ESS2222H

Tectonics and Planetary Dynamics
Lecture 10
Thermal Convection
Navier-Stokes Equations

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Thermal Convection

Conservation of Fluid in 2D

The flow entering a small volume $\delta x\, \delta y$ at x in $x - dir$: $\rho u\, \delta y$

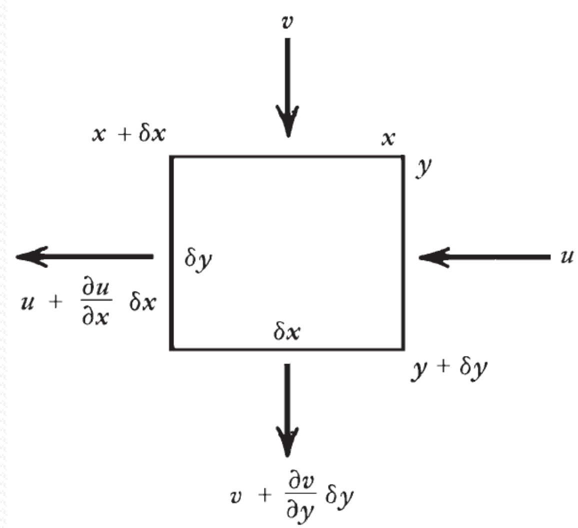
The flow leaving a small volume $\delta x\, \delta y$ at $x + \delta x$ in $x - dir$: $\rho u(x + \delta x)\, \delta y$

The flow rate per unit area: $\rho u(x + \delta x)$

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^{\,2} + \ldots$$ *Taylor expansion*

$$f(x+\delta x) \approx f(x) + \frac{f'(x)}{1!}(x + \delta x - x) + \ldots$$

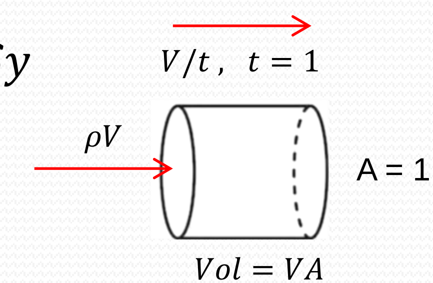
$$u(x+\delta x) = u(x) + \frac{\partial u}{\partial x}\delta x$$



Flow across the surfaces of an infinitesimal rectangular element.

The net flow passing through the volume (entering and leaving) in x-direction:

$$\rho\left[u(x) + \frac{\partial u}{\partial x}\delta x - u(x)\right]\delta y = \rho \frac{\partial u}{\partial x}\delta x \delta y$$



Similarly in $y - dir$:

$$\rho\left[v(x) + \frac{\partial v}{\partial y}\delta y - v(y)\right] = \rho \frac{\partial v}{\partial y}\delta y \delta x$$

Schubert, Turcotte & Olson, Mantle convection in the Earth and Planets, 2001; Turcotte & Schubert, Geodynamics, 2002

Conservation of Fluid in 2D

The total net flow through the volume:
:

$$\rho \frac{\partial u}{\partial x} \delta x \delta y + \rho \frac{\partial v}{\partial y} \delta y \delta x = \rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] \delta y \delta x$$

If the flow is steady (time-independent), and there are no density variations, then there will be no net flow into or out of the volume.
The conservation of fluid or continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \mathrm{div}\,\mathrm{V} = \nabla \cdot V = 0$$ *Mass Conservation*

In 3D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$ $\qquad V \equiv V(u, v, w)$

In spherical coordinate :

$$\nabla \cdot V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{rSin(\theta)} \frac{\partial}{\partial \theta}\left(V_\theta Sin(\theta)\right) + \frac{1}{rSin(\theta)} \frac{\partial V_\varphi}{\partial \varphi}$$

Momentum Equation

Elemental Force Balance

The forces acting on the small volume element are:
1) Pressure force
2) Viscous force
3) Gravity force (body force)
4) Inertial force

The Earth's mantle behave as a **highly viscous fluid** on geologic time scales.

The viscosity of mantle is ~ $\mu = 10^{21} P.s$
Its density is ~ $\rho = 4000\ kg/m^3$
And its thermal diffusivity is ~ $\kappa = 10^{-6}\ m/s$

The Prandtl number: $Pr = \frac{\upsilon}{\kappa}$

$\kappa = \frac{K}{\rho C_P}$ m^2/s *thermal diffusivity*

$\nu = \frac{\mu}{\rho}$ m^2/s *kinematic viscosity*

μ	$Pa.s$	*dynamic viscosity*
K	$W/(mK)$	*thermal conductivity*
C_P	$J/(kgK)$	*specific heat*
ρ	kg/m^3	*density*

$Pr_{Earth} \sim 10^{23}$

Momentum Equation

At high Prandtl numbers the inertial forces can be neglected:

$$\frac{\partial V}{\partial t} \approx 0$$

$$F_P + F_{visc} + F_g = 0$$

The force acting at x in $x - dir$ on $\delta y - element$: $p(x)\ \delta y$

The force acting at $x + \delta x$ in $x - dir$ on $\delta y - element$: $p(x + \delta x)\ \delta y$

The net pressure force on the element in the x-dir. per unit area of the fluid element:

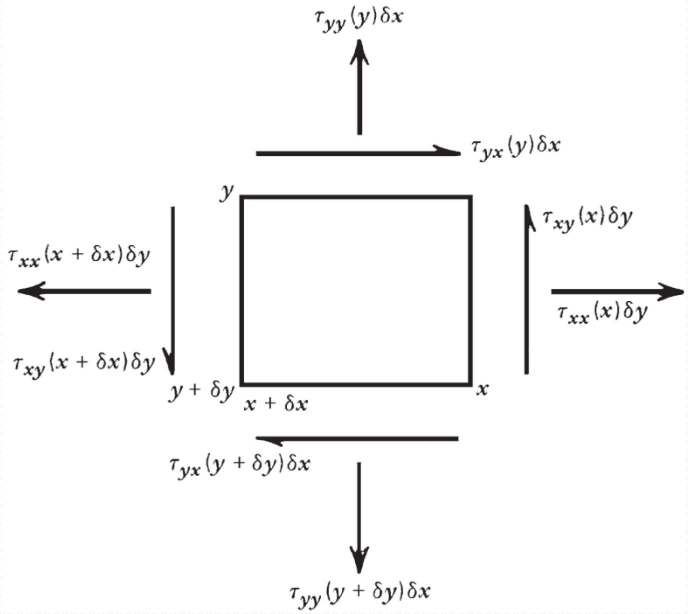
$$\frac{p(x)\delta y - p(x+\delta x)\delta y}{\delta x \delta y} = -\frac{p(x+\delta x) - p(x)}{\delta x}$$

By virtue of a simple Taylor series expansion → $-\frac{\partial p}{\partial x}$

Similarly for the pressure force on the element in the y-dir. Per unit area of the element: $-\frac{\partial p}{\partial y}$

Momentum Equation

The gravitational force acting on the volume element:



$$F_g = mg = \rho\delta x\delta y\, g$$

$$F_g = \rho\, g \qquad \textit{the net g} - \textit{force per unit volume}$$

Viscus Foces

The net viscous force in the x-dir. Per unit volume:

$$\frac{\tau_{xx}(x+\delta x)\delta y - \tau_{xx}(x)\delta y}{\delta x\delta y} + \frac{\tau_{yx}(y+\delta y)\delta x - \tau_{yx}(y)\delta x}{\delta x\delta y}$$

Expanding $\tau_{xx}(x+\delta x)$ and $\tau_{yx}(y+\delta y)$ around x and y, respectively (using Taylor series expansion):

$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y}$$

Momentum Equation

Similarly, the net viscous force in the y-dir. Per unit volume of the element:

$$\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}$$

For an ideal Newtonian viscous fluid, the viscous stresses are linearly proportional to the velocity gradients:

$$\tau_{xx} \equiv \tau_{11} = 2\mu \frac{\partial u}{\partial x}, \qquad \tau_{yy} \equiv \tau_{22} = 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{xy} \equiv \tau_{12} = 2\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \qquad \tau_{xy} = \tau_{yx}$$

In general:

$$\tau_{ij} = 2\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where μ is the dynamic viscosity.

Momentum Equation

The application of

$$\begin{cases} \tau_{xx} \equiv \tau_{11} = 2\mu \dfrac{\partial u}{\partial x} \\ \tau_{yy} \equiv \tau_{22} = 2\mu \dfrac{\partial v}{\partial y} \\ \tau_{xy} \equiv \tau_{12} = 2\mu \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \right) \end{cases}$$

in

$$\begin{cases} \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} & \textit{the net viscous force in the x-dir. Per unit volume} \\ \dfrac{\partial \tau_{yy}}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} & \textit{the net viscous force in the y-dir. Per unit volume} \end{cases}$$

yields to;

$2\mu \dfrac{\partial^2 u}{\partial x^2} + \mu \left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 v}{\partial x \partial y} \right)$ and $2\mu \dfrac{\partial^2 v}{\partial y^2} + \mu \left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} \right)$, respectively.

These expressions can be simplified using the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2}$$

$$2\mu \frac{\partial^2 u}{\partial x^2} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \rightarrow \qquad \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$2\mu \frac{\partial^2 v}{\partial y^2} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) \rightarrow \qquad \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

Momentum Equation

The force balance equations for an incompressible fluid ($\nabla \cdot V = 0$):

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$
$$-\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g = 0$$

To **eliminate the hydrostatic** pressure variation in Equation:
$P = p - \rho g y$

The pressure P is the pressure generated by fluid flow. With this substitution:

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$
$$-\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0$$

In compact form and in 3D-geometry:

$-\nabla P + \nabla^2 V = 0$ *Momentum Conservation*

Momentum Equation

The continuity equation for compressible fluid

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$ *Mass Conservation*

U: Characteristic Vel. **C: Sound Vel.**

Gough (1969)

If $M^2 = \frac{U^2}{C^2} \leq 1$ → $\frac{\partial}{\partial t} \rightarrow 0$ → $\nabla.(\rho V) = 0$ *M: Mach number*

$$\nabla.(\rho V) = \rho\nabla.V + V\nabla\rho$$

Momentum Equation in More General Form

$$F_{inertial} = F_P + F_{visc} + F_g$$

$$F_{inertial} = \rho\left(\frac{\partial V}{\partial t} + V\nabla \cdot V\right)$$

$$\rho\left(\frac{\partial V}{\partial t} + V\nabla \cdot V\right) = -\nabla p + \nabla^2 V + \rho\bar{g}$$

The Stream Function

Incompressible 2D-Fluid

Solving the Momentum Equation Using Stream Function Method

Define $u = -\frac{\partial \psi}{\partial y}$, $v = +\frac{\partial \psi}{\partial x}$

Substituting in **continuity** equation:

$-\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$ which shows the stream function ψ satisfies the continuity equation.

Substituting in **momentum** equations:

$$\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right) = 0$$

$$-\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial y^2 \partial x}\right) = 0$$

The Stream Function

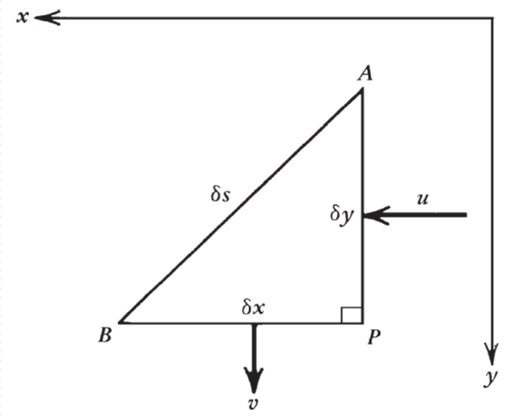
For a single differential equation for ψ, the pressure can be eliminated from these equations by taking the partial derivative of these equations with respect to *y* and x, respectively:

Substituting in momentum equations:

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0$$ *biharmonic equation*

The Stream Function

Ex. – The flow accord AB?



The flow across *AB* can be calculated from the flows across *AP* and *PB* because conservation of mass.

The volumetric flow rate across A*P* into the triangle per unit distance normal to the figure is: $u\delta y$

similarly the flow rate across *PB* out of the triangle is: $v\delta x$

The net flow rate out of *PAB* is thus: $-u\delta y + v\delta x$

This must be equal to the volumetric flow rate into *PAB* across *AB*.

$u = -\frac{\partial\psi}{\partial y}$

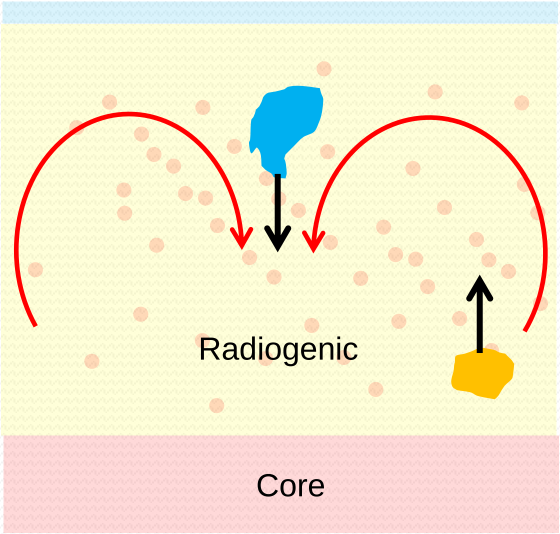
$v = +\frac{\partial\psi}{\partial x}$ $\rightarrow$ $-u\delta y + v\delta x = \frac{\partial\psi}{\partial y}\delta y + \frac{\partial\psi}{\partial x}\delta x \equiv d\psi(x,y)$ *the volumetric flow rate between A and B*

$\int_A^B d\psi = \psi_A - \psi_B$ *for A and B at arbitrary distance*

Thermal Convection

Plate tectonics is the consequence of **thermal convection** in the mantle, driven largely by **radiogenic** heat sources and the **cooling** of the Earth.

Thermal convection is the consequence of a change in density by a change in temperature (thermal expansion). This situation is **gravitationally instable** and the cool fluid tends to **sink** and the hot fluid **rises**.



Density variations caused by thermal expansion lead to the **buoyancy forces** that drive thermal convection.

$$\rho = \rho_0 + \rho' \quad \rightarrow \quad \rho' = \rho - \rho_0 \qquad F_B = \rho' g$$

$$F_g = \rho_0 g \quad \rightarrow F_g + F_B$$

$\rho_0 g \quad \rightarrow \quad \rho_0 g + \rho' g$ *in momentum conservation equation*

$\rho' \ll \rho_0$ ρ_0: *reference density*

Thermal Convection

In all other respects, however, the density variations are sufficiently small so that they can be neglected. This is known as the ***Boussinesq approximation***.

It allows us to use the **incompressible** conservation of fluid equation:

$$\nabla \cdot V = 0$$

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$-\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g = 0 \quad \rightarrow \quad -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g + \rho' g = 0$$

Eliminating the hydrostatic pressure by introducing:

$$P = p - \rho_0 g y$$

Thermal Convection

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$-\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho' g = 0$$

$$\rho' = -\rho_0 \alpha (T - T_0)$$

α: *volumetric coefficient of thermal expansion*

T_0: *reference temperature*

$$-\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \underbrace{g\rho_0 \alpha (T - T_0)}_{\text{Buoyancy force per unit volume}} = 0$$

To find the velocity field from momentum conservation equations, we **need the temperature T.** Therefore we require the **heat equation (energy equation)** that governs the variation of temperature.

Heat transfer = Conduction + Convection

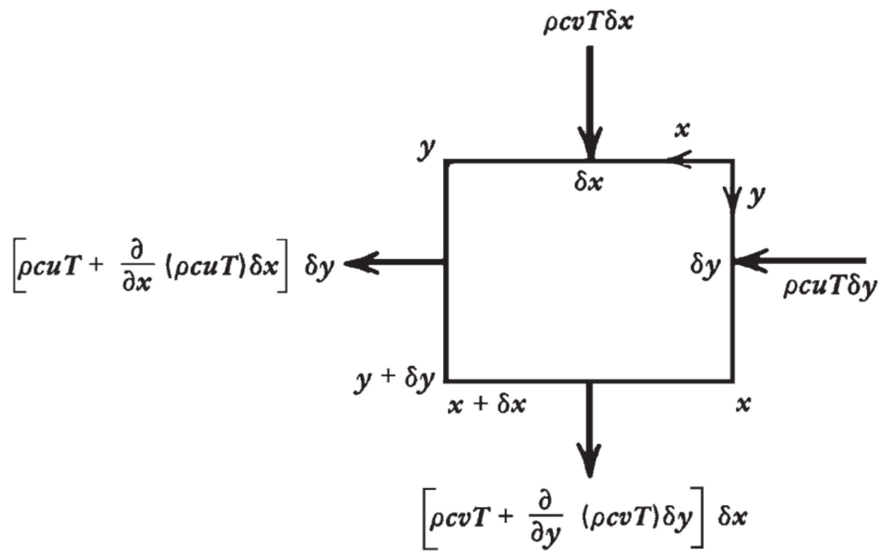
Thermal Convection

Thermal energy per unit volume: $\rho c T$ ($\rho c T u$ *energy flux or energy flow per unit area*)
Amount of heat transported across δy at x: $\rho c T u \delta y$ per unit time, crossing δy

Heat flux at $x + \delta x$: $\rho c T u + \frac{\partial(\rho c T u)}{\partial x}\delta x$

The **net energy advected** out of the elemental volume per unit time and per unit depth due to flow in the x direction is thus:

$$\left[\left[\rho c T u + \frac{\partial(\rho c T u)}{\partial x}\delta x\right] - \rho c T u\right]\delta y$$
$$= \frac{\partial(\rho c T u)}{\partial x}\delta x \delta y$$



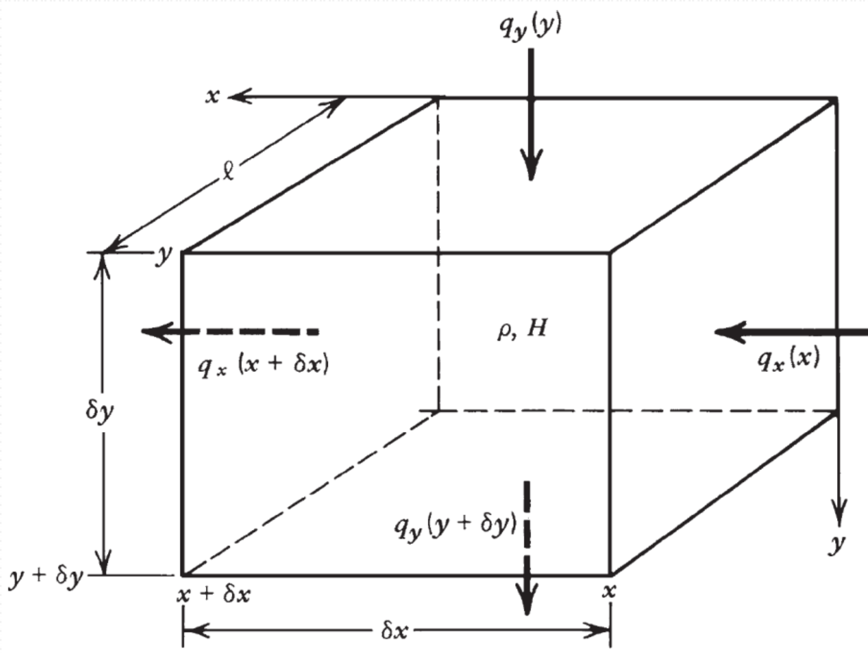
Similarly in y direction:

$$\left[\left[\rho c T v + \frac{\partial(\rho c T v)}{\partial y}\delta y\right] - \rho c T v\right]\delta x$$
$$= \frac{\partial(\rho c T v)}{\partial y}\delta x \delta y$$

Thermal Convection

The **net rate of heat advection out of the element** by flow in both directions is:

$$\left[\frac{\partial(\rho c T u)}{\partial x} + \frac{\partial(\rho c T v)}{\partial y}\right] \delta x \delta y$$



Heat Conduction

Heat flux in x direction at x: $q_x(x)$
Heat flux in x direction at $x + \delta x$: $q_x(x + \delta x)$

Heat flux in y direction at y: $q_y(y)$
Heat flux in y direction at $y + \delta y$: $q_y(y + \delta y)$

The net heat flow rate out of the element is:

$$[q_x(x + \delta x) \ - q_x(x)\ ]\delta y + [q_y(y + \delta y) \ - q_y(y)\ ]\delta x$$
$$= \frac{\partial q_x}{\partial x}\delta x \delta x + \frac{\partial q_y}{\partial y}\delta x \delta x = \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right)\delta x \delta x$$ *using Taylor expansion*

Thermal Convection

Steady State

In steady state

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0$$

In the presence of internal heating the rate of heat generation in the element is: $\rho H \delta x \delta x$

and

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = \rho H$$ and in 3D: $$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = \rho H$$

Fourier's Law of Conduction

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y}, \qquad q_z = -k\frac{\partial T}{\partial z}$$ *for isotropic medium*

→ $$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \rho H$$ or $$-k\nabla^2 T = \rho H$$

Thermal Convection

We had obtained the advection term:

$$\left[\frac{\partial(\rho cTu)}{\partial x}+\frac{\partial(\rho cTv)}{\partial y}\right]\delta x\delta y$$

And the conduction term:

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)\delta x\delta y$$

Energy Conservation

The **combined transport** of energy out of the elemental volume by conduction and convection must be **balanced** by the **change in the energy content** of the element. The thermal energy of the fluid is ρcT **per unit volume**. Thus, this quantity changes at the rate:

$$\frac{\partial(\rho cT)}{\partial t}\delta x\delta y$$

Thermal Convection

By combining the effects of conduction, convection, and *thermal inertia*, we obtain:

$$\underbrace{\frac{\partial(\rho c T)}{\partial t}}_{\text{Rate of change in heat}} - \underbrace{k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{\text{Conduction}} + \underbrace{\frac{\partial(\rho c T u)}{\partial x} + \frac{\partial(\rho c T v)}{\partial y}}_{\text{Convection}} = 0$$

energy balance

For constant ρ and c and noting that

$$\frac{\partial(Tu)}{\partial x} + \frac{\partial(Tv)}{\partial y} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}$$

$$\nabla \cdot V = 0$$

The thermal diffusion is defined as:

$$\kappa = \frac{K}{\rho c}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t} + \mathrm{V} \cdot \nabla T = \kappa \nabla^2 T$$

Thermal Convection

In this derivation we have neglected:

a) frictional heating in the fluid associated with the resistance to flow
b) compressional heating associated with the work done by pressure forces in moving the fluid

Navier-Stokes Equations

Navier-Stokes Equations

We saw that the **force balance** on an small volume element of fluid leads to the equation for **conservation of momentum:**

$$F_P + F_{visc} + F_g = 0$$

$$-\frac{\partial p}{\partial x_i} + -\frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$ *i^th^ component*

According to Newton's second law of motion, any **imbalance of forces** on the fluid parcel results in an **acceleration** of the elemental parcel:

$$\underbrace{\rho \frac{Du_i}{Dt}}_{\textit{Inertial}} = \underbrace{-\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}}_{\textit{Surface}} + \underbrace{\rho g_i}_{\textit{Body}} = 0, \quad i = 1,2,3$$

forces

Body forces:
Gravity force
Electromagnetic force
Centrifugal force
Coriolis force

where $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_i}\frac{dx_i}{dt} \equiv \frac{\partial u}{\partial t} + (u \cdot \nabla)u$ *(total derivative)*

$$F = ma \quad \rightarrow \quad \rho \frac{Du_i}{Dt} \qquad mass \times acceleration$$

Navier-Stokes Equations

$$\nabla f = \frac{\partial f}{\partial x_1}\hat{\imath} + \frac{\partial f}{\partial x_2}\hat{\jmath} + \frac{\partial f}{\partial x_3}\hat{k}$$

Note that:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} \equiv \frac{\partial f}{\partial t} + u \cdot \nabla f$$ *for scalar function f*

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_i}\frac{dx_i}{dt} \equiv \frac{\partial u}{\partial t} + (u \cdot \nabla)u$$ *for the velocity vector u*

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i}\frac{dx_i}{dt} \equiv \frac{\partial F}{\partial t} + (u \cdot \nabla)F$$ *for vector function F*

Total derivative
Material derivative
Substantial derivative
Lagrangian derivative

Note also that:

$$\frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial f}{\partial x_i}u_i = \frac{\partial f}{\partial x_1}u_1 + \frac{\partial f}{\partial x_2}u_2 + \frac{\partial f}{\partial x_3}u_3 = \left(\frac{\partial f}{\partial x_1}\hat{\imath} + \frac{\partial f}{\partial x_2}\hat{\jmath} + \frac{\partial f}{\partial x_3}\hat{k}\right) \cdot \left(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}\right) = \nabla f \cdot u$$

$$(u \cdot \nabla)F = \left(u_i \frac{\partial}{\partial x_i}\right)F \qquad \rightarrow \qquad (u \cdot \nabla)\, F_k = \left(u_i \frac{\partial}{\partial x_i}\right)F_k$$

Do not confuse total derivative with mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$ *Mass Conservation*

Navier-Stokes Equations

Also

$$(u \cdot \nabla)F = \left[\left(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}\right) \cdot \left(\hat{\imath}\frac{\partial}{\partial x_1} + \hat{\jmath}\frac{\partial}{\partial x_1} + \hat{k}\frac{\partial}{\partial x_1}\right)\right]\left(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}\right) =$$

$$\left(u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}\right)\left(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}\right) =$$

$$\left(u_1\frac{\partial F_1}{\partial x_1} + u_2\frac{\partial F_1}{\partial x_2} + u_3\frac{\partial F_1}{\partial x_3}\right)\hat{\imath} + \left(u_1\frac{\partial F_2}{\partial x_1} + u_2\frac{\partial F_2}{\partial x_2} + u_3\frac{\partial F_2}{\partial x_3}\right)\hat{\jmath} + \left(u_1\frac{\partial F_3}{\partial x_1} + u_2\frac{\partial F_3}{\partial x_2} + u_3\frac{\partial F_3}{\partial x_3}\right)\hat{k}$$

Navier-Stokes Equations

In the **absence of flow,** the only surface force is the **pressure force:**

$$-\frac{\partial p}{\partial x_i}$$

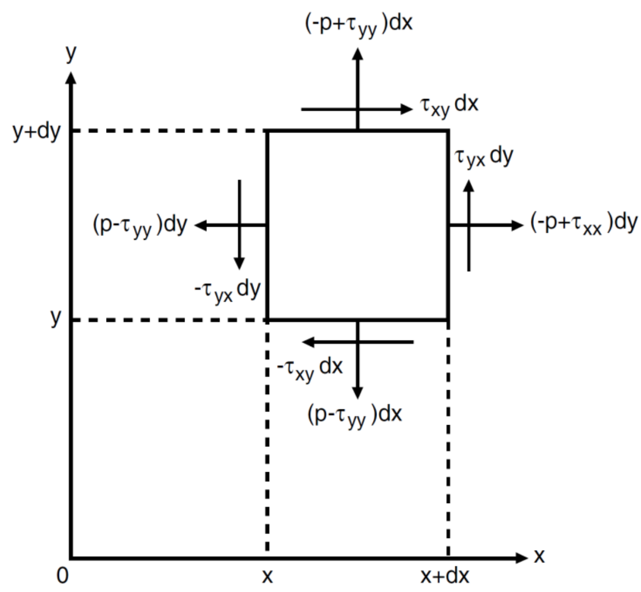
With flow, additional **deviatoric** forces act on the surface of an elemental parcel:

$$\frac{\partial \tau_{ij}}{\partial x_j} \sim$$ *gradient of the velocities*

In 2D

$$\underbrace{\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)} = -\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}$$

$$\underbrace{\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)}_{\text{inertial term}} = -\frac{\partial p}{\partial y}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\rho g$$



$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$ *strain tensor*

Constitutive Law - Newtonian Fluid

Newtonian fluid is a fluid for which the **dependence** of τ_{ij} on ε_{ij} is **linear**.

In addition if the medium is also **isotropic** (the constants of proportionality in the deviatoric stress–strain rate relation are **independent of the orientation** of coordinate system axes), then

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

where μ is the **dynamic viscosity**, and λ is the **second viscosity.**

$$\rightarrow \quad \frac{\tau_{ii}}{3} = \left(\lambda + \frac{2}{3}\mu\right)\varepsilon_{ii} \equiv k_B\varepsilon_{ii} \quad \rightarrow \quad \lambda = \left(k_B - \frac{2}{3}\mu\right) \qquad (\delta_{ii} = \delta_{11} + \delta_{11} + \delta_{11} = 3)$$

where $\boldsymbol{k_B}$ is called **bulk viscosity**, a **measure of dissipation** under compression or expansion.

Navier-Stokes Equations

Combining these two equations:

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \left(k_B - \frac{2}{3}\mu\right)\varepsilon_{kk}\delta_{ij} \quad \rightarrow \quad \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(k_B - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

And the momentum equation:

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(k_B - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] + \rho g_i = 0$$

For many fluids k_B is very small. $\boldsymbol{k_B \approx 0}$ ***Stokes assumption***

Constitutive Law with $\boldsymbol{k_B \approx 0}$

With $k_B \approx 0$, the constitutive or rheological law connecting deviatoric stress and strain rate becomes:

$$\tau_{ij} = 2\mu\varepsilon_{ij} - \frac{2}{3}\mu\varepsilon_{kk}\delta_{ij} \quad \rightarrow \quad \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$$

Navier-Stokes Equations

The Navier–Stokes equation

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right] + \rho g_i = 0$$

Incompressible flow

For incompressible flow $\nabla \cdot u = \frac{\partial u_k}{\partial x_k} = 0$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \rho g_i = 0$$

If the dynamic viscosity (μ) is constant:

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\left[\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i}\cancel{\frac{\partial u_j}{\partial x_j}}\right)\right] + \rho g_i = 0$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \rho g_i = 0$$

Energy Equation

For a simple case, in the absence of internal heating and viscous dissipation, and where the density and heat capacity were constants, we had obtained the energy balance equation as:

$$\frac{DT}{Dt} = \kappa \nabla^2 T \qquad \text{or} \qquad \frac{\partial T}{\partial t} + \mathrm{V} \cdot \nabla T = \kappa \nabla^2 T$$

In the presence of viscous dissipation and internal heat sources, and variable thermal conductivity, the energy conservation relation can be written as:

$$\underbrace{\rho c_P \frac{DT}{Dt}}_{\text{Rate of Change In energy}} = \underbrace{\nabla \cdot (K \nabla T)}_{\text{By conduction}} + \underbrace{\Phi}_{\text{By dissipation}} + \underbrace{\rho H}_{\text{By internal heating}} \qquad \textit{Energy balance}$$

Where H is the rate of internal heat production per unit mass and $\Phi \equiv \tau_{ij} \frac{\partial u_i}{\partial x_j}$ is viscous dissipative heat.

Change in velocity

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{\text{By time}} + \underbrace{u \cdot \nabla T}_{\text{By advection}}$$

Navier-Stokes Equations

For a **Newtonian** fluid:

Since $\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(k_B - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij},$ then

$$\Phi = \left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(k_B - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\frac{\partial u_i}{\partial x_j}$$

$$\rightarrow \Phi = k_B\left(\frac{\partial u_k}{\partial x_k}\right)^2 + 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}\right]^2$$

If the fluid is incompressible:

$$\Phi = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2$$

a) The **bulk viscosity** $k_B = \left(\lambda + \frac{2}{3}\mu\right)$ leads to dissipation due to **volume changes** in a deforming fluid.
b) The dynamic viscosity μ leads to dissipation **through shear.** Note that there are **no volume changes** associated with the bracketed tensor in the second term.

i.e., $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} = 0$ *for i=j*

Navier-Stokes Equations

It can be shown that for **compressible flow**:

$$\rho c_P \frac{DT}{Dt} - \alpha T \frac{DP}{Dt} = \nabla \cdot (K \nabla T) + \Phi + \rho H$$

$$\rho c_P \left(\frac{\partial T}{\partial t} + u \cdot \nabla T \right) - \alpha T \left(\frac{\partial P}{\partial t} + u \cdot \nabla P \right) = \nabla \cdot (K \nabla T) + \Phi + \rho H$$

$$\rho c_P \left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P} \frac{\partial P}{\partial t} + u \cdot \left(\nabla T - \frac{\alpha T}{\rho c_P} \nabla P \right) \right] = \nabla \cdot (K \nabla T) + \Phi + \rho H$$

$$\rho c_P \left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P} \frac{\partial P}{\partial t} + u \cdot (\nabla T - \nabla T_S) \right] = \nabla \cdot (K \nabla T) + \Phi + \rho H$$

where $\nabla T_S \equiv \frac{\alpha T}{\rho c_P} \nabla P$ *Adiabatic temperature gradient*

$$\nabla P = 0\hat{\imath} + 0\hat{\jmath} + \frac{\partial P}{\partial z} \hat{k} \approx \frac{\partial(-\rho g z)}{\partial z} \hat{k} \rightarrow \nabla T_S \approx -\frac{g\alpha}{c_P} T$$

and $\Phi \equiv \tau_{ij} \frac{\partial u_i}{\partial x_j}$

$$\boldsymbol{\rho c_P \left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P} \frac{\partial P}{\partial t} + u \cdot (\nabla T - \nabla T_S) \right] = \nabla \cdot (K \nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j} + \rho H}$$

Basic Equations

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho V) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P\left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P}\frac{\partial P}{\partial t} + u \cdot \ (\nabla T - \nabla T_S)\right] = \nabla \cdot (K \nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho = \rho_r\left[1 - \alpha(T - T_r) + \frac{1}{K_T}(P - P_r)\right] + \Delta\rho_i(\Gamma_i - \Gamma_{ri})$$ *i =1,2,3*

$$\Gamma_i = \frac{1}{2}[1 + \tanh(\pi_i)]$$

$$\pi_i = \frac{d_i - d - \gamma_i(T - T_i)}{h_i}$$

State Equation

$\rho(P,T) = ?$

Taylor expansion

$$\rho(P,T) = \rho_r + \left(\frac{\partial\rho_r}{\partial T}\right)_P (T - T_r) + \left(\frac{\partial\rho_r}{\partial P}\right)_T (P - P_r) \qquad where \quad \rho_r \equiv \rho(P_r, T_r)$$

P_r: *hydrostatic pressure*
T_r: *adiabatic temperature*

But

$$\alpha = \frac{1}{\rho_r}\left(\frac{\partial\rho_r}{\partial T}\right)_P \text{ , and } \quad K_T = \rho_r\left(\frac{\partial P}{\partial\rho_r}\right)_T$$

$$\rho(P,T) = \rho_r\left[1 - \alpha(T - T_r) + \frac{(P-P_r)}{K_T}\right]$$

State Equation

$$\nabla\rho_r(S, P_H) = \left(\frac{\partial\rho_r}{\partial S}\right)_{P_H} \nabla S + \left(\frac{\partial\rho_r}{\partial P_H}\right)_S \nabla P_H$$

Since reference state is adiabatic: $\nabla S = 0$

and since $\nabla P_H = 0 + 0 + \frac{dP_H}{dz}\hat{k} = -\rho_r g\hat{k}$

and $\nabla\rho_r(S, P_H) = 0 + 0 + \frac{d\rho_r}{dz}\hat{k}$

$\frac{1}{\rho_r}\frac{d\rho_r}{dz} = -g\left(\frac{\partial\rho_r}{\partial P_H}\right)_S = -\frac{g\rho_r}{K_s}$ where $K_S = \rho_r\left(\frac{\partial P_H}{\partial\rho_r}\right)_S$

Gruneisen's parameter is defined as: $\Gamma = \frac{\alpha K_S}{\rho C_P} = \frac{\alpha K_T}{\rho C_V}$

then:

$\frac{1}{\rho_r}\frac{d\rho_r}{dz} = -\frac{g\alpha}{\Gamma C_P} = -\frac{1}{\Gamma H_T}$ where $H_T = \frac{C_P}{g\alpha}$ *scale height*

This is the well-known **Adams-Williamson** relation for a chemically homogeneous adiabatic density distribution under hydrostatic pressure (Birch 1952).

State Equation

Spiegel & Veronis (1960) gave criteria for the applicability of the **Boussinesq** approximation to compressible fluids:

$\frac{d}{H_T} \ll 1$ *for shallow layers,* $d: characteristc\ length$

For constant Γ: $\frac{1}{\rho_r}\frac{d\rho_r}{dz} = -\frac{1}{\Gamma H_T}$ → $\int \frac{d\rho_r}{\rho_r} = -\int_d^z \frac{dz}{\Gamma H_T}$

$\ln\left[\frac{\rho_r(z)}{\rho_0(z)}\right] = \frac{d-z}{\Gamma H_T}$ → $\rho_r(z) = \rho_0\, exp(d-z)/\Gamma H_T$

where $\rho_0 = \rho_r(z = d)$ is the density at the upper surface (*bottom: z=0*).

$\Gamma \approx 1.1$
$\rho_r(z = 0) = \ \rho_0\, exp(d/\Gamma H_T)$
$\rho_r(z = d) = \rho_0$
$\Delta\rho \approx 0\ if\ \frac{d}{H_T} \ll 1$

State Equation

We also have:

$\frac{1}{K_T} = \frac{\alpha}{\rho_r C_v}$ and $C_V = \frac{C_P}{1+\alpha\Gamma T_r}$

$$\begin{cases} \rho(P,T) = \rho_r\left[1 - \alpha(T - T_r) + \frac{(P-P_r)}{K_T}\right] \\ \rho_r(z) = \rho_0\, exp(d - z)/\Gamma H_T \end{cases}$$

$\rightarrow \rho(P,T) = \rho_r[1 - \alpha(T - T_r) + \alpha[(1 + \alpha\Gamma T_r)/(\Gamma\rho_r C_P)](P - P_r)]$

Note that $\rho_r = \rho_0\, exp(d - z)/\Gamma H_T$

In non-dimensional form $\rho_r{}' = exp[(1 - z')D/\Gamma]$ where $D = d/H_T$

$z' = z/d, \quad \rho' = \rho/\rho_0,$

State Equation

The non-dimensional form can be written as:

$$\rho(P,T) = \rho_r\left[1 - \mu(T - T_r) + \mu D(1/(\Gamma\rho_r))(P - P_r) + \mu^2 D[\Gamma\,(T_r + T_0)/(\Gamma\rho_r)](P - P_r)\right]$$

$\mu = \alpha\Delta T$, $D = d/H_T$

For liquids: $\mu \ll 1$
For shallow depths: $D \ll 1$
For dilute gases: $\mu \approx 1$

For Boussinesq approximation: $\mu \ll 1$, $D \ll 1$

$$\rho(P,T) = \rho_r[1 - \mu(T - T_r)]$$

For anelastic liquid approximation: $\mu \ll 1$, $D \sim 1$

$$\rho(P,T) = \rho_r[1 - \mu(T - T_r) + \mu D(1/(\Gamma\rho_r))(P - P_r)]$$

Basic Equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P\left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P}\frac{\partial P}{\partial t} + u \cdot \ (\nabla T - \nabla T_S)\right] = \nabla \cdot (K \nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho(P,T) =$$
$$\rho_r\left[1 - \mu(T - T_r) + \mu D(1/(\Gamma \rho_r))(P - P_r) + \mu^2 D[\Gamma\,(T_r + T_0)/(\Gamma \rho_r)](P - P_r)\right]$$

Anelastic Equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P\left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P}\frac{\partial P}{\partial t} + u \cdot \ (\nabla T - \nabla T_S)\right] = \nabla \cdot (K \nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho(P,T) =$$
$$\rho_r\left[1 - \mu(T - T_r) + \mu D(1/(\Gamma \rho_r))(P - P_r) + \mu^2 D[\Gamma\,(T_r + T_0)/(\Gamma \rho_r)](P - P_r)\right]$$

Anelastic Liquid Equations

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho V) = 0$$

$$\rho \left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i \right) = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P \left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P} \frac{\partial P}{\partial t} + u \cdot \ (\nabla T - \nabla T_S) \right] = \nabla \cdot (K \nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho(P, T) =$$
$$\rho_r \left[1 - \mu(T - T_r) + \mu D(1/(\Gamma \rho_r))(P - P_r) + \mu^2 D[\Gamma\,(T_r + T_0)/(\Gamma \rho_r)](P - P_r) \right]$$

Truncated Anelastic Liquid Equations

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho V) = 0$$

$$\rho \left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P \left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P} \frac{\partial P}{\partial t} + u \cdot \ (\nabla T - \nabla T_S) \right] = \nabla \cdot (K \nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho(P, T) =$$
$$\rho_r \left[1 - \mu (T - T_r) + \mu D (1/(\Gamma \rho_r))(P - P_r) + \mu^2 D [\Gamma \, (T_r + T_0)/(\Gamma \rho_r)](P - P_r) \right]$$

Extended Boussinesq Equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P\left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P}\frac{\partial P}{\partial t} + u \cdot\ (\nabla T - \nabla T_S)\right] = \nabla \cdot (K\nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho(P,T) =$$
$$\rho_r\left[1 - \mu(T - T_r) + \mu D(1/(\Gamma\rho_r))(P - P_r) + \mu^2 D[\Gamma\,(T_r + T_0)/(\Gamma\rho_r)](P - P_r)\right]$$

$$\rho_r = \rho_{surf} = Const.$$

Boussinesq Equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i = 0$$

$$\rho c_P\left[\frac{\partial T}{\partial t} - \frac{\alpha T}{\rho c_P}\frac{\partial P}{\partial t} + u \cdot\ (\nabla T - \nabla T_S)\right] = \nabla \cdot (K\nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho H$$

$$\rho(P,T) =$$
$$\rho_r\left[1 - \mu(T - T_r) + \mu D(1/(\Gamma\rho_r))(P - P_r) + \mu^2 D[\Gamma\,(T_r + T_0)/(\Gamma\rho_r)](P - P_r)\right]$$

$$\rho_r = \rho_{surf} = Const.$$

Energy Regime

$$Q_{Surf} = Q_{Sec,Mant} + Q_{Rad} + Q_{CMB}, \qquad Q_{Sec,Mant} = \left(MC\frac{dT}{dt}\right)_{Mantle}$$

$$Q_{CMB} = Q_{Sec,Core} + Q_L + Q_G \qquad Q_{Sec,Core} = \left(MC\frac{dT}{dt}\right)_{Core}$$

Q_{Surf} *Surface heat flow (W)*

$Q_{Sec,Mant}$ *Secular cooling of mantle*

Q_{Rad} *Radiogenic heat*

Q_{CMB} *CMB heat flow*

$Q_{Sec,Core}$ *Secular cooling of core*

Q_L *Latent heat flow from the inner core boundary due to solidification*

Q_G *Gravitational heat flow from the inner core boundary due to solidification*

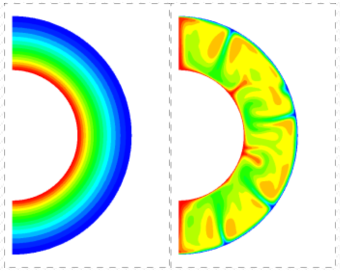
Rayleigh number

The Rayleigh number (Ra) for a fluid is a **dimensionless** number associated with buoyancy-driven flow that characterises the fluid's flow regime. **Lower** values denote **laminar** flow; and **higher** values denote **turbulent** flow. Below a **threshold** value (**critical Rayleigh number**), there is no fluid motion and heat transfer is by conduction rather than convection.

$$Ra_T = \frac{\rho g \alpha T d^3}{\nu K}$$

$$Ra_H = \frac{g \alpha H d^5}{\nu \kappa K}$$

$Ra_T = 10^7 \; for\ mantle$



$low\ Ra$ $\quad$ $High\ Ra$

ρ: $density\ (kg/\,m^3)$

ν: $Kinematic\ viscosity (m^2/s)$ $\qquad$ $\mu = \nu\rho$: $dynamic\ viscosity\ \left(Pa \cdot s;\ N \cdot s/m^2;\ kg/(m \cdot s)\right)$

d: $characteristc\ length\ (m)$

g: $gravity\ acc.\ (m/\,s^2)$

T: $temperature (Kelvin)$

α: $thermal\ expansivity (1/K)$

K: $thermal\ cond.\ (W/\,m\,/K)$

κ: $thermal\ diffusivity (m^2/s)$ $\quad \kappa = \frac{K}{\rho C_P} \left(\frac{W/\,m\,/K}{(kg/\,m^3)((J/\,kg\,/K))} = m^2/s \right)$

Prandtl number

The Prandtl number (Pr) is a dimensionless number, defined as the ratio of **momentum diffusivity** to **thermal diffusivity** (after the German physicist Ludwig Prandtl)

$$Pr = \frac{\nu}{\kappa} = \frac{viscous\ diff.\ rate}{thermal\ diff.\ rate} \qquad Pr \sim \frac{10^{20}}{10^{-6}} = 10^{26}\ for\ mantle$$

$\nu: Kinematic\ viscosity (m^2/s)$

$\kappa: thermal\ diffusivity (m^2/s) \quad \kappa = \frac{K}{\rho C_P} \left(\frac{W/m/K}{(kg/m^3)((J/kg/K))} = m^2/s \right)$

For mantle $\nu \gg \kappa \quad Pr \rightarrow \infty$

$Pr \ll 1: the\ thermal\ diffusivity\ dominates$

$Pr \gg 1: the\ momentum\ diffusivity\ dominates$

Ex. – In liquid mercury the **heat conduction** is more **significant** compared to **convection.**
For engine oil, **convection** is very **effective** in transferring **energy** (compared to pure conduction), so **momentum** diffusivity is **dominant**.

Nusselt Number

The Nusselt number (Nu) is the ratio of convective to conductive heat transfer at a boundary in a fluid. **Convection** includes both **advection** (fluid motion) and **diffusion** (conduction).

$$Nu = \frac{h}{K/d} = \frac{convective\ heat\ transfer\ coef.}{conductive\ heat\ transfer\ coef.}$$

$d: characteristc\ length\ (m)$
$h: convective\ heat\ transfer\ coef.\left(h = \frac{q}{\Delta T}\right)$
$q: heat\ flux\ (W/m^2)$
$\Delta T: temparature\ difference\ (Kelvin)$

$Nu = 1,\ pure\ conduction$
$Nu = 1 - 10,\ slug\ flow$
$Nu = 100 - 1000,\ turbulent\ flow$

Reynolds Number

The Reynolds number (Re) is a **measure** of the **flow patterns** in a fluid. **Laminar** flow (sheet-like) has **low Reynolds** number, while **turbulent** flow has **higher** values of Reynolds number.

$$Re = \frac{inertial\ forces}{viscous\ forces} = \frac{ma}{\tau A} = \frac{(\rho V)\cdot du/dt}{\mu\ du/dy\cdot A} = \frac{\rho d^3\cdot du/dt}{\mu du/dy\ d^2}$$
$$= \frac{\rho d\cdot dy/dt}{\mu} = \frac{\rho du}{\mu} = \frac{ud}{\nu}$$

d: $characteristc\ length\ (m)$
u: $velocity\ (m/s)$
μ: $dynamic\ viscosity\ (Pa\cdot s)$

Dimensionless Numbers

Peclet Number

The Peclet number is defined to be the ratio of the rate of advection to the rate of diffusion.

$$Pe = \frac{rate\ of\ advection}{rate\ of\ diffusion} = \frac{ud}{\kappa} = Re \times Pr$$

d: $characteristc\ length\ (m)$
u: $velocity\ (m/s)$
κ: $thermal\ diffusivity (m^2/s)$

Mach Number

The Mach number is ratio of convective velocity to sound velocity.

$$M = \frac{convective \ \ velocity}{sound \ \ velocity} = \frac{u}{c}$$

u: $velocity\ (m/s)$
c: $velocity\ (m/s)$

M^2 << 1 → a separation of time scales → elastic vibrations irrelevant on convective time scales.