha\rangle) = \left\{(\langle\alpha|X^{\dagger})\cdot|\beta\rangle\right\}^{*}$$

$$= \langle\alpha|X^{\dagger}|\beta\rangle^{*}$$

For a Hermitian X;

$$\langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^{*}$$

or;

$$\overbrace{\underbrace{\int (F\psi_1)^{*}\psi_2\,d\tau = \int \psi_1^{*}F^{\dagger}}_{\text{Hermitian adjoint requirement (def.)}}\psi_2\,d\tau = \int}^{F \text{ being Hermitian (def.)}} \psi_1^{*}F\psi_2\,d\tau$$

1.3 Base Kets and Matrix Representation:

Eigenkets of an Observable:

Theorem: The eigenvalues of a Hermitian op. A are real. The eigenkets of A corresponding to different eigenvalues are orthogonal.

Proof: $$A|a'\rangle = a'|a'\rangle \qquad (1)$$

$$A|a''\rangle = a''|a''\rangle$$

$$A|a''\rangle \xleftrightarrow{DC} \langle a''|A^{+} = \langle a''|A \qquad \text{(because } A \text{ is Hermitian)}$$

$$a''|a''\rangle \longleftrightarrow a''^{*}\langle a''|$$

$$\longrightarrow \langle a''|A = a''^{*}\langle a''| \qquad (2)$$

$$(1) \rightarrow \quad \langle a''|A|a'\rangle = a'\langle a''|a'\rangle.$$

$$(2) \rightarrow \quad \langle a''|A|a'\rangle = a''^{*}\langle a''|a'\rangle$$

$$0 = (a' - a''^{*})\langle a''|a'\rangle$$

if $a' = a''$, since $\langle a'|a'\rangle \neq 0 \longrightarrow (a' - a'^{*}) = 0$

$\longrightarrow a' = a'^{*}$ (we have used the fact that $|a'\rangle \neq$ null)

Now if $a' \neq a''$

$$a' - a''^* = a' - a'' \neq 0$$ using the first part of the Theorem.

$$\longrightarrow \langle a'' | a' \rangle = 0$$

On physical grounds: An observable has real eigenvalues

$\longrightarrow$ This is the reason that we talk about Hermitian observables in quantum mechanics

Orthonormality: $\langle a' | a'' \rangle = \delta_{a'a''}$

Eigenkets as Base kets:

Fundamental Expansion Postulate:

Every physical quantity can be represented by a Hermitian op. A with eigenkets $|a_1\rangle, |a_2\rangle \dots |a_n\rangle \dots$ and every physical state by sum

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle \qquad (1)$$

Where

$$c_{a'} = \langle a' | \alpha \rangle \qquad \text{(by orthonormality)}$$

(2)

Complete set:

Acc. to the mentioned postulate, every physical op. has a sufficient number of eigenfunctions to represent an arbitrary state.

Such a set of orthonormal eigenfuncs. is said to be complete. (complete set by construction).

Now; (1)(2) $\rightarrow$ $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$ (3)

$\langle a'|\alpha\rangle$: Projection of $|\alpha\rangle$ along $|a'\rangle$

(3) is valid $\forall\, |\alpha\rangle$ $\rightarrow$ $\sum_{a'} |a'\rangle\langle a'| = I$

completeness relation or closure

Application:

Ex. If $|\alpha\rangle$ is normalized, i.e. $\langle\alpha|\alpha\rangle = 1$

$\rightarrow \langle\alpha|\alpha\rangle = \langle\alpha| \cdot \left(\sum_{a'} |a'\rangle\langle a'|\right) \cdot |\alpha\rangle = \sum_{a'} |\langle a'|\alpha\rangle|^2$

$= \sum_{a'} |C_{a'}|^2 = 1$

Projection op.:

$$(|a'\rangle\langle a'|)\cdot|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle = c_{a'}|a'\rangle$$

$c_{a'}|a'\rangle$: portion of $|\alpha\rangle$, parallel to $|a'\rangle$

$$\rightarrow \quad \Lambda_{a'} \equiv |a'\rangle\langle a'| \qquad \text{Projection op.}$$

$$\rightarrow \quad \sum_{a'} \Lambda_{a'} = I$$

Matrix Representation;

$$\sum_{a'} |a'\rangle\langle a'| = I \quad \rightarrow \quad X = \sum_{a''}\sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'|$$

The number of $\langle a''|X|a'\rangle$: N^2

N: dimensionality of the ket space.

$$X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots \\ \vdots & & \ddots \end{pmatrix}$$

(is represented by)

Now $\quad \langle a''|X|a'\rangle = \langle a'|X^{\dagger}|a''\rangle^{*} \qquad (1)$

Since $X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^\dagger$ (2) Hermitian adjoint operation

(1)(2) $\longrightarrow$ Hermitian adjoint operation is related to the complex conjugate transposed.

For a Hermitian op.: $\langle a''|B|a'\rangle = \langle a'|B|a''\rangle^*$

(Notice: $B \neq A$ in general)

The way we arranged $\langle a''|X|a'\rangle$ into a square matrix is in conformity with the **usual** matrix multiplication.

$$Z = XY$$

$$\langle a''|Z|a'\rangle = \sum_{a'''} \langle a''|X|a'''\rangle\langle a'''|Y|a'\rangle \qquad (\quad) = (\quad)(\quad)$$

Ket relation $|\gamma\rangle = X|\alpha\rangle$ in terms of baskets:

$$\langle a'|\gamma\rangle = \langle a'|X|\alpha\rangle = \sum_{a''} \langle a'|X|a''\rangle\langle a''|\alpha\rangle$$

$$\longrightarrow |\gamma\rangle = X|\alpha\rangle$$

where

$$|\gamma\rangle \doteq \begin{pmatrix} \langle a^{(1)}|\gamma\rangle \\ \langle a^{(2)}|\gamma\rangle \\ \vdots \\ \vdots \end{pmatrix} \qquad X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \cdots \\ \vdots & \end{pmatrix} \qquad |\alpha\rangle \doteq \begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \vdots \end{pmatrix}$$

Likewise; given

$$\langle\gamma| = \langle\alpha| X$$

$$\rightarrow \langle\gamma|a'\rangle = \sum_{a''} \langle\alpha|a''\rangle\langle a''|X|a'\rangle$$

$$\rightarrow \langle\gamma| \doteq \left(\langle\gamma|a^{(1)}\rangle, \langle\gamma|a^{(2)}\rangle, \dots\right)$$

$$= \left(\langle a^{(1)}|\gamma\rangle^*, \langle a^{(2)}|\gamma\rangle^*, \dots\right)$$

The inner product:

$$\langle\beta|\alpha\rangle = ?$$

$$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle = \left(\langle a^{(1)}|\beta\rangle^*, \langle a^{(2)}|\beta\rangle^*, \dots\right)\begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \end{pmatrix}$$

$$\langle\alpha|\beta\rangle = \sum_{a'} \langle\alpha|a'\rangle\langle a'|\beta\rangle = \left(\langle a^{(1)}|\alpha\rangle^*, \langle a^{(2)}|\beta\rangle^*, \dots\right)\begin{pmatrix} \langle a^{(1)}|\beta\rangle \\ \langle a^{(2)}|\beta\rangle \\ \vdots \end{pmatrix}$$

We see $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$

which is consistant with the fundamental property of the inner product.

Finally for the outer product:

$$|\beta\rangle\langle\alpha| \doteq \begin{pmatrix} \langle a^{(1)}|\beta\rangle\langle a^{(1)}|\alpha\rangle^* & \langle a^{(1)}|\beta\rangle\langle a^{(2)}|\alpha\rangle^* & \dots \\ \langle a^{(2)}|\beta\rangle\langle a^{(1)}|\alpha\rangle^* & \langle a^{(2)}|\beta\rangle\langle a^{(2)}|\alpha\rangle^* & \dots \\ \vdots & \vdots & \end{pmatrix}$$

Using the eigenkets of an observable A:

$$A = \sum_{a''} \sum_{a'} |a''\rangle \langle a''|A|a'\rangle \langle a'|$$

But square matrix $\langle a''|A|a'\rangle$ is diagonal

$$\langle a''|A|a'\rangle = \langle a'|A|a'\rangle \delta_{a'a''} = a' \delta_{a'a''}$$

$$\rightarrow \quad A = \sum_{a'} a' |a'\rangle \langle a'| = \sum_{a'} a' \Lambda_{a'}$$

Spin $\frac{1}{2}$ system:

The base kets: $|S_z, \pm\rangle$ or briefly $|\pm\rangle$

The identity op.:

$$I = \sum_{a'} |a'\rangle\langle a'| \longrightarrow I = |+\rangle\langle +| + |-\rangle\langle -|$$

Acc. to; $A = \sum_{a'} a' |a'\rangle\langle a'|$

$$\longrightarrow S_z = \frac{\hbar}{2}\left[(|+\rangle\langle +|) - (|-\rangle\langle -|)\right]$$

A check:

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

We define two non-Hermitian ops.:

$$S_+ \equiv \hbar|+\rangle\langle -| \quad , \quad S_- \equiv \hbar|-\rangle\langle +|$$

where $S_+|-\rangle = \hbar|+\rangle \qquad S_+|+\rangle = 0$

$S_-|+\rangle = \hbar|-\rangle \qquad S_-|-\rangle = 0$

Conclusion: S_+, raises spin component by one unit of $\hbar$.

If the spin component can not be raised any further, we get a null state.

We will show later $S_\pm = S_x \pm i S_y$

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S_+ \doteq \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- \doteq \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In constructing the matrix representation of the angular momenta ops., it is customary to label the column (row) indices in descending order.

1-4 Measurements, Observables, and the Uncertainty Relations

Measurements; (quantum theory):

P.A.M. Dirac: A measurement always causes the system jumps into an eigenstate of the dynamical variable that is being measured.

Before a measurement of observable A,

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \quad \text{(the system assumed to be)}$$

When a measurement is performed;

$$|\alpha\rangle \xrightarrow[\text{(the system is thrown into)}]{\text{A measurement}} |a'\rangle \qquad \text{one of the eigenstates of } A$$

Ex.

A silver atom with an arbitrary spin orientation will change into $|S_z;+\rangle$ or $|S_z;-\rangle$ when subjected to $SG\hat{z}$.

Thus:

A measurement usually changes the state.

The only exception:

If the system is already in one of the eigenstates of the observable being measured;

$$|a'\rangle \xrightarrow[\text{(no change)}]{\text{A measurement}} |a'\rangle$$

When a measurement causes, $|\alpha\rangle \longrightarrow |a'\rangle$, it is said that A is measured to be a'.

Point: We don't know in advance into which of the various $|a'\rangle$'s the system will be thrown as the result of A measurement.

However, we postulate;

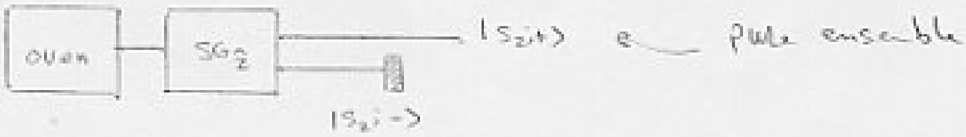
$$\text{Probability for jumping into } |a'\rangle = |\langle a'|\alpha\rangle|^2 \qquad (1)$$

where $|\alpha\rangle$: normalized

Although we have been talking about a single physical system, to determine probability (1) empirically, we must consider a great number of measurements performed on an ensemble (That is, a collection of identically prepared physical systems, all characterized by the same ket $|\alpha\rangle$.).

Such an ensemble is known as a pure ensemble.

Ex.



Ex.

system before the measurement: $|a'\rangle$

For A measurement, acc. to (1) $\quad (|a'\rangle \xrightarrow{A} |a'\rangle)$

$$P = |\langle a'|a'\rangle|^2 = 1$$

and for $a'' \neq a'$ $\quad (|a'\rangle \xrightarrow{A} |a''\rangle)$

$$P = |\langle a''|a'\rangle|^2 = 0$$

Remark:

$$P_{a'} \geq 0 \qquad \sum_{a'} P_{a'} = 1$$

$$(1) \rightarrow \sum_{a'} |\langle a'|\alpha\rangle|^2 = \sum_{a'} \langle\alpha|a'\rangle\langle a'|\alpha\rangle = \langle\alpha|\alpha\rangle = 1$$

Def. - Expectation Value

$$\langle A\rangle_\alpha \equiv \langle\alpha|A|\alpha\rangle \qquad \text{(average measured value)}$$

$$\langle A\rangle = \sum_{a'}\sum_{a''} \langle\alpha|a''\rangle\langle a''|A|a'\rangle\langle a'|\alpha\rangle$$

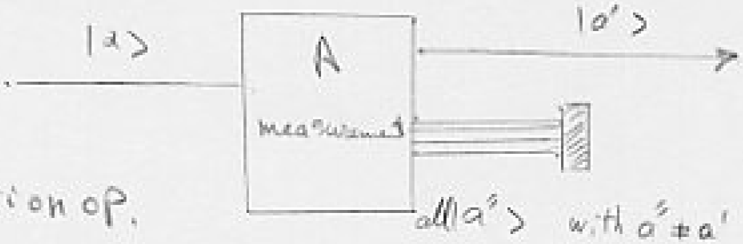
$$= \sum_{a'}\sum_{a''} a'\delta_{a'a''} \langle\alpha|a''\rangle\langle a'|\alpha\rangle$$

$$= \sum_{a'} \underbrace{a'}_{\text{measured value } a'} \underbrace{|\langle a'|\alpha\rangle|^2}_{\text{probability}}$$

Selective Measurement (filtration):

Mathematically;

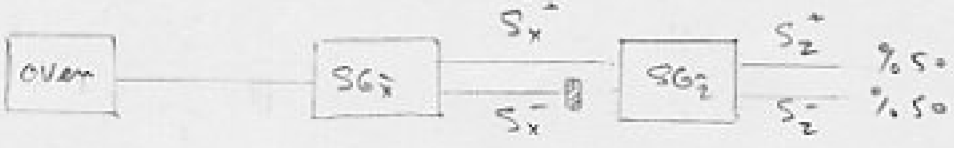
such a selective measurement amounts to applying the projection op. $\Lambda_{a'}$ to $|\alpha\rangle$



$$\Lambda_{a'}|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle$$

Spin $\frac{1}{2}$ System (once again):

We use: $\begin{cases} 1- \text{sequential SG experiments} \\ 2- \text{Postulates of Q.M.} \end{cases}$



$$|\langle a'|\alpha\rangle| \sim |\langle S_z;+|S_x;+\rangle| = \frac{1}{\sqrt{2}}$$

$$|\langle a''|\alpha\rangle| \sim |\langle S_z;-|S_x;+\rangle| = \frac{1}{\sqrt{2}}$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \sim |S_x;+\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}} e^{i\delta_1}|S_z;-\rangle$$

δ_1: real

The overall phase (common to both $|S_z;+\rangle$ and $|S_z;-\rangle$) is <u>immaterial</u>

By convention: the coeff of $|S_z;+\rangle$; <u>real</u> and <u>positive</u>

Also;

$$\langle S_x;-|S_x;+\rangle \overset{\text{must}}{=\!=} 0 \qquad \text{(orthogonality)} \quad (1)$$

Because $|S_x;+\rangle$ alternative and $|S_x;-\rangle$ alternative are <u>mutually</u> exclusive.

$$(1) \rightarrow |S_x;-\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle - \frac{1}{\sqrt{2}} e^{i\delta_1}|S_z;-\rangle \qquad (2)$$

When we have, again, chosen the coeff of $|S_z;+\rangle$ to be: real and Positive (by convention)

Now $S_x = ?$

Using $A = \sum_{a'} a' |a'\rangle\langle a'|$

$$\rightarrow S_x = \frac{\hbar}{2}\left[\left(|S_x;+\rangle\langle S_x;+|\right) - \left(|S_x;-\rangle\langle S_x;-|\right)\right]$$

$$= \frac{\hbar}{2}\left[e^{-i\delta_1}|S_z;+\rangle\langle S_z;-| + e^{i\delta_1}|S_z;-\rangle\langle S_z;+|\right] \qquad (3)$$

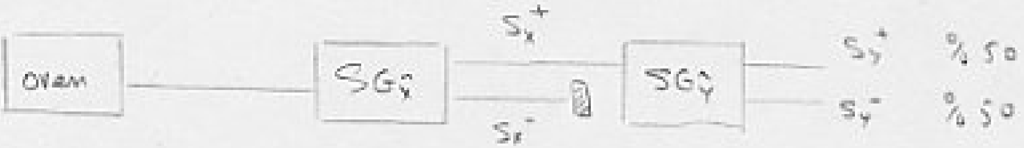
Note that S_x is Hermitian just as it must be.

By a similar argument:

$$|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{1}{\sqrt{2}} e^{i\delta_2}|S_z;-\rangle \qquad (4)$$

$$S_y = \frac{\hbar}{2}\left[e^{-i\delta_2}|S_z;+\rangle\langle S_z;-| + e^{i\delta_2}|S_z;-\rangle\langle S_z;+|\right] \qquad (5)$$

$\delta_1 = ?$ $\delta_2 = ?$



$$|\langle a'|\alpha\rangle| \sim |\langle S_y;\pm|S_x;+\rangle| = \frac{1}{\sqrt{2}} \qquad (6)$$

$$|\langle a'|\alpha\rangle| \sim |\langle S_y;\pm|S_x;-\rangle| = \frac{1}{\sqrt{2}} \qquad (7)$$

which __is not__ surprizing in view of the __invariance__ of physical systems under __rotations__.

$(2)\,(4)$ in $(6)\,(7)$ $\longrightarrow \frac{1}{2}\left|1 \pm e^{i(\delta_1-\delta_2)}\right| = \frac{1}{\sqrt{2}}$

$\longrightarrow \delta_1 - \delta_2 = \frac{\pi}{2}, \; -\frac{\pi}{2}$

$\longrightarrow$ The matrix elements of $S_x \doteq (\;\;)$ and $S_y \doteq (\;\;)$ __can not__ all be __real__.

If S_x : real $\longrightarrow$ S_y : purely imaginary (and vice versa).

Once again we see that the introduction of complex-numbers in Q.M. is __essential__.

It is convenient to take the S_x matrix elements = real

$\rightarrow \delta_1 = 0$

Note: if we take $\delta_1 = \pi$ $\rightarrow$ the positive x-axis would be oriented in the opposite dir.

with $\delta_1 = 0$ $\longrightarrow$ $\delta_2 = \pm \frac{\pi}{2}$

Given x- and z-axes, we have two alternative in choosing y-dir

It can be shown $\delta_2 = \frac{\pi}{2}$ corresponds to right hand coord.-sys.

$$\rightarrow \quad |S_x, \pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$$
$$|S_y, \pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right) \qquad (8)$$

and
$$S_x = \frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right]$$
$$S_y = \frac{\hbar}{2}\left[-i\left(|+\rangle\langle-|\right) + i\left(|-\rangle\langle+|\right)\right] \qquad (9)$$

Only the relative phase between $|+\rangle$ and $|-\rangle$ in (8) is of physical significance.

(9) $\rightarrow$ $S_{\pm} = S_x \pm i S_y$ (10)

It can be shown:

$[S_i, S_j] = i \epsilon_{ijk} \hbar S_k$ (11) (Simplest generalization of the Ang. Momentum commutation relations)

$\{S_i, S_j\} = \frac{1}{2} \hbar^2 \delta_{ij}$ (12) (special property of spin $\frac{1}{2}$ systems)

$S \cdot S = S^2 = S_x^2 + S_y^2 + S_z^2$

(12) $\rightarrow$ $S^2 = \left(\frac{3}{4}\right) \hbar^2 I$ (for spin $\frac{1}{2}$)

Obviously $[S^2, S_i] = 0$ (valid also for spins $> \frac{1}{2}$)

Compatible Observable:

$$[A, B] = 0 \qquad \text{compatible}$$

$$[A, B] \neq 0 \qquad \text{incompatible}$$

Ex. $[S^2, S_z] = 0$ compatible

$[S_x, S_z] \neq 0$ incompatible

First we consider compatible case $[A, B] = 0$

Assumption: The ket space is spanned by $\{|a'\rangle\}$.

We may regard also the same ket space " " $\{|b'\rangle\}$.

$$|a'\rangle \xleftrightarrow[\text{relation}]{?} |b'\rangle$$

Before answering to this question we state some difficulties in the case where there are degeneracies:

1- If there is degeneracy;

$|a'\rangle$ notation is not sufficient to give a complete description.

2- In the theorem on orthogonality of different eigenkets we assumed there is no degeneracy.

3 - The whole concept that Ket space is spanned by $\{|a'\rangle\}$ appears to run into difficulty when:

$$\text{dim. of Ket space} > \text{number of distinct eigenvalues of } A$$

Fortunately, in practical applications in Q.M., it is usually the case that in such a situation the eigenvalues of some other commuting observable, say B can be used to label the degenerate eigenkets.

Theorem: Suppose $\begin{cases} [A,B]=0 \\ \text{eigenvalues of } A\text{: nondegenerate} \end{cases}$

$\longrightarrow$ $\langle a''|B|a'\rangle$ are all diagonal.

Proof: $\langle a''|[A,B]|a'\rangle = (a''-a')\langle a''|B|a'\rangle = 0$

(remember A and B are Hermitian)

$\langle a''|B|a'\rangle = 0$ unless $a''=a'$

$$\langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a'\rangle$$

$\rightarrow\ A \doteq \begin{pmatrix} \text{diagonal} \end{pmatrix} \qquad B \doteq \begin{pmatrix} \text{diagonal} \end{pmatrix}$

(with the same base kets $\{|a'\rangle\}$)

Now;

$$B = \sum_{a''}\sum_{a'} |a''\rangle\langle a''|B|a'\rangle\langle a'|$$

$$= \sum_{a''}\sum_{a'} |a''\rangle\langle a'|B|a'\rangle\langle a'|\ \delta_{a'a''} = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$$

$$B|a'\rangle = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|a'\rangle = (\langle a'|B|a'\rangle)\,|a'\rangle$$

$\rightarrow\ b' \equiv \langle a'|B|a'\rangle$ eigenvalue

$\rightarrow\ |a'\rangle$ is simultaneous eigenket of A and B.

$\xrightarrow[\text{notation}]{}\ |a', b'\rangle$

We have seen if $[A,B]=0 \rightarrow A$ and B have simultaneous eigenkets.

Even though we have proved it for nondegenerate case, it is true if there is an n-fold degeneracy; that is,

$$A|a'^{(i)}\rangle = a'|a'^{(i)}\rangle \qquad \text{for } i=1,2,\cdots n$$

where $|a'^{(i)}\rangle$: n-mutually orthonormal eigenkets of A, all with the same eigenvalue a'.

To see this, all we need to do, is to construct appropriate linear combination of $|a'^{(i)}\rangle$ that diagonalize the B op. -

i.e. $c_1|a'^{(1)}\rangle + c_2|a'^{(2)}\rangle + \cdots + c_n|a'^{(n)}\rangle$?

(see p 42')

Now, $A|a',b'\rangle = a'|a',b'\rangle$

$B|a',b'\rangle = b'|a',b'\rangle$

Remark: When there is no degeneracy the notation $|a',b'\rangle$ is somewhat superfluous because it is clear fr. $b' = \langle a'|B|a'\rangle$, if we specify a', we necessarily know b'.

The notation $|a',b'\rangle$ is much <u>powerful</u> when there are <u>degeneracies</u>.

<u>Ex.</u>

Eigenvalue of L^2 : $l(l+1)\hbar^2$

" " L_z : $m_l\hbar$ $\quad (m_l = -l, \cdots +l)$

L^2 and L_z are compatible ; $[L^2, L_z] = 0$

An orbital ang. momentum state is completely characterized by <u>both</u> l and <u>m_l</u> .

the state $= |l, m_l\rangle$

Because if $\quad$ the state $= |l=1\rangle$

m_l can take $-1, 0, +1$

and if $\quad$ the state $= |m_l=1\rangle$

l can take $1, 2, 3, 4, \ldots$

Notation: We may indicate all the required parameters with a collective index K';

$$|k'\rangle = |a', b'\rangle$$

Generalization:

$$[A,B] = [B,C] = [A,C] = \cdots = 0 \quad \text{(mutually)} \qquad (1)$$

Assume the commuting observable set $\{A, B, C, \ldots\}$ is maximal, (i.e. we cannot add any more observables to this set without violating (1).

We may have degeneracies for some of these observables, but if we specify a combination $(a', b', c', \ldots)$, then the corresponding simultaneous eigenket of $A, B, C, \ldots$ is uniquely specified.

Again;

$$|k'\rangle = |a', b', c', \dots\rangle$$

$$\langle k''|k'\rangle = \delta_{k'k''} = \delta_{a'a''}\,\delta_{b'b''}\,\delta_{c'c''}\cdots$$

$$\sum_{k'} |k'\rangle\langle k'| = \sum_{a'}\sum_{b'}\sum_{c'} |a', b', c', \dots\rangle\langle a', b', c', \dots| = I$$

A and B Measurements; (for compatible case).

First; A-measurement $\longrightarrow a'$

Second; B- " $\longrightarrow b'$

Third; A- " $\longrightarrow a'$

i.e. B-measurement does <u>not</u> destroy the previous information

This is obvious when there is <u>no</u> degeneracy for <u>A</u>

$$|\alpha\rangle \xrightarrow{A} |a', b'\rangle \xrightarrow{B} |a', b'\rangle \xrightarrow{A} |a', b'\rangle$$

In the case of degeneracy for A;

$$|\alpha\rangle \xrightarrow{A} \sum_{i=1}^{n} c_{a'}^{(i)} |a', b'^{(i)}\rangle$$

{ $|a'\rangle$ one of the degenerate states

n: deg of degeneracy

$$\sum_{i=1}^{n} C_{a'}^{(i)} |a', b'^{(i)}\rangle \xrightarrow{B} |a', b'^{(j)}\rangle$$

$$|a', b'^{(j)}\rangle \xrightarrow{A} |a', b'^{(j)}\rangle \quad \text{(with } a' \text{ eigenvalue)}$$

Incompatible Observables:

The incompatible ops. $[A,B] \neq 0$ do not have a complete set of simultaneous eigenkets.

Proof: To show this let us assume the converse to be true.

Then;
$$AB|a',b'\rangle = Ab'|a',b'\rangle = a'b'|a',b'\rangle$$
$$BA|a',b'\rangle = Ba'|a',b'\rangle = a'b'|a',b'\rangle$$

$$\rightarrow AB|a',b'\rangle = BA|a',b'\rangle \quad \rightarrow [A,B] = 0$$

in contradiction to the assumption.

So in general, $|a',b'\rangle$ does not make sense for incompatible observables.

If $[A,B]=0$

$$AB|a'\rangle = BA|a'\rangle = a'B|a'\rangle \rightarrow A(B|a'\rangle) = a'(B|a'\rangle) \quad (1)$$

If $|a'\rangle$'s are non-degenerate, $\rightarrow$ $a' \xleftrightarrow[\text{only}]{\text{Corresponds}} |a'\rangle \quad (2)$

(1)(2) $\rightarrow$ $B|a'\rangle \sim |a'\rangle$

$\rightarrow B|a'\rangle = b'|a'\rangle$

$\rightarrow |a'\rangle$: simultaneous eigenket of A and B

Now consider two-fold degeneracy;

i.e. $A|a'^{(1)}\rangle = a'|a'^{(1)}\rangle$

$A|a'^{(2)}\rangle = a'|a'^{(2)}\rangle$

we can only assert $\rightarrow$ $\begin{cases} B|a'^{(1)}\rangle = b_{11}|a'^{(1)}\rangle + b_{12}|a'^{(2)}\rangle \\ B|a'^{(2)}\rangle = b_{21}|a'^{(1)}\rangle + b_{22}|a'^{(2)}\rangle \end{cases}$

Acc. the Dirac's explanation about the measurement of $[A,B]=0$ observables.

However we can take a linear combination of $|a'^{(1)}\rangle$ and $|a'^{(2)}\rangle$ such that:

$B|a'(\mathrm{I})\rangle = b_1|a'(\mathrm{I})\rangle$

$B|a'(\mathrm{II})\rangle = b_2|a'(\mathrm{II})\rangle$

For example, $B(|a'^{(1)}\rangle + \lambda|a'^{(2)}\rangle) = (b_{11}+\lambda b_{21})|a'^{(1)}\rangle + (b_{12}+\lambda b_{22})|a'^{(2)}\rangle \quad (3)$

On the other hand $\rightarrow$ $\quad // \quad = b_{1,2}(|a'^{(1)}\rangle + \lambda|a'^{(2)}\rangle) \quad (4)$

(3)(4) $\rightarrow \dfrac{b_{12}+\lambda b_{22}}{b_{11}+\lambda b_{21}} = \lambda \rightarrow$ Two values for $\lambda \xrightarrow[\text{Corresponds to}]{} b_{1,2}$

Remark: One may use diagonalization method to find simultaneous eigenkets.

An interesting <u>exception</u>:

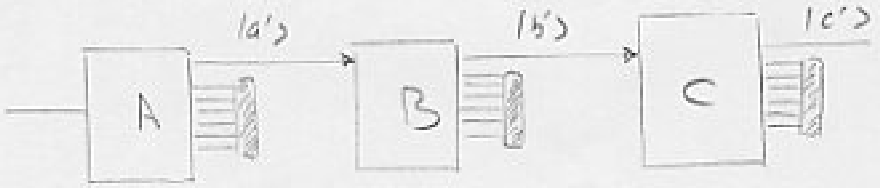
It may happen that there exists a <u>subspace</u> of the ket space such that $AB|a',b'\rangle = BA|a',b'\rangle$ holds $\forall$ elements of this subspace, even $[A,B] \neq 0$

<u>Ex.</u> For $l=0$ (one-dim. subspace)

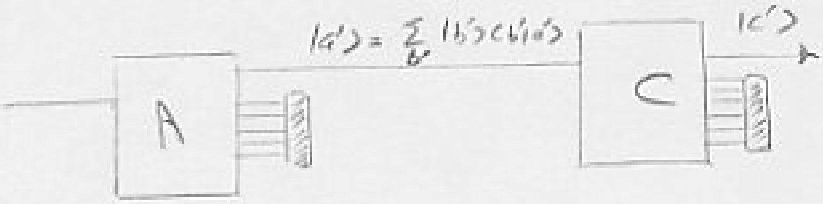
Even though $[L_x, L_z] \neq 0$ but $\begin{cases} L_x\,|l=0\rangle = 0\,|l=0\rangle \\ L_z\,|l=0\rangle = 0\,|l=0\rangle \end{cases}$

Some of the peculiarities Associated with incompatible Observables:

also, $|a'\rangle = \sum_{b'} |b'\rangle\langle b'|a'\rangle$ ($|b'\rangle$ one of these)



Experiment (a)



" (b)

(Sequence of Selective measurements)

For Fig. (a):

$$P_{c'}(b') = |\langle c'|b'\rangle|^2 \, |\langle b'|a'\rangle|^2$$ the probability of obtaining $|c'\rangle$

(where $\langle a'|a'\rangle = 1$)

$$P_{c'} = \sum_{b'} P_{c'}(b') = \sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2$$

$$= \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle$$

Total probability of obtaining $|c'\rangle$ Taking into account all possible b' routes.

For Fig. (b):

$$P'_{c'} = |\langle c'|a'\rangle|^2 = \left|\sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\right|^2$$

$$= \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle$$

These two Probabilities are different.

This is a quantum mechanical Phenomenon.

When $P_{c'} \overset{?}{=} P'_{c'}$

Answer: if $[A,B]=0$ or $[B,C]=0$ (in the absence of degeneracy)

Proof. - For $[A,B]=0$ case

In Fig. (a): $|a'\rangle \to |a',b'\rangle$, $|b'\rangle \to |a',b'\rangle$

$$P_{c'} = \sum_{b''} \underbrace{|\langle c'|a',b''\rangle|^2 \, |\langle a',b''|a',b'\rangle|^2}_{l',m'' \quad l',m'' \; l',m'} = |\langle c'|a',b'\rangle|^2$$

→ Example

In Fig (b): $|a'\rangle \to |a',b'\rangle$

$$P'_{c'} = |\langle c'|a',b'\rangle|^2 = |\sum_{a'',b''} \langle c'|a'',b''\rangle\langle a'',b''|a',b'\rangle|^2$$

Since there is no degeneracy $\xrightarrow{a''=a'} b'=b''$

The Uncertainty Relation:

We define an op. $\Delta A \equiv A - \langle A \rangle I$

$$\langle (\Delta A)^2 \rangle = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

dispersion of A
or variance
or mean square Deviation

Note:
If the state is an <u>eigenstate</u> of A $\longrightarrow \langle (\Delta A)^2 \rangle = 0$

<u>Ex.</u> The state $= |S_z ; + \rangle$

$$\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4}$$

where $\langle (\Delta S_z)^2 \rangle = 0$

Roughly speaking: The dispersion of an observable characterizes "fuzziness".

Now; $$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geqslant \frac{1}{4} | \langle [A, B] \rangle |^2$$

To prove this we state 3-Lemmas;

Lemma 1 -

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$ Schwarz inequality

which is analogous to $|a|^2|b|^2 \geq |a.b|^2$ in real Euclidian space

or; $\left(\int |f|^2 d\tau\right)\left(\int |g|^2 d\tau\right) \geq \left|\int f^* g\, d\tau\right|^2$

Proof: First note; $(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$

λ: any complex number

In particular the inequality holds for the choice;

$$\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$$

$$\rightarrow \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle - |\langle\alpha|\beta\rangle|^2 \geq 0$$

Lemma 2 -

The expectation value of a Hermitian op. is purely real.

Proof. - $\langle a''|B|a'\rangle = \langle a'|B|a''\rangle^*$ for Hermitian op.

$$\rightarrow \langle\alpha|B|\alpha\rangle = \langle\alpha|B|\alpha\rangle^* \quad ; \text{real}$$

Lemma 3 —

The expectation value of an anti-Hermitian op. ($C = -C^\dagger$) is purely imaginary.

Proof. — $\langle a'' | B | a' \rangle = -\langle a' | B | a'' \rangle^*$

$\Rightarrow \langle \alpha | B | \alpha \rangle = -\langle \alpha | B | \alpha \rangle^* \longrightarrow$ purely imaginary

Now;

using Lemma 1, with $\begin{cases} |\alpha\rangle = \Delta A |\text{state}\rangle \\ |\beta\rangle = \Delta B | \;\text{"}\; \rangle \end{cases}$

$\rightarrow \begin{cases} \langle \alpha | = \langle \text{stat} | \Delta A^\dagger = \langle \text{state} | \Delta A \\ \langle \beta | = \langle \text{"} | \Delta B^\dagger = \langle \;\text{"}\; | \Delta B \end{cases}$

$\Rightarrow \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geqslant |\langle \Delta A \, \Delta B \rangle|^2 \qquad (1)$

We note that; $\Delta A \, \Delta B = \frac{1}{2} [\Delta A, \Delta B] + \frac{1}{2} \{\Delta A, \Delta B\}$

But $[\Delta A, \Delta B] = [A, B]$

and $([A,B])^\dagger = (AB - BA)^\dagger = BA - AB = -[A,B]$

anti Hermitian

In contrast;

$\{\Delta A, \Delta B\}$: Hermitian

$$\rightarrow \langle \Delta A \Delta B \rangle = \tfrac{1}{2} \langle [A,B] \rangle + \tfrac{1}{2} \langle \{\Delta A, \Delta B\} \rangle$$

purely imaginary

purely real

$$\rightarrow |\langle \Delta A \Delta B \rangle|^2 = \tfrac{1}{4} |\langle [A,B] \rangle|^2 + \tfrac{1}{4} |\langle \{\Delta A, \Delta B\} \rangle|^2 \quad (2)$$

$$(2) \text{ in } (1) \rightarrow \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \tfrac{1}{4} |\langle [A,B] \rangle|^2 + \tfrac{1}{4} |\langle \{\Delta A, \Delta B\} \rangle|^2 \quad (3)$$

$$\rightarrow \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \tfrac{1}{4} |\langle [A,B] \rangle|^2 \quad (4)$$

The omission of the last term in (3) makes the inequality stronger.

1.5 Change of Basis (or change of Representation)

Transformation operator:

Suppose $[A,B]\neq 0$

The ket space can be viewed as being spanned by $\begin{cases}\text{either the set } \{|a'\rangle\} \\ \text{or by " " } \{|b'\rangle\}\end{cases}$

Ex. Spin $\frac{1}{2}$ system: $\begin{cases}|S_z;\pm\rangle \\ |S_x;\pm\rangle\end{cases}$

The two different sets of base kets, of course, span the same ket space.

We are interested in finding out $\{|a'\rangle\} \underset{\text{relation}}{\overset{?}{=\!=\!=}} \{|b'\rangle\}$

Def. — The basis with the $\{|a'\rangle\}$ base eigenkets is called A representation (or A diagonal representation).

Our basic task is to construct a tr. op. that:

$$U|a^{(l)}\rangle = |b^{(l)}\rangle \qquad (U=?)$$

Theorem: Given two sets of base kets, both satisfying orthonormality and completeness, there exists a unitary op. U such that:

$$|b^{(l)}\rangle = U|a^{(l)}\rangle \qquad l=1,\dots N$$

Def. - By a unitary op., we mean:

$$U^{+}U = I \quad \text{as well as} \quad UU^{+} = I$$

(Note: $U^{+}UU^{-1} = U^{-1} \rightarrow U^{+} = U^{-1}$)

Proof. - We prove this theorem by explicit construction. We assert that the op.

$$U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$$

will do the job.

Clearly: $U|a^{(\ell)}\rangle = |b^{(\ell)}\rangle$ (by orthonormality of $\{|a\rangle\}$)

Furthermore; $U^{+}U = \sum_k \sum_\ell |a^{(\ell)}\rangle\langle b^{(\ell)}|b^{(k)}\rangle\langle b^{(k)}| =$

$$= \sum_k |a^{(k)}\rangle\langle \cancel{b}^{(k)}_{a}| = I$$

(by orthonormality of $\{|b\rangle\}$ and completeness of $\{|a\rangle\}$)

Similarly, we can show $UU^{+} = I$

Remark: If A: Hermitian $\rightarrow UAU^{-1}$ is Hermitian (similarity tr.)

Remark: If $\begin{cases} |\alpha'\rangle = U|\alpha\rangle \\ |\beta'\rangle = U|\beta\rangle \end{cases} \longrightarrow \langle\alpha'|\beta'\rangle = \langle\alpha|U^{+}U|\beta\rangle = \langle\alpha|\beta\rangle$

Transformation Matrix:

$$\begin{cases} U = \sum_k |b^{(k)}\rangle\langle a^{(k)}| \\ U|a^{(l)}\rangle = |b^{(l)}\rangle \end{cases} \longrightarrow \langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle \qquad (1)$$

$$\rightarrow U \doteq \begin{pmatrix} \langle a^{(1)}|U|a^{(1)}\rangle & \cdots \\ & \\ & \end{pmatrix} = \begin{pmatrix} \langle a^{(1)}|b^{(1)}\rangle & \cdots \\ & \\ & \end{pmatrix}$$

Matrix representation of U (transformation matrix) from $\{|a'\rangle\} \xrightarrow{\text{to}} \{|b'\rangle\}$

Remark: We recall the rotation matrix in 3-dims.

$$(\hat{x}, \hat{y}, \hat{z}) \xrightarrow{R} (\hat{x}', \hat{y}', \hat{z}')$$

base vectors

$$R = \begin{pmatrix} \hat{x}\cdot\hat{x}' & \hat{x}\cdot\hat{y}' & \cdots \\ \hat{y}\cdot\hat{x}' & \hat{y}\cdot\hat{y}' & \\ \vdots & & \end{pmatrix}$$

Now, given, $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \qquad (2)$

$\langle b'|\alpha\rangle = ?$ $\qquad \left(|\alpha\rangle = \sum_{b'} |b'\rangle\langle b'|\alpha\rangle \right)$

$(2) \rightarrow \langle b^{(k)}|\alpha\rangle = \sum_l \langle b^{(k)}|a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle$

(1) in (2) $\rightarrow \langle b^{(k)}|\alpha\rangle = \sum_l \langle a^{(k)}|U^\dagger|a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle \qquad (3)$

$(3) \rightarrow (\text{New}) = U^\dagger (\text{old})$

$$\text{or } (3) \longrightarrow \begin{pmatrix} \langle b^{(1)}|\alpha\rangle \\ \langle b^{(2)}|\alpha\rangle \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle a^{(1)}|U^{\dagger}|a^{(1)}\rangle & \cdots \\ & \\ & \\ & \end{pmatrix} \begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \vdots \end{pmatrix}$$

New $\qquad\qquad\qquad\qquad$ old

The relation between the old and the new matrix elements of an op. X;

$$\langle b^{(k)}|X|b^{(l)}\rangle = \sum_m \sum_n \langle b^{(k)}|a^{(m)}\rangle\langle a^{(m)}|X|a^{(n)}\rangle\langle a^{(n)}|b^{(l)}\rangle$$

$$= \sum_m \sum_n \langle a^{(k)}|U^{\dagger}|a^{(m)}\rangle\langle a^{(m)}|X|a^{(n)}\rangle\langle a^{(n)}|U|a^{(l)}\rangle$$

or $\qquad X' = U^{\dagger} X U \qquad$ similarity tr.

Def. $- \qquad \mathrm{tr}(X) = \sum_{a'} \langle a'|X|a'\rangle$

$$\sum_{a'} \langle a'|X|a'\rangle = \sum_{a'} \sum_{b'} \sum_{b''} \langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle$$

$$= \sum_{a'} \sum_{b'} \sum_{b''} \langle b''|a'\rangle\langle a'|b'\rangle\langle b'|X|b''\rangle$$

$$= \sum_{b'} \sum_{b''} \langle b''|b'\rangle\langle b'|X|b''\rangle = \sum_{b'} \langle b'|X|b'\rangle$$

$\longrightarrow$ The trace of an op. X is independent of representation.

We can also prove;

$$tr(XY) = tr(YX)$$

$$tr(U^{+}XU) = tr(X)$$

$$tr(|a'\rangle\langle a''|) = \delta_{a'a''} \qquad (\text{i.e. } X = |a'\rangle\langle a''|)$$

$$tr(|b'\rangle\langle a'|) = \langle a'|b'\rangle$$

$$\left(\text{Ex. } \; X = |+\rangle\langle -| \right)$$
$$X' = |+\rangle\langle +|$$

Diagonalization:

Assume, $\langle a''|B|a'\rangle$ are Known, we are intrested to find:

$\{b'\}$ and $\{|b'\rangle\}$.

This prob. $\underbrace{\text{is equivalent}}$ to find a unitary matrix U that

diagonalizes B.

$$B|b'\rangle = b'|b'\rangle \longrightarrow \sum_{a'} \langle a''|B|a'\rangle\langle a'|b'\rangle = b'\langle a''|b'\rangle$$

or
$$\sum_{a^{(j)}} \langle a^{(i)}|B|a^{(j)}\rangle\langle a^{(j)}|b^{(\ell)}\rangle = b^{(\ell)}\langle a^{(i)}|b^{(\ell)}\rangle$$

$$\longrightarrow \begin{pmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & B_{22} & \cdots \\ & & \end{pmatrix} \begin{pmatrix} C_1^{(\ell)} \\ C_2^{(\ell)} \\ \vdots \end{pmatrix} = b^{(\ell)} \begin{pmatrix} C_1^{(\ell)} \\ C_2^{(\ell)} \\ \vdots \end{pmatrix}$$

$$B_{ij} = \langle a^{(i)}|B|a^{(j)}\rangle \qquad C_k^{(\ell)} = \langle a^{(k)}|b^{(\ell)}\rangle \qquad i,j,k = 1, \cdots N$$

Nontrivial sols. of $C_k^{(\ell)}$ are possible if the characteristic equ.;

$$\det(B - \lambda I) = 0$$

$$\lambda_1 = b^{(1)}, \quad \lambda_2 = b^{(2)}, \; \cdots \; \lambda_N = b^{(N)}$$

Knowing $b^{(\ell)}$ $\xrightarrow{\text{we can solve}}$ for the corresponding $C_k^{(\ell)}$ <u>up to</u> an <u>overall const.</u> to be determined from the <u>normalization</u> cond.

Comparing
$$\begin{cases} \langle a^{(k)} | U | a^{(\ell)} \rangle = \langle a^{(k)} | b^{(\ell)} \rangle \\ \qquad\qquad C_k^{(\ell)} = \langle a^{(k)} | b^{(\ell)} \rangle \end{cases}$$

$$\rightarrow U \doteq \begin{pmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \\ & & \end{pmatrix} \qquad \text{where } \{|a'\rangle\} \xrightarrow{U} \{|b'\rangle\}$$

For this procedure (diagonalization) the Hermicity of B is important (i.e. $U^\dagger U = I \rightarrow (c_1^{\ell *}, c_2^{\ell *} \cdots) \begin{pmatrix} c_1^{\ell'} \\ c_2^{\ell'} \\ \vdots \end{pmatrix} = \delta_{\ell\ell'}$ for Hermitian op)

<u>Ex.</u> $S_+ = \hbar |+\rangle\langle -| = S_x + iS_y \qquad S_+ \doteq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_z$

S_+ is <u>non-Hermitian</u> and <u>cannot</u> be diagonalized by any <u>unitary</u> matrix.

Remark: In the formalism used, we have assumed the eigenkets are <u>orthonormal</u>. Otherwise the formalism <u>can not</u> be applied immediately.

Unitary Equivalent Observables;

Theorem: Consider again two sets of orthonormal basis $\{|a'\rangle\}$ and $\{|b'\rangle\}$ connected by the;

$$U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$$

Knowing U, we may construct a unitary transform of A, UAU^{-1}; then A and UAU^{-1} are said to be unitary equivalent Observables.

$$A|a^{(\ell)}\rangle = a^{(\ell)}|a^{(\ell)}\rangle \rightarrow UAU^{-1}U|a^{(\ell)}\rangle = a^{(\ell)}U|a^{(\ell)}\rangle \quad (1)$$

$$(1) \rightarrow UAU^{-1}|b^{(\ell)}\rangle = a^{(\ell)}|b^{(\ell)}\rangle \quad (2)$$

Conclusion;

$|b'\rangle$'s are eigen kets of UAU^{-1} with exactly the same eigenvalues as the A eigenvalues.

$\rightarrow \left\{\begin{array}{l} A \text{ and} \\ UAU^{-1} \end{array}\right.$ have identical spectra.

Since $B|b^{(l)}\rangle = b^{(l)}|b^{(l)}\rangle$ (3)

(2)(3) $\rightarrow$ B and UAU^{-1} are simultaneously diagonalizable.

Question: $UAU^{-1} \overset{?}{=} B$ (See P 64/3)

Answer; Yes, in cases of Physical interest.

Ex. spin $\frac{1}{2}$ system

$A = S_z \quad B = S_x$

$S_z|S_z, \pm\rangle = \pm\frac{\hbar}{2}|S_z, \pm\rangle, \quad S_x|S_x; \pm\rangle = \pm\frac{\hbar}{2}|S_x, \pm\rangle$

$US_zU^{-1} = S_x$ $\qquad$ $U =$ rotation op around the y-axis by angle $\frac{\pi}{2}$.

1.6. Position, Momentum, and Translation.

Continuous Spectra

Ex. P_z: z-component of momentum op.

It has continuous eigenvalues; $-\infty < P_z < \infty$ any real value

Dimentionality $= \infty$

Eigenvalue equn;

$$\xi|\xi'\rangle = \xi'|\xi'\rangle$$ in continuous spectrum case

Some replacements:

i) $\langle a'|a''\rangle = \delta_{a'a''}$ $\longrightarrow$ $\langle \xi'|\xi''\rangle = \delta(\xi'-\xi'')$

Kronecker symbol — Dirac's δ-fun

ii) $\sum_{a'} |a'\rangle\langle a'| = I$ $\longrightarrow$ $\int d\xi' |\xi'\rangle\langle\xi'| = I$

iii) $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$ $\longrightarrow$ $|\alpha\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$

iv) $\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1$ $\longrightarrow$ $\int d\xi' |\langle\xi'|\alpha\rangle|^2 = 1$

v) $\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle$ $\longrightarrow$ $\langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle$

vi) $\langle a''|A|a'\rangle = a'\,\delta_{a'a''}$ $\longrightarrow$ $\langle\xi''|A|\xi'\rangle = \xi'\,\delta(\xi''-\xi')$

Position Eigenkets and
Position Measurements:

We have earlier mentioned that a <u>measurement</u> in Q.M. is essentially a <u>filtering process</u>.

We extend this idea to continuous spectra case.

Ex. Specific example; position op. in 1-dim;

$$X|x'\rangle = x'|x'\rangle \qquad \text{(Position)}$$

$\{|x'\rangle\}$ are postulated to form a complete set.

An arbitrary state: $|\alpha\rangle = \int_{-\infty}^{\infty} dx' |x'\rangle\langle x'|\alpha\rangle$

Using ideal detector;

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' |x'\rangle\langle x'|\alpha\rangle \xrightarrow{X\text{-Measurement}} |x'\rangle$$

The detector clicks when; $|\alpha\rangle \xrightarrow{\text{jumps}} |x'\rangle$

the probability $= |\langle x'|\alpha\rangle|^2$

But there is no such a detector;

A realistic detector;

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'' |x''\rangle\langle x''|\alpha\rangle \xrightarrow{X\text{-Measurement}} \int_{x'-\frac{\Delta}{2}}^{x'+\frac{\Delta}{2}} dx'' |x''\rangle\langle x''|\alpha\rangle$$

$|\langle x'|\alpha\rangle|^2 dx'$: the probability that the detector clicks

$dx' \to \Delta$ (i.e. when $x'-\frac{\Delta}{2} <$ Position $< x'+\frac{\Delta}{2}$)

where we have assumed $\langle x'' | \alpha \rangle$ does not change appreciably within the narrow interval.

The probability of recording the particle somewhere between $-\infty$ and ∞ is given by $\int_{-\infty}^{\infty} dx' \, |\langle x' | \alpha \rangle|^2$

If $\langle \alpha | \alpha \rangle = 1 \quad \rightarrow \quad \int_{-\infty}^{\infty} dx' \langle \alpha | x' \rangle \langle x' | \alpha \rangle = 1$

$\longrightarrow \qquad \langle x' | \alpha \rangle \equiv \psi_\alpha(x')$ wave func. for physical state represented by $|\alpha\rangle$

It is assumed in nonrelativistic Q.M. the position eigenkets $|x'\rangle$ are complete (spin ignored).

Extension to 3-dim.;

$$|\alpha\rangle = \int d^3x' \, |\mathbf{x}'\rangle \langle \mathbf{x}' | \alpha \rangle$$

$\mathbf{x}'$: stands for x', y', z'

$|\mathbf{x}'\rangle \equiv |x', y', z'\rangle$

$x|\mathbf{x}'\rangle = x'|\mathbf{x}'\rangle, \quad y|\mathbf{x}'\rangle = y'|\mathbf{x}'\rangle, \quad z|\mathbf{x}'\rangle = z'|\mathbf{x}'\rangle$

i.e. $|\mathbf{x}'\rangle$ simultaneous eigenket of observables x, y, z

$[x_i, x_j] = 0 \qquad$ (where $x_1 = x$, $x_2 = y$, $x_3 = z$)

Translation:

the phase factor $=1$ (by convention)

$$\mathcal{T}(d\mathbf{x}')\,|\mathbf{x}'\rangle = |\mathbf{x}'+d\mathbf{x}'\rangle$$

infinitesimal translation op.

well localized state

another well localized state

An arbitrary state;

$$|\alpha\rangle \longrightarrow \mathcal{T}(d\mathbf{x}')|\alpha\rangle = \mathcal{T}(d\mathbf{x}')\int d^3x'\,|\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle$$

$$= \int d^3x'\,|\mathbf{x}'+d\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle = \int d^3x'\,|\mathbf{x}'\rangle\langle\mathbf{x}'-d\mathbf{x}'|\alpha\rangle$$

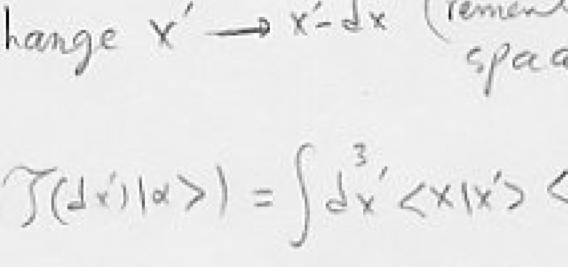
where we have change $\mathbf{x}' \to \mathbf{x}'-d\mathbf{x}'$ (remember integration is over all space.

Now; $\langle\mathbf{x}|\left(\mathcal{T}(d\mathbf{x}')|\alpha\rangle\right) = \int d^3x'\,\langle\mathbf{x}|\mathbf{x}'\rangle\langle\mathbf{x}'-d\mathbf{x}'|\alpha\rangle = \langle\mathbf{x}-d\mathbf{x}|\alpha\rangle$

Alternative Approach;

An infinitesimal translation of the physical system $\underset{\text{equivalent}}{\sim}$ Change in coord. system in opposite dir. $-d\mathbf{x}'$.

Physically;



The meaning of this equivalent approach;

$\longrightarrow$ How the <u>same</u> state ket would look to another observer whose coord. sys. is shifted by $-d\mathbf{x}'$

We will <u>not</u> use this approach.

Properties of $\mathcal{T}(dx')$:

i) $\mathcal{T}^{\dagger}(dx')\,\mathcal{T}(dx') = I$ unitary

This is imposed by probability conservation;

If $\langle\alpha|\alpha\rangle = 1$ $\xrightarrow[\text{to require}]{\text{it is reasonable}}$ $\langle\alpha|\,\mathcal{T}^{\dagger}(dx')\,\mathcal{T}(dx')\,|\alpha\rangle = 1$

This cond. is guaranteed if; $\mathcal{T}^{\dagger}(dx')\,\mathcal{T}(dx') = I$

Quite generally:

The norm of a ket is preserved under unitary transformation.

ii) $\mathcal{T}(dx'')\,\mathcal{T}(dx') = \mathcal{T}(dx' + dx'')$ (we demand)

iii) $\mathcal{T}(-dx') = \mathcal{T}^{-1}(dx')$ We expect; opposite dir tr. = inverse of the original tr.

iv) $\lim_{dx'\to 0} \mathcal{T}(dx') = I$ (of first order in dx')

Now; if we take; $\mathcal{T}(dx') = I - iK\cdot dx'$

where K_x, K_y, K_z : Hermitian ops.

→ all the properties listed are satisfied.

First property: $\mathcal{T}^{+}(dx')\mathcal{T}(dx') = (I + iK^{+}\cdot dx')(I - iK\cdot dx')$

$= I - i(K - K^{+})\cdot dx' + O[(dx')^2]] = I$

(it is true if $K = K^{+}$)

Second property: $\mathcal{T}(dx'')\mathcal{T}(dx') = (I - iK\cdot dx'')(I - iK\cdot dx')$

$\simeq I - iK\cdot(dx' + dx'') = \mathcal{T}(dx' + dx'')$

Third property: $\mathcal{T}(-dx') = I + iK\cdot dx'$

For unitary op. $\mathcal{T}^{-1}(dx') = \mathcal{T}^{+}(dx') = I + iK\cdot dx'$

Forth property; $\lim_{dx'\to 0} \mathcal{T}(dx') = \lim_{dx'\to 0} (I - iK\cdot dx') = I$

Fundamental relation between K-op. and x-op.:

Note that:

$x\,\mathcal{T}(dx')|x'\rangle = x|x' + dx'\rangle = (x' + dx')|x' + dx'\rangle$

$\mathcal{T}(dx')\,x|x'\rangle = x'\,\mathcal{T}(dx')|x'\rangle = x'|x' + dx'\rangle$

$[x, \mathcal{T}(dx')]|x'\rangle = dx'|x' + dx'\rangle \simeq dx'|x'\rangle$

error; Second order

Since this is true $\forall |x'\rangle$ and $\{|x'\rangle\}$ is a complete set;

$$\longrightarrow \quad [\mathbf{x}, \mathcal{T}(d\mathbf{x}')] = d\mathbf{x}' I$$

number

$$\rightarrow \quad -i\, \mathbf{x}\, \mathbf{K}\cdot d\mathbf{x}' + i\, \mathbf{K}\cdot d\mathbf{x}'\, \mathbf{x} = d\mathbf{x}' I$$

vector (but not op.)

Now choose $d\mathbf{x}'$ in $\hat{x}_j$-dir
and form scalar product with $\hat{x}_i$;

$$-i\, \mathbf{x}\cdot\hat{x}_i (\mathbf{K}\cdot d\mathbf{x}') + i(\mathbf{K}\cdot d\mathbf{x}')\mathbf{x}\cdot\hat{x}_i = d\mathbf{x}'\cdot\hat{x}_i$$

$$-i\, x_i (K_j dx'_j) + i(K_j dx'_j)\, x_i = dx'_j\, \delta_{ij}$$

$$\rightarrow \quad i\,[K_j, x_i] = \delta_{ij} \quad \rightarrow \quad [x_i, K_j] = i\delta_{ij} I$$

Transformations of states
and Observables

The laws of nature are believed to be invariant under certain space-time symmetry operations, including displacement, rotations, and transformations between frames of reference in uniform relative motion.

Corresponding to each such space-Time tr. $\longrightarrow$

$$A \longrightarrow A' \quad , \quad |\Psi\rangle \longrightarrow |\Psi'\rangle$$

a) If $A|\varphi_n\rangle = a_n|\varphi_n\rangle \xrightarrow{\text{tr.}} A'|\varphi_n'\rangle = a_n|\varphi_n'\rangle$

The eigenvalues of A and A' are the same, because A' represents an essentially similar observable to A differing only by tr. to another frame of ref.

b) If $|\Psi\rangle = \sum_n c_n|\varphi_n\rangle$

$$\xrightarrow{\text{tr}} |\Psi'\rangle = \sum_n c_n'|\varphi_n'\rangle$$

Now $|c_n|^2 \overset{\text{must}}{=\!=} |c_n'|^2$

i.e. $|\langle\varphi_n|\Psi\rangle|^2 = |\langle\varphi_n'|\Psi'\rangle|^2$

Because they express the equality of possibilities for equivalent events in the two frames of ref.

Theorem (Wigner)

Any mapping of the vector space onto itself that preserves the value $|\langle\varphi|\psi\rangle|$ may be implemented by an op. U:

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$$
$$|\varphi\rangle \to |\varphi'\rangle = U|\varphi\rangle$$

with U being <u>either</u> unitary (linear) <u>or</u> antiunitary (antilinear).

Case (a): U is unitary; $(UU^\dagger = U^\dagger U = I)$

$$\langle\varphi'|\psi'\rangle = (\langle\varphi|U^\dagger)(U|\psi\rangle) = \langle\varphi|\psi\rangle$$

$\rightarrow$ A unitary tr. preserves the <u>complex</u> value of an inner product, not merely its <u>absolute</u> value.

Case (b): U is antilinear $\longrightarrow$ $U c|\psi\rangle = c^* U|\psi\rangle$

U is antiunitary $\longrightarrow$ $\langle\varphi'|\psi'\rangle = \langle\varphi|\psi\rangle^*$

Only linear ops. can describe continuous trs. ·

Suppose, for example,

$U(l)$: a diplacement through the dist. l

This can be done by two successive displacements;

$$U(l) = U(l/2)\,U(l/2)$$

Since the product of two antilinear ops. is linear,

$\longrightarrow$ $U(l)$: linear (whether or not $U(l/2)$ is linear)

Also a continuous op. can not change discontinuously from linear to antilinear as a func. of l,

$\longrightarrow$ the op. must be linear for all l.

Now; $|\varphi\rangle \xrightarrow{tr.} |\varphi'\rangle = U|\varphi\rangle$

Also $A \xrightarrow{tr} A'$

From the invariance of the laws of the nature;

The relationship of A to $|\varphi\rangle$ $\underset{\text{same}}{\underbrace{\text{the}}}$ The relationship of A' to $|\varphi'\rangle$

In particular;

$$A|\varphi_n\rangle = a_n|\varphi_n\rangle \xrightarrow{\qquad} A'|\varphi_n'\rangle = a_n|\varphi_n'\rangle$$

(1)

$$A'(U|\varphi_n\rangle) = a_n(U|\varphi_n\rangle) \rightarrow U^{-1}A'U|\varphi_n\rangle = a_n|\varphi_n\rangle \quad (2)$$

$$(1)(2) \rightarrow (A - U^{-1}A'U)|\varphi_n\rangle = 0 \quad \forall |\varphi_n\rangle \in \{|\varphi_n\rangle\}$$

$$\rightarrow (A - U^{-1}A'U) = 0 \rightarrow A' = UAU^{-1}$$

Now consider a family of unitary ops. $U(s)$.

$$\text{Let} \begin{cases} U(0) = I \\ U(s_1+s_2) = U(s_1)U(s_2) \end{cases}$$

This is possible (without proof)

s: parameter

$$U(s) = I + \frac{dU}{ds}\Big|_{s=0} s + O(s^2) \qquad s\text{: infinitesimal}$$

$$UU^+ = I + \left(\frac{dU}{ds} + \frac{dU^+}{ds}\right)\Big|_{s=0} s + O(s^2)$$

Since $UU^+ = I \longrightarrow \frac{dU}{ds}\Big|_{s=0} = iK$ with $K = K^+$

K: the generator of the family of unitary ops.

because it determines $U(s)$, not only for infinitesimal s but for all s.

This can be shown by differentiating:

$$U(s_1+s_2) = U(s_1)\,U(s_2)$$

$$\frac{\partial}{\partial s_2} U(s_1+s_2)\Big|_{s_2=0} = U(s_1)\frac{d}{ds_2}U(s_2)\Big|_{s_2=0}$$

$$\rightarrow \frac{dU(s)}{ds}\Big|_{s=s_1} = U(s_1)\, ik$$

The sol. for the first-order diff. equ. with initial cond $U(0)=I$

$$\rightarrow U(s) = e^{iks}$$

Thus, the op. for any finite tr. is determined by the generator of infinitesimal trs.

Momentum as a Generator of Translation:

$$\tau(dx')\,\psi(x') = \psi(x'-dx')$$

$$(I - ik\cdot dx')\,\psi(x') = \psi(x'-dx')$$

$$〃 \qquad = \psi(x') - (dx'\cdot\nabla)\,\psi(x') + \cdots$$

$$〃 \qquad = \Big(I - \frac{i}{\hbar}(dx'\cdot P)\Big)\,\psi(x') + \cdots$$

vector   operator

where $P \equiv -i\hbar\nabla$

$$\longrightarrow \quad k = P/\hbar$$

Then; $[x_i, k_j] = i\delta_{ij} \longrightarrow [x_i, P_j] = i\hbar\,\delta_{ij}$

Therefore; $\langle(\Delta x)^2\rangle\langle(\Delta P_x)^2\rangle \geq \hbar^2/4$

For finite tr.;

$$\tau(\Delta x'\hat{x})\,|x'\rangle = |x' + \Delta x'\hat{x}\rangle$$

$$\tau(\Delta x'\hat{x}) = \lim_{N\to\infty}\Big(I - i\frac{P_x}{\hbar}\frac{\Delta x'}{N}\Big)^N = e^{-i\frac{P_x\Delta x'}{\hbar}}$$

Generally; $\tau(\bar{\rho}) = e^{-i\frac{\bar{P}\cdot\bar{\rho}}{\hbar}}$

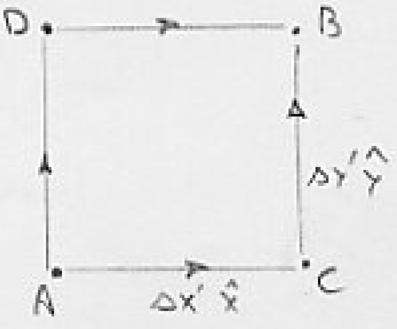
Fundamental Property;

Successive translations in different dirs. commute.

$$\tau(\Delta y'\hat{y})\,\tau(\Delta x'\hat{x}) = \tau(\Delta x'\hat{x} + \Delta y'\hat{y})$$

$$\tau(\Delta x'\hat{x})\,\tau(\Delta y'\hat{y}) = \tau(\;〃 \;+\; 〃\;)$$

(1)



$$[\mathcal{T}(\Delta y'\hat{y}), \mathcal{T}(\Delta x'\hat{x})] = \left[\left(I - \frac{iP_y\Delta y'}{\hbar} - \frac{P_y^2(\Delta y')^2}{2\hbar^2} + \cdots\right), \left(I - \frac{iP_x\Delta x'}{\hbar} - \frac{P_x^2(\Delta x')^2}{2\hbar^2} + \cdots\right)\right]$$

$$\simeq -\frac{(\Delta x')(\Delta y')[P_y, P_x]}{\hbar^2}$$

(up to second order)

Remark: Taking into account the higher term is led to the same conclusion

Requirement (1) requires; $[P_x, P_y] = 0$

or , $[P_i, P_j] = 0$

Def. – Whenever the generators of transformations commute, the corresponding group is said to be Abelian.

The translation group in 3-dims. is Abelian.

$[P_i, P_j] = 0 \longrightarrow |P'\rangle \equiv |P_x', P_y', P_z'\rangle$ simultaneous eigenket

$P_x|P'\rangle = P_x'|P'\rangle$ . . . . .

Also;

$$\mathcal{T}(dx')|P'\rangle = \left(1 - \frac{iP\cdot dx'}{\hbar}\right)|P'\rangle = \underbrace{\left(1 - \frac{iP'\cdot dx'}{\hbar}\right)}_{\text{eigenvalue}}|P'\rangle$$

eigenvalue

$\rightarrow |P'\rangle$: eigenket of $\mathcal{T}(dx')$

(complex, because $\mathcal{T}(dx')$ though unitary, is not Hermitian)

It was anticipated because,

$[P, \mathcal{T}(dx')] = 0$

The Canonical Commutation Relations:

$[x_i, x_j] = 0 \qquad [P_i, P_j] = 0 \qquad [x_i, P_j] = i\hbar\delta_{ij}$

P.A.M Dirac: Fundamental quantum conditions

or : Canonical commutation relations

or : Fundamental " "

In 1925, P.A.M Dirac observed that the various Q. Mechanical relations can be obtained from the corresponding classical relations in the following way;

$$\underset{\substack{\text{classical}\\ \text{poisson}\\ \text{bracket}}}{[\ ,\ ]_{\text{classical}}} \longrightarrow \underset{\text{commutators in Q.M.}}{\frac{[\ ,\ ]}{i\hbar}}$$

Def. — Poisson bracket; $[A(q,P), B(q,P)] \equiv \sum_s \left(\frac{\partial A}{\partial q_s}\frac{\partial B}{\partial P_s} - \frac{\partial A}{\partial P_s}\frac{\partial B}{\partial q_s} \right)$

Ex. — $[x_i, P_j]_{cl.} = \delta_{ij} \longrightarrow \frac{[x_i, P_j]}{i\hbar} = \delta_{ij}$

$\longrightarrow [x_i, P_j] = i\hbar\delta_{ij}$

Dirac's rule, is plausible, because $\left\{ \begin{array}{l} \text{the classical Poisson brackets} \\ \text{and Q. Mechanical commutators} \end{array} \right.$ satisfy similar algebraic properties.

$[A,A]=0$

$[A,B]=-[B,A]$

$[A,c]=0$ (c: number)

$[A+B,C]=[A,C]+[B,C]$

$[A,BC]=[A,B]C+B[A,C]$

$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$ Jacobi identity

Important differences;

1- Dim. of classical Poisson bracket differs from that of Q. mechanical commutator.

2- Poisson bracket of <u>real</u> fns. of q's and p's is <u>purely real</u>, while commutator of two Hermitian op. is <u>anti-Hermitian</u>.

To take care of these differences the factor <u>$i\hbar$</u> is inserted.

Remark: In derivation of commutator relations we used direct derivations (not Dirac's approach of $[\,,\,]_{cl.}\to\frac{[\,,\,]}{i\hbar}$)
This is more <u>powerful</u> because it can be <u>generalized</u> to situations where <u>observables</u> have <u>no classical</u> analogues (like spin).

1.7 – Wave Funs. in Position and Momentum Space;

Position-Space Wave Func

Consider <u>one</u>-dim. case;

$$x|x'\rangle = x'|x'\rangle$$

$$\langle x''|x'\rangle = \delta(x''-x')$$

Also $|\alpha\rangle = \int dx'\, |x'\rangle\langle x'|\alpha\rangle$

and $|\langle x'|\alpha\rangle|^2 dx'$: the probability for particle to be found in a narrow interval dx' around x'.

In our formalism, the inner product $\langle x'|\alpha\rangle = \psi_\alpha(x')$

Now consider the inner product;

$$\langle \beta|\alpha\rangle = \int dx'\, \langle\beta|x'\rangle\langle x'|\alpha\rangle = \int dx'\, \psi_\beta^*(x')\, \psi_\alpha(x')$$

by completeness postulate

The overlap between two wave funcs.

$\langle\beta|\alpha\rangle$: The probability amplitude for state $|\alpha\rangle$ to be found in state $|\beta\rangle$

Now we interpret the expression;

$$|\alpha\rangle = \sum_{a'} |a'\rangle \langle a'|\alpha\rangle$$

$$\langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle \langle a'|\alpha\rangle$$

In the notation of wave mechanics;

$$\psi_\alpha(x') = \sum_{a'} C_{a'} \, u_{a'}(x')$$

$u_{a'}(x') = \langle x'|a'\rangle$ eigenfunc of A

Now, $\langle \beta|A|\alpha\rangle$ in terms of the wave funcs. of $|\alpha\rangle$ and $|\beta\rangle$;

$$\langle \beta|A|\alpha\rangle = \int dx' \int dx'' \langle \beta|x'\rangle \langle x'|A|x''\rangle \langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \, \psi_\beta^*(x') \langle x'|A|x''\rangle \psi_\alpha(x'')$$

Now If $A = x^2$ (in H of Simple Harmonic Osc.)

$$\langle x'|x^2|x''\rangle = (\langle x'|).(x''^2|x''\rangle) = x'^2 \delta(x'-x'')$$

Then $\langle \beta|x^2|\alpha\rangle = \int dx' \langle \beta|x'\rangle x'^2 \langle x'|\alpha\rangle = \int dx' \psi_\beta^*(x') x'^2 \psi_\alpha(x')$

In general;

$$\langle \beta|f(x)|\alpha\rangle = \int dx' \psi_\beta^*(x') f(x') \psi_\alpha(x')$$

Momentum Operator in the Position Basis:

$\{|x'\rangle\}$: the basis

$$\mathcal{T}(\Delta x')|\alpha\rangle = \left(I - \frac{iP\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx'\, \mathcal{T}(\Delta x')|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' |x'+\Delta x'\rangle\langle x'|\alpha\rangle = \int dx' |x'\rangle\langle x'-\Delta x'|\alpha\rangle$$

$$= \int dx' |x'\rangle\left(\langle x'|\alpha\rangle - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

Taylor expansion

$$\longrightarrow P|\alpha\rangle = \int dx' |x'\rangle\left(-i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right) \quad (1)$$

Remark:
$\langle\alpha|P|\alpha\rangle = \int dx'\, \Psi_\alpha^*(x')\left(-i\hbar\frac{\partial}{\partial x'}\right)\Psi_\alpha(x')$

$$\rightarrow \langle x'|P|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle \quad (2)$$

Taking $|\alpha\rangle = |x''\rangle$

$$\langle x'|P|x''\rangle = -i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')$$

$$(1) \rightarrow \langle\beta|P|\alpha\rangle = \int dx' \langle\beta|x'\rangle\left(-i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$= \int dx'\, \Psi_\beta^*(x')\underbrace{\left(-i\hbar\frac{\partial}{\partial x'}\right)}_{P\text{-op.}}\Psi_\alpha(x')$$

in our formalism this is not a postulate, rather it has been derived using the basic properties of momentum.

By repeatedly applying (2):

$$\langle x'|P^n|\alpha\rangle = (-i\hbar)^n\frac{\partial^n}{\partial x'^n}\langle x'|\alpha\rangle$$

$$\langle\beta|P^n|\alpha\rangle = \int dx'\, \Psi_\beta^*(x')(-i\hbar)^n\frac{\partial^n}{\partial x'^n}\Psi_\alpha(x')$$

Momentum-Space wave Func.

$\{|p'\rangle\}$: basis

Consider one-dim. space;

$p|p'\rangle = p'|p'\rangle$ and $\langle p'|p''\rangle = \delta(p'-p'')$

Also, $|\alpha\rangle = \int dp'\, |p'\rangle\langle p'|\alpha\rangle$

$|\langle p'|\alpha\rangle|^2 dp'$: The probability that a measurement of p gives eigenvalue p' within narrow interval dp'

$\langle p'|\alpha\rangle = \phi_\alpha(p')$ momentum-space wave func.

If $\langle\alpha|\alpha\rangle = 1 \longrightarrow \int dp'\, \langle\alpha|p'\rangle\langle p'|\alpha\rangle = \int dp'\, |\phi_\alpha(p')|^2 = 1$

Now; $\{|x'\rangle\} \xleftrightarrow[\text{connection}]{?} \{|p'\rangle\}$

We remember in the discrete case

$\{|a'\rangle\}$ and $\{|b'\rangle\}$ are related by

$\langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle$

Likewise we expect the information is contained in

$\langle x'|p'\rangle$ transformation func. from x-rep. $\xrightarrow{\text{to}}$ p-rep.

In (2P71) $\xrightarrow{|\alpha\rangle = |p'\rangle}$ $\langle x'|p|p'\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|p'\rangle$

or $\quad p'\langle x'|p'\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|p'\rangle$

The sol. of this differential equ for $\langle x'|p'\rangle$ is

$$\langle x'|p'\rangle = N e^{\frac{ip'x'}{\hbar}}$$

N: normalization const.

(Plane wave)

Even though $\langle x'|p'\rangle$ is a func. of x' and p', we can temporarily regard it as a fuc of x' with p' fixed.

$\xrightarrow[\text{it can be viewed}]{\text{Then}}$ $\langle x'|p'\rangle$: the probability amplitude for the momentum eigenstate $|p'\rangle$ specified by p' to be found at the state $|x'\rangle$ specified by x'.

In other words: $\langle x'|p'\rangle$: wave func. for momentum eigenstate $|p'\rangle$ (in x-space)

The normalization cnst. N;

$$\langle x'|x''\rangle = \int dp' \langle x'|p'\rangle\langle p'|x''\rangle$$

$$\delta(x'-x'') = |N|^2 \int dp' \, e^{\frac{ip'(x'-x'')}{\hbar}} = 2\pi\hbar |N|^2 \, \delta(x'-x'')$$

Choosing N: real, Positive (by convention);

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip'x'}{\hbar}}$$

Now, we show $\langle x'|\alpha\rangle \xleftrightarrow{\text{relation}} \langle p'|\alpha\rangle$

position-space wave func — momentum-space wave func

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle \langle p'|\alpha\rangle$$

$$\langle p|\alpha\rangle = \int dx' \langle p'|x'\rangle \langle x'|\alpha\rangle$$

$$\begin{cases} \psi_\alpha(x') = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int dp' \, e^{\frac{ip'x'}{\hbar}} \, \phi_\alpha(p') \\ \phi_\alpha(p') = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int dx' \, e^{-\frac{ip'x'}{\hbar}} \, \psi_\alpha(x') \end{cases}$$

Fourier's transforms

Gaussian Wave Packets:

$$\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}\, e^{\left(ikx' - \frac{x'^2}{2d^2}\right)}$$

Sinusoidal wave with wave number k modulated by a Gaussian profile centered at origin.

Probability density $= |\langle x'|\alpha\rangle|^2 \xrightarrow{\text{rapidly}} 0$ for $|x'| > d$

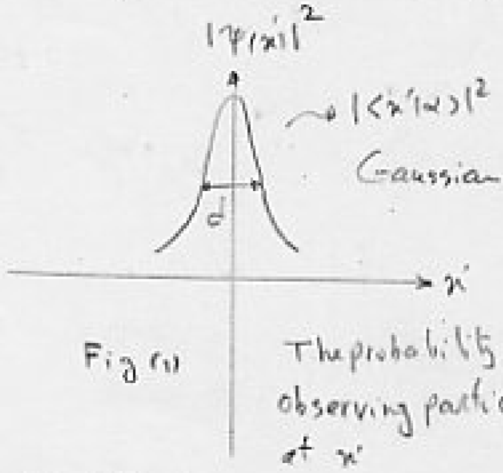


Fig (1) The probability of observing particle at x'

Now; $\langle x\rangle = ?$ $\langle x^2\rangle = ?$
$\langle p\rangle = ?$ $\langle p^2\rangle = ?$

$$\langle x\rangle = \int_{-\infty}^{\infty} dx' \langle\alpha|x'\rangle x' \langle x'|\alpha\rangle = \int_{-\infty}^{\infty} dx'\, |\langle x'|\alpha\rangle|^2 x' = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx'\, x'^2 |\langle x'|\alpha\rangle|^2 = \left(\frac{1}{\sqrt{\pi}d}\right)\int_{-\infty}^{\infty} dx'\, x'^2\, e^{\frac{-x'^2}{d^2}} = \frac{d^2}{2}$$

$$\rightarrow \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{d^2}{2} \qquad (1)$$

$$\langle p\rangle = \hbar k \qquad \langle p^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar^2}{2d^2} \qquad (2)$$

Now we check Heisenberg uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \frac{\hbar^2}{4}$$

$$(1)(2) \longrightarrow \langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle = \frac{\hbar^2}{4} \qquad \text{indep. of } d$$

So for a Gaussian wave packet we have an equality relation.

For this reason Gaussian wave packet is called Minimum Uncertainty wave packet.

In momentum space;

$$\langle p'|\alpha\rangle \equiv \varphi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx'\, e^{\frac{-ip'x'}{\hbar}}\, \psi_\alpha(x')$$

$$\rightarrow \langle p'|\alpha\rangle = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\int_{-\infty}^{\infty} dx'\, e^{\left(-\frac{ip'x'}{\hbar} + ikx' - \frac{x'^2}{2d^2}\right)}$$

$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\; e^{\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}}$$

Remark: We may obtain $\langle p\rangle$ and $\langle p^2\rangle$ using $\langle p'|\alpha\rangle$ in momentum space. (the op. P is just P in this space)

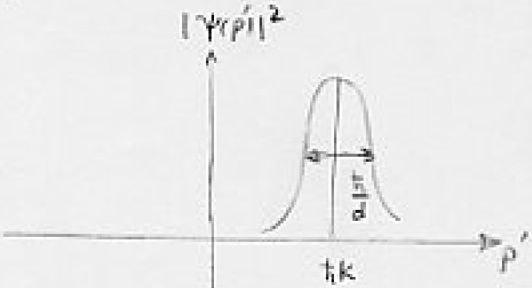


Fig.(2) The probability of observing particle with momentum p'

Width (in Fig.(1)) $= d$

" (" " (2)) $= \frac{\hbar}{d}$

Width(1) $\cdot$ Width(2) $= \hbar$ const.

This is another way of expressing

$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4}$ const.

$\longrightarrow$ The wider the spread in p-space

$\longrightarrow$ the narrower the spread in x-space

(and Vice Versa)

Ex. (extreme example)

$d \to \infty$ $\quad \langle x'|\alpha\rangle \longrightarrow$ plane wave extending over all space

$\to$ $|\langle x'|\alpha\rangle|^2 \longrightarrow$ const everywhere

while; $\quad \langle p'|\alpha\rangle \to \delta$-func. like, sharply peaked at $\hbar k$.

$d \to 0$ $\quad \langle x'|\alpha\rangle \longrightarrow \delta$-func. (well localized)

$\langle p'|\alpha\rangle \longrightarrow$ const.

An extremely well localized (in x-space) state is to be regarded as a superposition momentum eigenstates with all possible values of momenta.

Even those mom. eigenstates whose $p \sim mc$ or $p > mc$ must be included in the superposition.

However at such high values of momentum, a description based on nonrelativistic quantum mechanics is bound to break down.

Despite this limitation our formalism, based on the existance of the position eigenket $|x'\rangle$, has a wide domain of applicablitiy.

Remark: The concept of a localized state in relativistic Q.M. is far mor intricate because of the possibility of negative energy states or pair creation.

Generalization to 3-dim.;

$$x|x'\rangle = x'|x'\rangle$$
$$p|p'\rangle = p'|p'\rangle$$

They obey; $\langle x'|x''\rangle = \delta^3(x'-x'')$

$\langle p'|p''\rangle = \delta^3(p'-p'')$

where $\delta^3(x'-x'') = \delta(x'-x'')\,\delta(y'-y'')\,\delta(z'-z'')$

$$\int d^3x'\,|x'\rangle\langle x'| = I$$

completeness relations

$$\int d^3p'\,|p'\rangle\langle p'| = I$$

$$|\alpha\rangle = \int d^3x'\,|x'\rangle\langle x'|\alpha\rangle \qquad |\alpha\rangle = \int d^3p'\,|p'\rangle\langle p'|\alpha\rangle$$

where $\langle x'|\alpha\rangle = \psi_\alpha(x')$ $\qquad \langle p'|\alpha\rangle = \varphi_\alpha(x')$

$$\langle \beta|p|\alpha\rangle = \int d^3x'\,\psi^*_\beta(x')(-i\hbar\nabla)\psi_\alpha(x')$$

$$\langle x'|p'\rangle = \left(\frac{1}{(2\pi\hbar)^{3/2}}\right) e^{\frac{ip'\cdot x'}{\hbar}}$$ translation func.

$$\psi_\alpha(x') = \left(\frac{1}{(2\pi\hbar)^{3/2}}\right)\int d^3p'\, e^{\frac{ip'\cdot x'}{\hbar}}\varphi_\alpha(p')$$

$$\varphi_\alpha(x') = (\quad " \quad)\int d^3x'\, e^{-\frac{ip'\cdot x'}{\hbar}}\psi_\alpha(x')$$

Acc. to $\langle\alpha|\alpha\rangle = \int_{-\infty}^{\infty} dx' |\langle x'|\alpha\rangle|^2 = 1$ (dim.-less) (in one-dim)

$\rightarrow$ dim. of $\langle x'|\alpha\rangle = \frac{1}{(\text{length})^{1/2}}$, In 3-dim case, dim. of $\langle x'|\alpha\rangle = \frac{1}{(\text{length})^{3/2}}$