ESS2222H

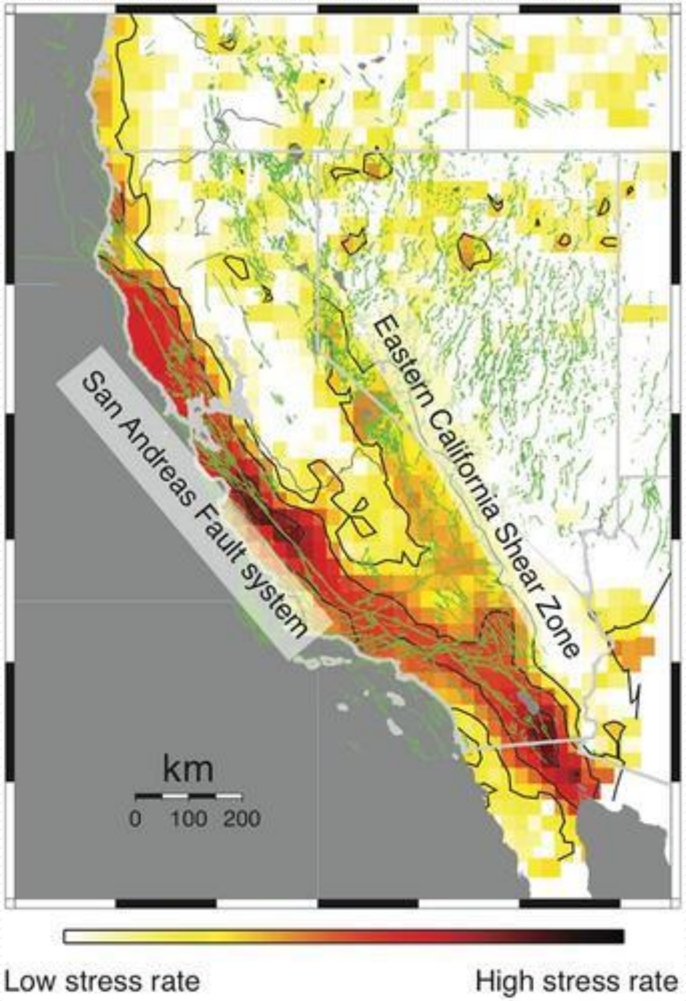
Tectonics and Planetary Dynamics
Lecture 4 – Stress, Strain - II

Hosein Shahnas

University of Toronto, Department of Earth Sciences,

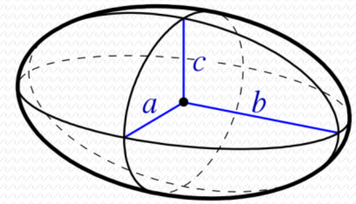
Stress, Deformation, and Strain

- ❑ **Stress & Strain Tensors**
- ❑ **Isobaric & Deviatoric Stress**
- ❑ **Principal Axes & Principal Stress**
- ❑ **Isotropic & Deviatoric Strain**
- ❑ **Mohr's Circle**



Stressing rate of the crust around California derived from two decades of geodetic measurements (USGS).

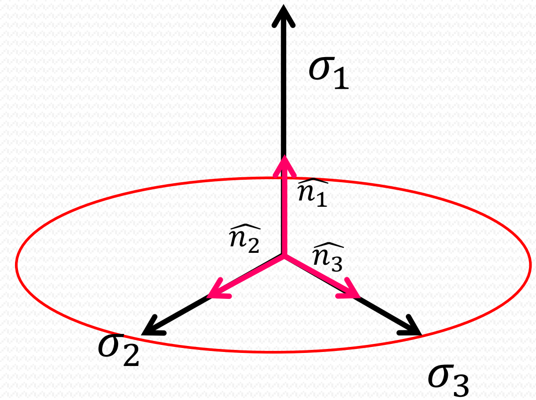
Symmetries



Axi-symmetric stress state

When two of the principal stresses are equal, only one of the principal directions will be unique.

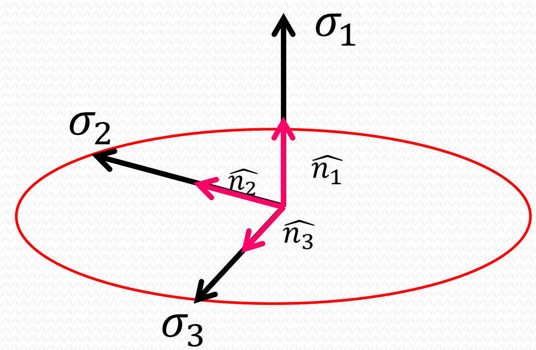
$\sigma_1, \quad \sigma_2 = \sigma_3$



Isotropic State of Stress (spherical symmetry)

When **all three components** of the principal stresses are **equal**, all directions are principal directions and the stress tensor has the form of

$$[\sigma_{ij}] = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$



In all coordinate systems

$\sigma_1 = \sigma_2 = \sigma_3 \equiv \sigma$

Two-Dimensional Problem

Two dimensional approximation

Geological problems involving stress can often be **approximated** to be **approximately two-dimensional.**

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

And any surface is defined by its unit normal and unit tangent

$$\hat{n}_i = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} Cos\ \theta \\ Sin\ \theta \end{pmatrix} \qquad \hat{t}_i = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} -Sin\ \theta \\ Cos\ \theta \end{pmatrix}$$

where n_i is the projection of $\hat{n}_i$ on $x_i - axis$

Using

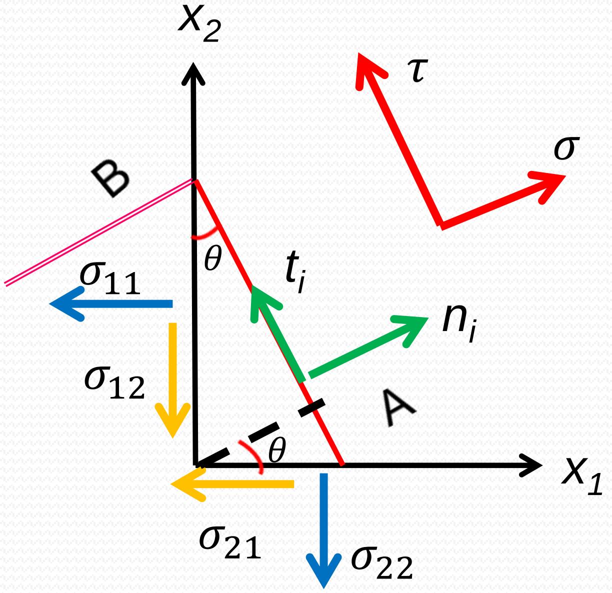
$T_i^n = \sigma_{ij}\, n_j$ **Cauchy's formula**

$\sigma = T_i^n n_i = \sigma_{ij}\, n_i\, n_j$ *normal stress*

We have

$$\sigma = \sigma_{11}\, n_1\, n_1 + \sigma_{12}\, n_1\, n_2 + \sigma_{21}\, n_2\, n_1 + \sigma_{22}\, n_2\, n_2$$

$$\sigma = \sigma_{11}\ Cos\ \theta^2 + \sigma_{12}\ Cos\ \theta\ \ Sin\ \theta + \sigma_{21} Sin\ \theta\ Cos\ \theta + \sigma_{22}\ Sin\ \theta^2$$



Two-Dimensional Problem

Using $Cos\,\theta\;\;Sin\,\theta \equiv \frac{1}{2}\,Sin\,2\theta$ identity and symmetry property ($\sigma_{12} = \sigma_{21}$):

$$\sigma = \sigma_{11}Cos^2\theta + 2\,\sigma_{12}\,Cos\,\theta\;\;Sin\,\theta + \;\sigma_{22}Sin^2\theta$$
$$\sigma = \sigma_{11}Cos^2\theta + \;\sigma_{12}\,Sin\,2\theta + \;\sigma_{22}Sin^2\theta$$

Using

$T_i^n = \sigma_{ij}\,n_j$ **Cauchy's formula**

$\tau = T_i^n t_i = \sigma_{ij}\,t_i\,n_j$ *shear stress*

We have

$$\tau = \sigma_{11}\,n_1\,t_1 + \;\sigma_{12}\,n_1\,t_2 + \;\sigma_{21}\,n_2\,t_1 + \;\sigma_{22}\,n_2\,t_2$$
$$\tau = -\,\sigma_{11}Sin\,\theta\;\;Cos\,\theta - \sigma_{12}Cos^2\theta - \;\sigma_{21}Sin^2\theta + \;\sigma_{22}Sin\,\theta\;\;Cos\,\theta$$

Using $Cos^2\theta = \frac{1}{2}(1 + Cos\,2\theta)$ and $Sin^2\theta = \frac{1}{2}(1 - Cos\,2\theta)$ identity

$$\sigma = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22})\;\;Cos\,2\theta + \sigma_{12}\,Sin\,2\theta$$
$$\tau = -\frac{1}{2}(\sigma_{11} - \sigma_{22})\,Sin\,2\theta + \sigma_{12}\,Cos\,2\theta$$

Two-Dimensional Problem

If the rotation coincides with the principal coordinates, then

$$\sigma = \sigma_x = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \ Cos\ 2\theta + \sigma_{12}\ Sin\ 2\theta$$

$$\boldsymbol{\tau = 0} = \frac{1}{2}(\sigma_{22} - \sigma_{11})\ Sin\ 2\theta + \sigma_{12}\ Cos\ 2\theta \quad \rightarrow \quad \theta = \frac{1}{2}\text{arctan}[2\sigma_{12}/(\sigma_{11} - \sigma_{22})]$$

Similarly writing the equations for plane perpendicular to the first plane:

with normal having angle $\theta + \frac{\pi}{2}$

$$\sigma = \sigma_{11} Sin^2\theta - \sigma_{12}\ Sin\ 2\theta + \sigma_{22} Cos^2\theta$$

$$\sigma = \sigma_y = \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \frac{1}{2}(\sigma_{11} - \sigma_{22}) \ Cos\ 2\theta - \sigma_{12}\ Sin\ 2\theta$$

$$\boldsymbol{\tau = 0} = \frac{1}{2}(\sigma_{11} - \sigma_{22})\ Sin\ 2\theta - \sigma_{12}\ Cos\ 2\theta \quad \rightarrow \quad 2\theta = \text{arctan}[2\sigma_{12}/(\sigma_{11} - \sigma_{22})]$$

Two-Dimensional Problem

With some algebra it can be shown:

$$Sin\ 2\theta = Sin(\arctan[2\sigma_{12}/(\sigma_{11} - \sigma_{22})]) = \frac{\frac{2\sigma_{12}}{(\sigma_{11}-\sigma_{22})}}{\sqrt{1+\left(\frac{2\sigma_{12}}{(\sigma_{11}-\sigma_{22})}\right)^2}}$$

$$Cos\ 2\theta = Cos(\arctan[2\sigma_{12}/(\sigma_{11} - \sigma_{22})]) = \frac{1}{\sqrt{1+\left(\frac{2\sigma_{12}}{(\sigma_{11}-\sigma_{22})}\right)^2}}$$

Eliminating θ from the main equations:

$$\sigma_x = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \left[\sigma_{12}^2 + \frac{1}{4}(\sigma_{11} - \sigma_{22})^2\right]^{1/2}$$

$$\sigma_y = \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \left[\sigma_{12}^2 + \frac{1}{4}(\sigma_{11} - \sigma_{22})^2\right]^{1/2}$$

Mohr’s Circle

The state of stress using the Mohr’s Circle (Otto Mohr)

Mohr's circle is a **two-dimensional graphical representation** of the transformation law for the Cauchy stress tensor.

In this method the **normal** and **shear** stresses acting on a **single plane** are represented by a single point on the Mohr circle.

The **normal** and shear **stresses** acting on **two perpendicular planes** are represented by two points, one at each end of a diameter on the Mohr circle.

We obtained two parametric equations (θ being parameter):

$$\sigma = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \; Cos\, 2\theta + \sigma_{12}\, Sin\, 2\theta$$

$$\tau = -\frac{1}{2}(\sigma_{11} - \sigma_{22})\, Sin\, 2\theta + \sigma_{12}\, Cos\, 2\theta$$

Eliminating $\boldsymbol{\theta}$ from these equations will yield the non-parametric equation of the Mohr circle:

Mohr's Circle

$$\left[\sigma - \frac{1}{2}(\sigma_{11}+\sigma_{22})\right]^2 = \frac{1}{4}(\sigma_{11}-\sigma_{22})^2 \ Cos^2 2\theta + (\sigma_{12} Sin2\theta\ )^2 + \frac{1}{2}(\sigma_{11}-\sigma_{22}) Cos2\theta\ \sigma_{12} Sin2\theta$$

$$\tau^2 = \frac{1}{4}(\sigma_{11}-\sigma_{22})^2 \ Sin^2 2\theta + (\sigma_{12} Cos2\theta\ )^2 - \frac{1}{2}(\sigma_{11}-\sigma_{22}) Cos2\theta\ \sigma_{12} Sin2\theta$$

+

$$\left[\sigma - \frac{1}{2}(\sigma_{11}+\sigma_{22})\right]^2 + \tau^2 = \frac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2$$ **Mohr's circle**

$$X^2 + Y^2 = R^2$$

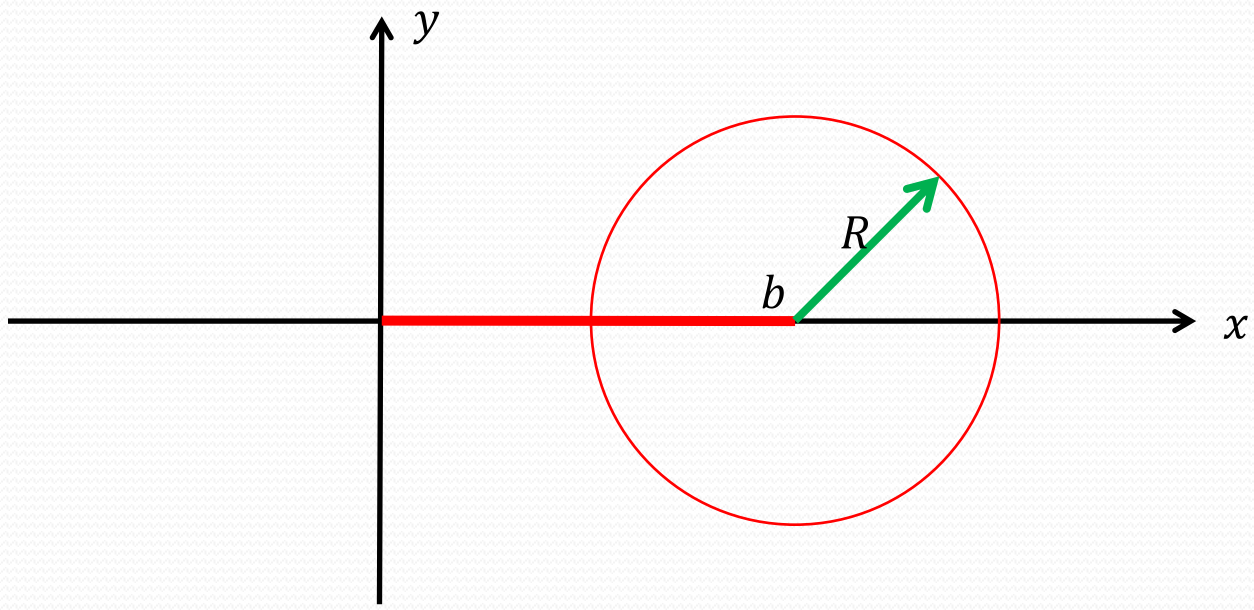
$$\boldsymbol{X} \equiv \sigma - \frac{1}{2}(\sigma_{11}+\sigma_{22}), \qquad \boldsymbol{Y} \equiv \tau, \qquad \boldsymbol{R} \equiv \sqrt{\frac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2}$$

$$\sigma_{ave} \equiv \frac{1}{2}(\sigma_{11}+\sigma_{22}) \quad \rightarrow \quad [\sigma - \sigma_{ave}]^2 + \tau^2 = R^2$$ *(Equation of circle)*

Mohr's Circle

$[\sigma - \sigma_{ave}]^2 + \tau^2 = R^2$ Mohr's circle centered at $(\sigma, \tau) = (\sigma_{ave}, 0)$

Note that



$$[y - b]^2 + x^2 = R^2$$

Mohr's Circle

$[\sigma - \sigma_{ave}]^2 + \tau^2 = R^2$ Mohr's circle centered at $(\sigma, \tau) = (\sigma_{ave}, 0)$

Stress Coordinates

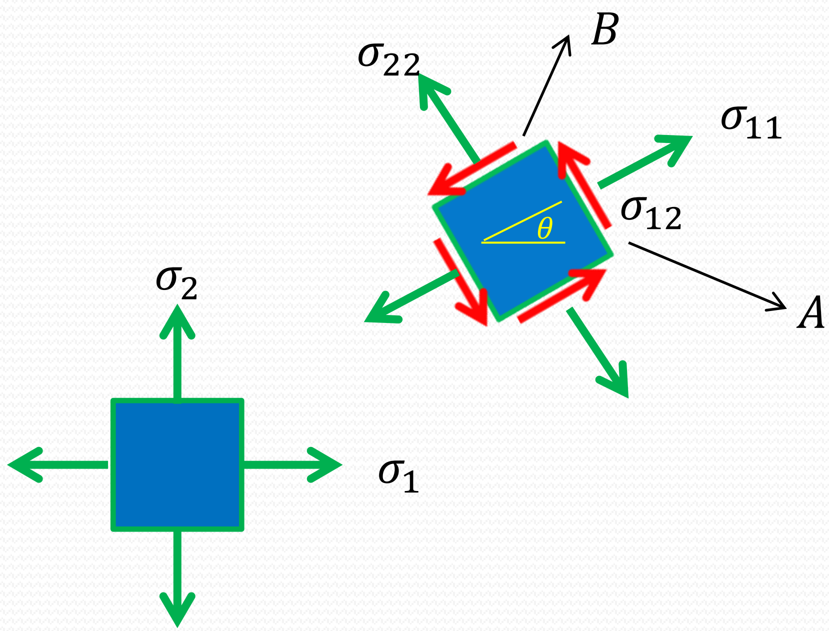
a) Normal stress σ: abscissa (horizontal)

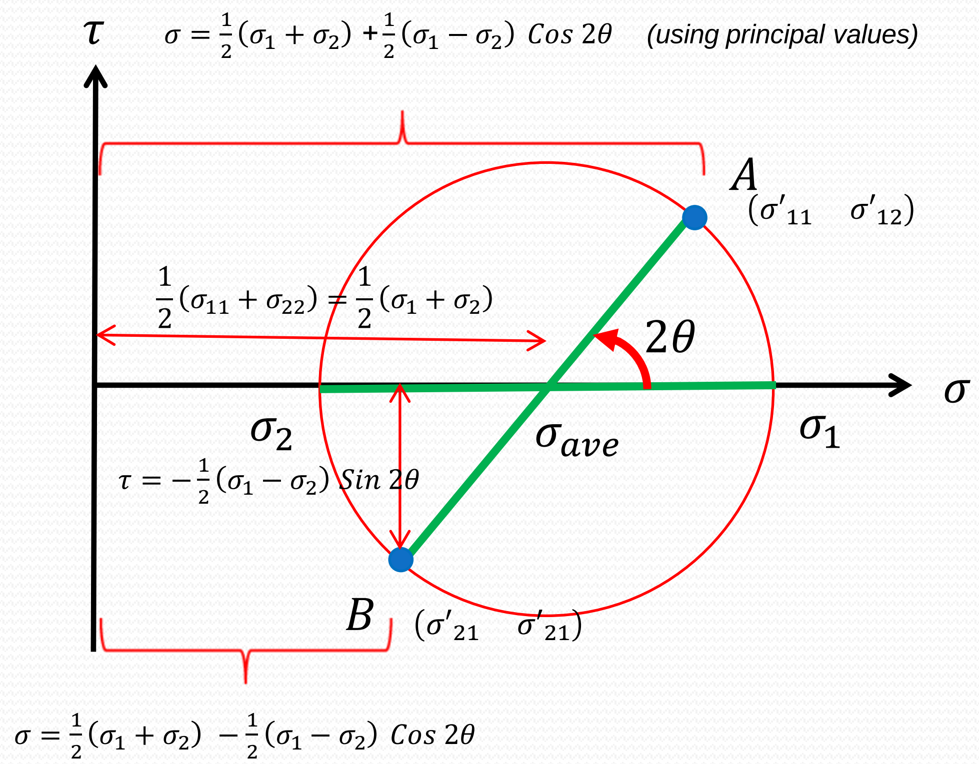
b) Shear stress τ: ordinate (vertical)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Note that

$$\boldsymbol{R} \equiv \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2} = \boldsymbol{\frac{1}{2}(\sigma_1 - \sigma_2)}$$





Ex –

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} -36 & 15 \\ 15 & 50 \end{pmatrix}$$

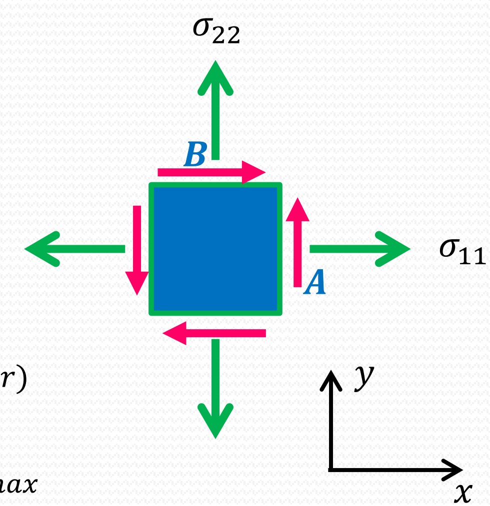
Steps to draw Mohr's circle

1 - $\sigma_{ave} = \frac{1}{2}(\sigma_{11} + \sigma_{22})$

2 - $R = \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2}$

3 - Plot circle centered at $(\sigma_{ave}, 0)$

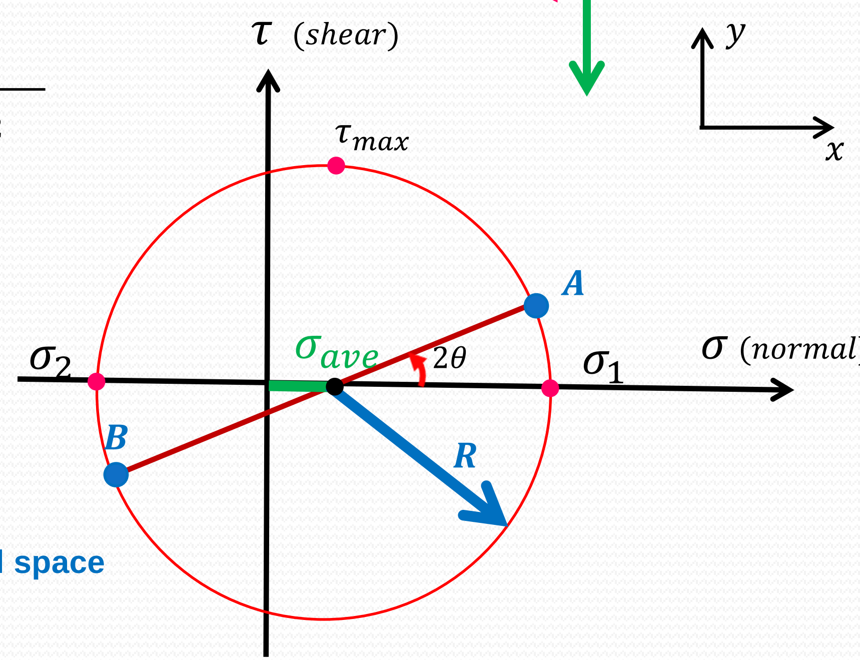
4 - Find A$(\sigma_{11}, \sigma_{12})$ and B$(\sigma_{22}, -\sigma_{21})$ on the Mohr's circle



$\boldsymbol{\sigma_{ave}} = \frac{1}{2}(-36 + 50) = 7$

$\boldsymbol{R} = \sqrt{\frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2} = \sqrt{\frac{1}{4}(-36 - 50)^2 + 15^2}$

$\boldsymbol{R} \approx 45.5$



Sign convention

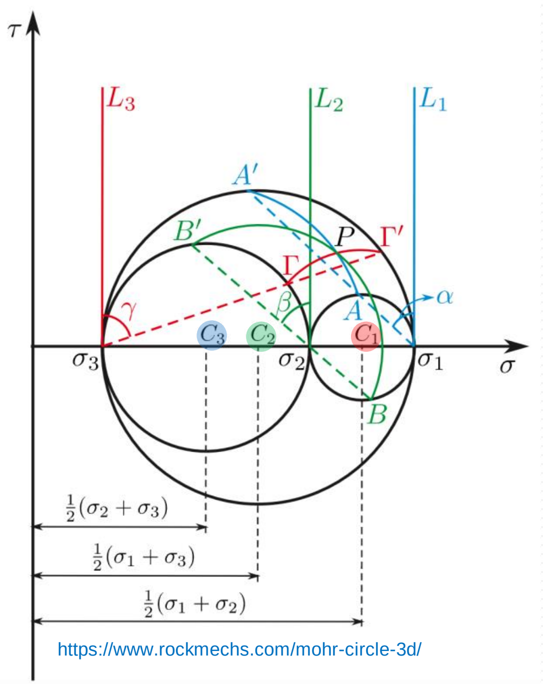
a) Shear stress upward
b) Rotation positive in counterclockwise
c) Reverse shear sign on Mohr's circle for the horizontal-face

Note that θ is the rotation angle to the principal axes in physical space corresponding to 2θ in Mohr's circle .

Mohr's Circle in 3D

Mohr Circle Diagram

$$[\sigma_{ij}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$



In principal coordinate

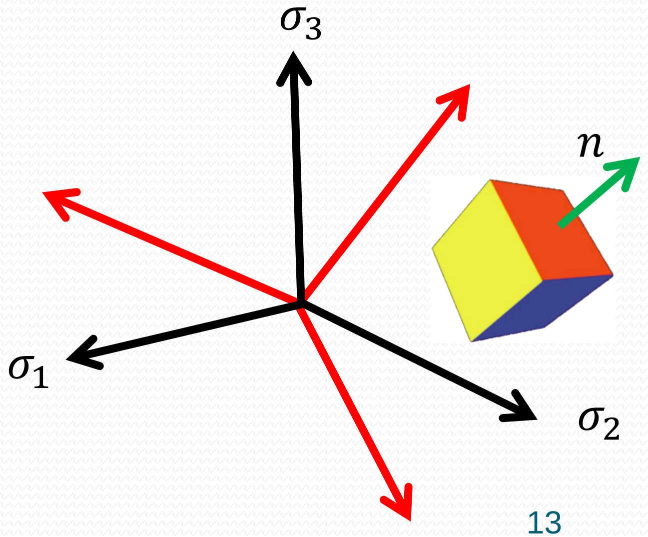
$$[\sigma_{ij}] = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$$C_1 = \frac{1}{2}(\sigma_1 + \sigma_2) \qquad R_1 = \frac{1}{2}(\sigma_1 - \sigma_2)$$

$$C_2 = \frac{1}{2}(\sigma_1 + \sigma_3) \qquad R_2 = \frac{1}{2}(\sigma_1 - \sigma_3)$$

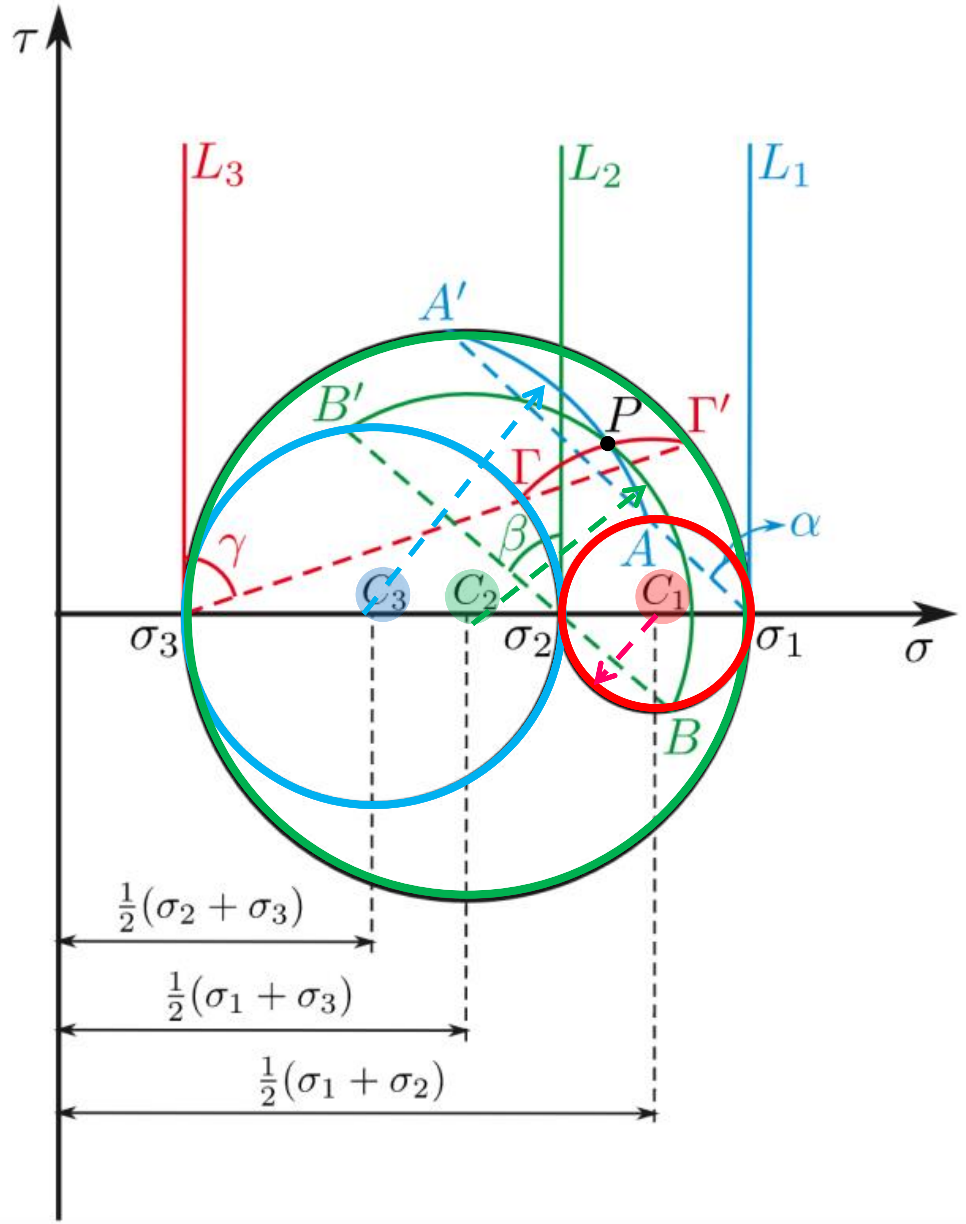
$$C_3 = \frac{1}{2}(\sigma_2 + \sigma_3) \qquad R_3 = \frac{1}{2}(\sigma_2 - \sigma_3)$$

$\boldsymbol{\alpha = Arc\ Cos(n_1)}$, $\boldsymbol{\beta = Arc\ Cos(n_2)}$, $\boldsymbol{\gamma = Arc\ Cos(n_3)}$



a) Draw L_1 parallel to τ passing through σ_1
b) Measure α from this line and draw AA'
c) Use center C_3 (which doesn't depend on σ_1 and draw arc AA'
d) Repeat in similar way for the other angles
e) The normal and shear components for the plane with normal n are given by the coordinates of the intersection point P

Mohr’s Circle in 3D



$$R_1 = \frac{1}{2}(\sigma_1 - \sigma_2)$$
$$R_2 = \frac{1}{2}(\sigma_1 - \sigma_3)$$
$$R_3 = \frac{1}{2}(\sigma_2 - \sigma_3)$$

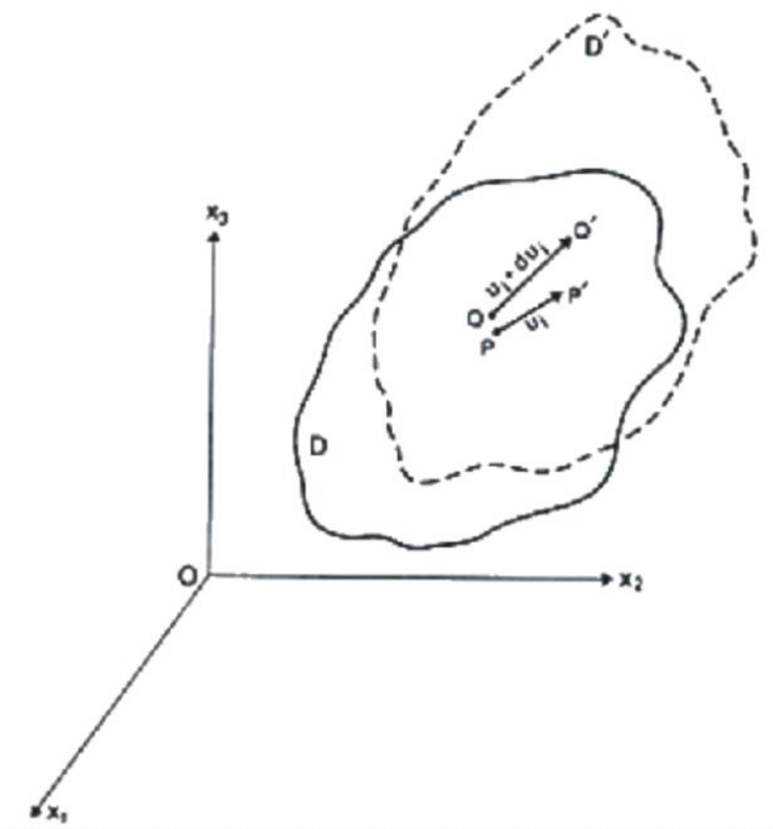
$$C_1 = \frac{1}{2}(\sigma_1 + \sigma_2)$$
$$C_2 = \frac{1}{2}(\sigma_1 + \sigma_3)$$
$$C_3 = \frac{1}{2}(\sigma_2 + \sigma_3)$$

The Infinitesimal Strain Tensor

Consider a continuous body occupying domain D and two neighbouring points $P(x_i)$ and $Q(x_i + dx_i)$

After displacement:

$P(x_i) \rightarrow P'(x_i + u_i)$
$Q(x_i + dx_i) \rightarrow Q'(x_i + dx_i + u_i + du_i)$
u_i and $u_i + du_i$ are displacements.



In general displacement of the particles results from:
1) Translation
2) Rotation
3) Deformation

Assume $u_i = u_i(x, y, z)$ and $\partial u_i / \partial x_j$ are small, then

$\mathrm{d}u_i \approx \frac{\partial u_i}{\partial x_j} \mathrm{d}x_j + \ldots$ (Taylor series)

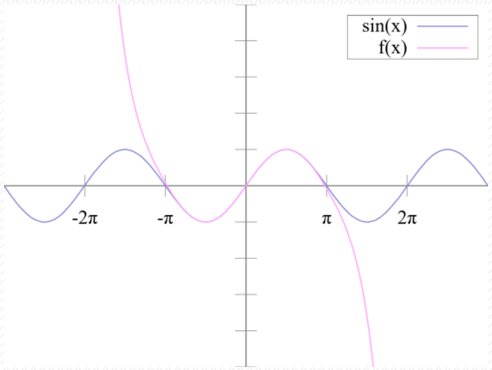
Strain Tensor

Remark: Taylor series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^{\,2} + \ldots = \sum_0^\infty \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example: Expansion of $Sin(x)$ around $x = 0$

$$Sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



$$f(x) - f(a) \approx \frac{f'(a)}{1!}(x-a) + \ldots$$

$$\mathrm{d}u_i \approx \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j + \ldots$$

Resolving the displacement into two parts:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

where

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ *infinitesimal strain tensor*

Strain Tensor

$$\omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$ *infinitesimal rigid body rotation*

Note that

$\varepsilon_{ij} = \varepsilon_{ji}$ *symmetric tensor*

$\omega_{ij} = -\omega_{ji}$ anti − *symmetric tensor*

Therefore

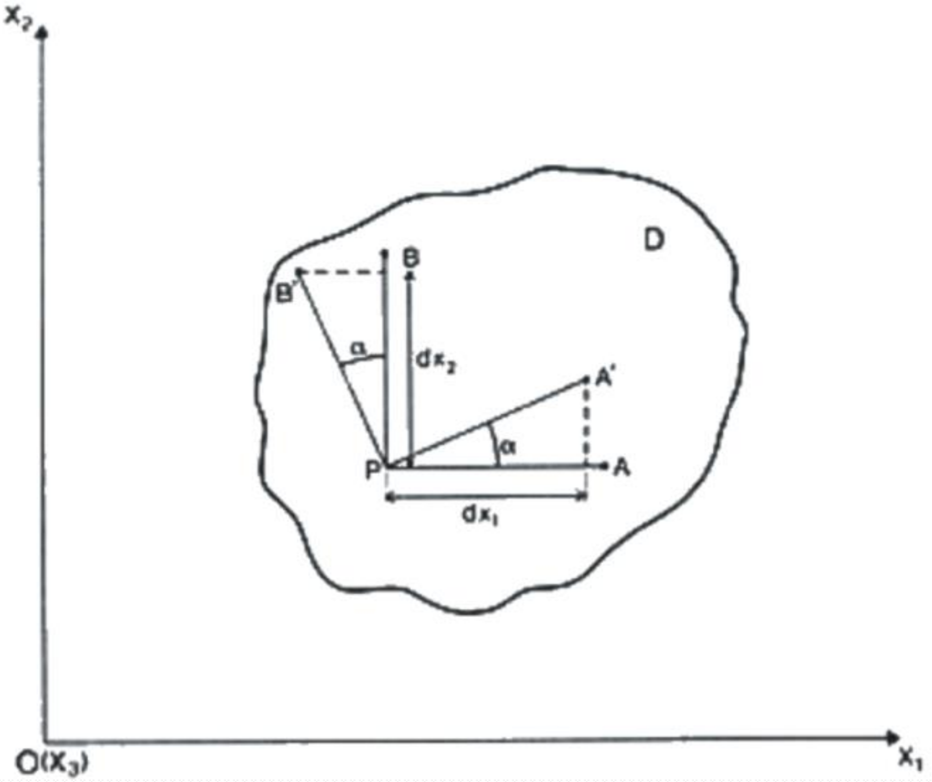
$$u_i + du_i = u_i + \frac{\partial u_i}{\partial x_j} dx_j =$$

$$u_i + \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

u_i: *rigid body translation*

ε_{ij}: *measure of deformation*

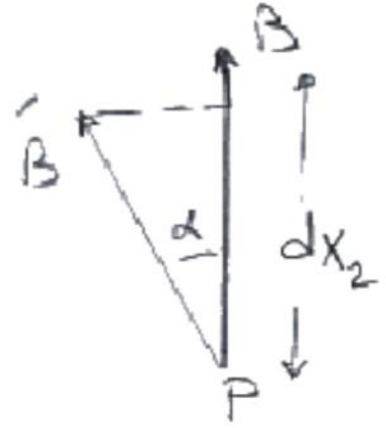
ω_{ij}: *measure of rotation*



Strain Tensor

We now show why ω_{ij} represents a rotation. Consider a small rotation about x_3 – axis, we can write

$$\alpha = \tan\alpha = \frac{du_2}{dx_1} = \frac{\frac{\partial u_2}{\partial x_1}dx_1}{dx_1} = \frac{\partial u_2}{\partial x_1}$$ (APA’ triangle)

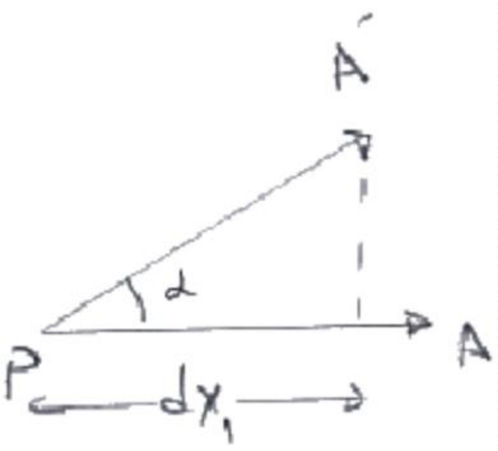


On the other hand

$$\alpha = \tan\alpha = -\frac{du_1}{dx_2} = - \frac{\frac{\partial u_1}{\partial x_2}dx_2}{dx_2} = - \frac{\partial u_1}{\partial x_2}$$

Combining two results

$$\alpha = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \omega_{12}$$



Strain Tensor

$$\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{pmatrix}$$

Strain Tensor

Strain tensor

Symmetric strain tensor with 6 independent components required to specify it at a point.

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

On-diagonal terms

Consider an on-diagonal term, say $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$. With reference to the figure suppose the volume is stretched (or compressed) in the direction of x_2, therefore:

$dx_2 \quad \rightarrow \quad dx_2 + du_2$

Initial final

In this case $u_2 = u_2(x_2\ only)$,

then $du_i = \frac{\partial u_i}{\partial x_j} dx_j \quad \rightarrow du_2 = \frac{\partial u_2}{\partial x_2} dx_2$

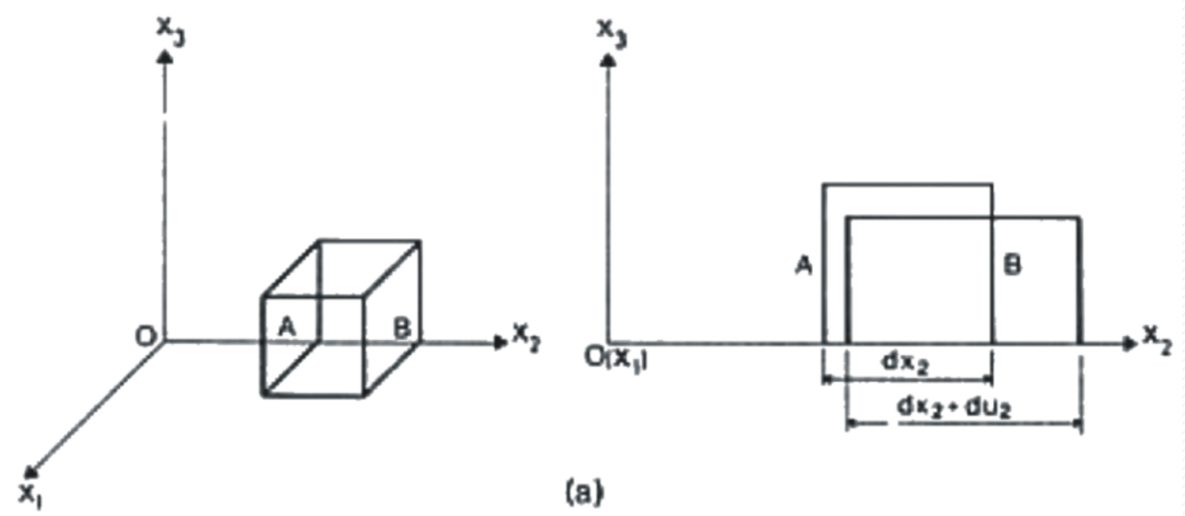
Strain Tensor

$\rightarrow dx_2 + du_2 = \left(1 + \frac{\partial u_2}{\partial x_2}\right) dx_2 = (1 + \varepsilon_{22}) dx_2$

Which shows that ε_{22} represents "the change in the length per unit length" (or elongation in $x_2 - dir$.).

Similarly:
ε_{11}: elongation in $x_1 - dir$.
ε_{33}: elongation in $x_3 - dir$.

$\varepsilon_{ii} > 0$: expansion
$\varepsilon_{ii} < 0$: *compression*



Off-diagonal terms

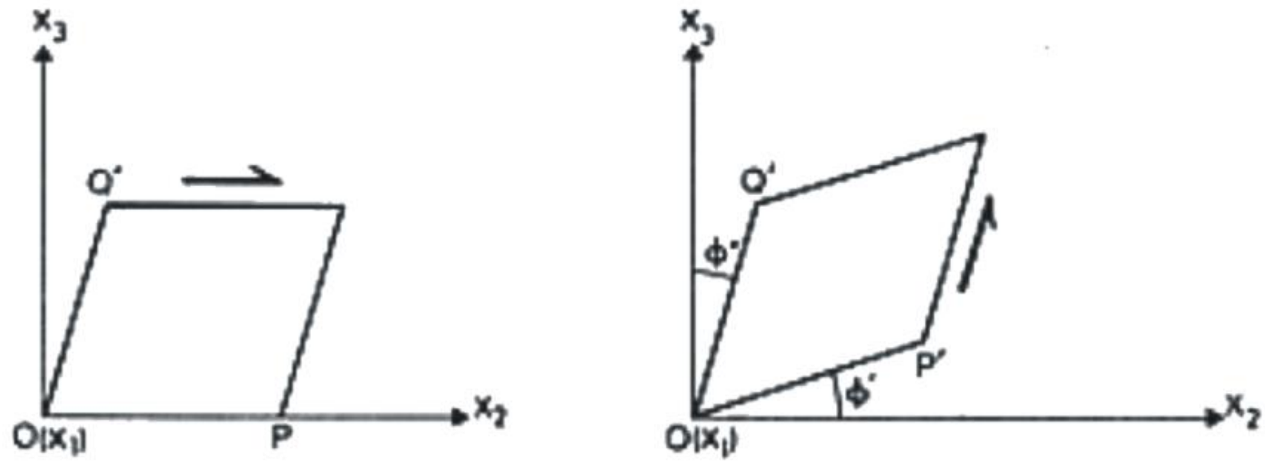
Off-diagonal terms represent **deformation**.

Suppose shear stress acting on the faces Normal to x_2 and x_3 axes. Dissolve the problem in two steps as shown in the figure

$$\phi' \approx \tan\phi' = \frac{\partial u_3}{\partial x_2}$$

$$\phi'' \approx \tan\phi'' = \frac{\partial u_2}{\partial x_3}$$

$\phi' + \phi'' = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} = 2\varepsilon_{23}$ change in the angle of lines



Strain Tensor

Cubical dilatation

Consider an element volume $V = dx_1 dx_2 dx_3$, after deformation → $\mathrm{V} + \mathrm{dV}$. The change in the volume is:

$$\mathrm{dV} = \left(dx_1 \frac{\partial u_1}{\partial x_1} + dx_1\right)\left(dx_2 \frac{\partial u_2}{\partial x_2} + dx_2\right)\left(dx_3 \frac{\partial u_3}{\partial x_3} + dx_3\right) - dx_1 dx_2 dx_3$$

Neglecting the higher order terms:

$$\mathrm{dV} = \left[dx_1 dx_2 dx_3 \; - \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_1 dx_2 dx_3\right] - dx_1 dx_2 dx_3$$

$$dV = (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) dx_1 dx_2 dx_3$$

$$\theta = \frac{dV}{V} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{kk} = \frac{\partial u_k}{\partial x_k}$$ *cubical dilatation*

Strain Tensor

Isotropic and deviatoric Strain

As in the case of stress, the strain tensor can be resolved into isotropic and deviatotic parts, i.e.,

The isotropic strain

$\varepsilon_{ij}^{0} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \varepsilon_0\,\delta_{ij}$ *where* $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ $\varepsilon_0 = \frac{1}{3}\varepsilon_{kk}$ *(mean normal stress)*

$$\varepsilon_{ij}^{0} = \begin{pmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 \end{pmatrix} \equiv \varepsilon_0\,\delta_{ij}$$

pure volume change

The deviatoric strain

$\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon_{ij}^{0}$ → $\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon_{ij}^{0}$

$$\varepsilon'_{ij} = \begin{pmatrix} \varepsilon_{11} - \varepsilon_0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_0 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_0 \end{pmatrix}$$

$(\varepsilon_{kk} = 0)$ *change of shape*

$\varepsilon'_{ij} = \varepsilon_{ij}$ for $i \neq j$ *the shear components of the strain deviator (angular deformation)*

Strain Tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} - \varepsilon_0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_0 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_0 \end{pmatrix}$$

Note that

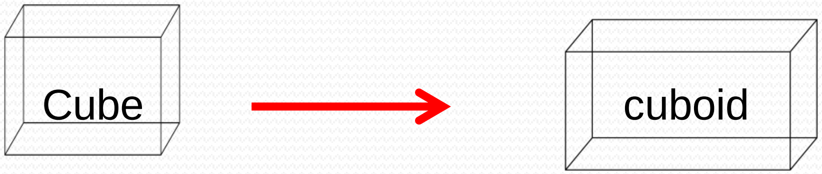
a) If $\varepsilon'_{ij} = 0 \ \forall \, i, j$

$$\varepsilon'_{ij} = \begin{pmatrix} \varepsilon_{11} - \varepsilon_0 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_0 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_0 \end{pmatrix} \rightarrow \varepsilon_{11} = \varepsilon_0, \quad \varepsilon_{22} = \varepsilon_0, \quad \varepsilon_{33} = \varepsilon_0$$

and off-diagonal elements = 0 *purely volumetric deformation*

b) If $\varepsilon_{kk} = 0$ *no volumetric deformation*

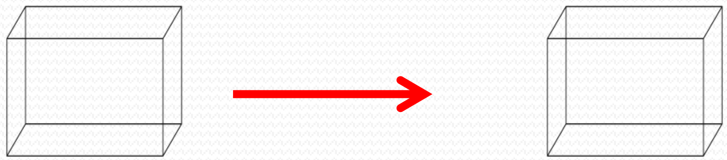
Ex. $\varepsilon_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



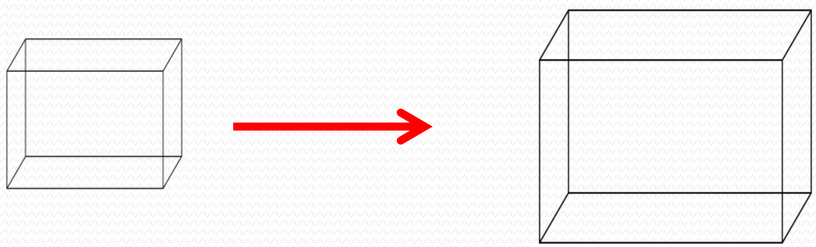
$\varepsilon_{kk} = 0$ *no volumetric deformation*

Strain Tensor

Ex. - $\varepsilon_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \varepsilon_{kk} = 0$ *no volumetric deformation*

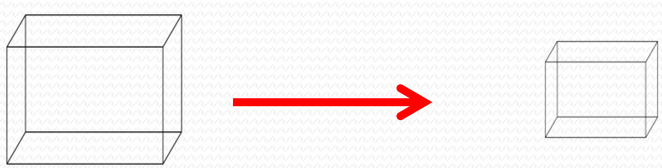


Ex. - $\varepsilon_{ij} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ $\quad \varepsilon_{kk} = 15$



Dilatation

Ex. - $\varepsilon_{ij} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$ $\quad \varepsilon_{kk} = -15$



Contraction

Strain Tensor

c) But if $\varepsilon_{ij} = 0$ for $i \neq j$ this is **not sufficient** condition for having purely volumetric deformation

Ex. - $\varepsilon_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *if rotated, we can find non-zero off-diagonal elements*

This means that the strain tensor (in a given co-ordinate) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ may result shape deformation provided that the all three elongations **are not equal**.

Different representations of the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

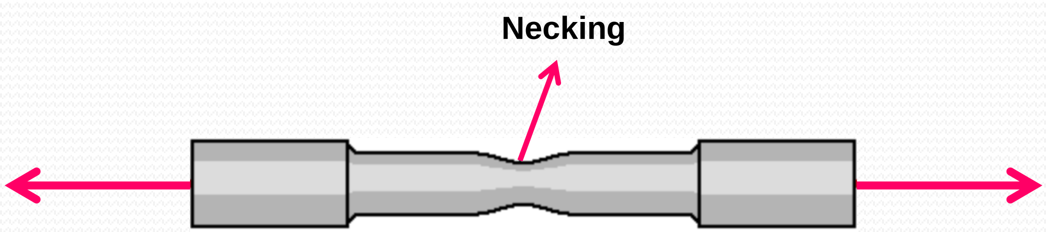
Similar representations may be used for the strain tensor.

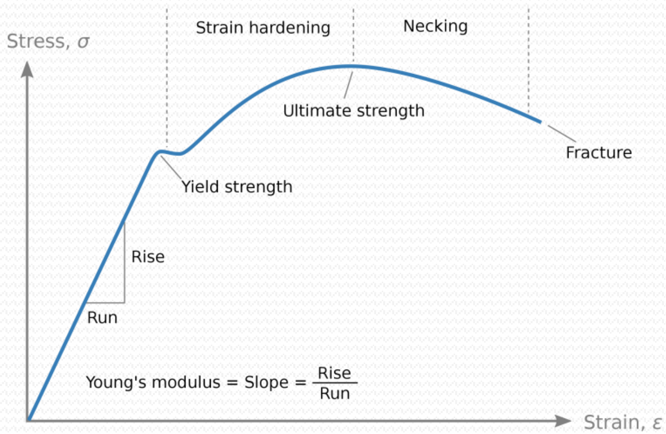
Young Modulus

Young's modulus

The Young modulus E (the modulus of elasticity) is a mechanical property that measures the tensile or compressive stiffness of a solid material when the force is applied lengthwise.

$$E = \frac{Stress}{Strain} = \frac{\sigma}{\varepsilon}$$





https://commons.wikimedia.org/w/index.php?curid=89891144

Vector and Tensor Rotation

Active transformation versus passive transformation

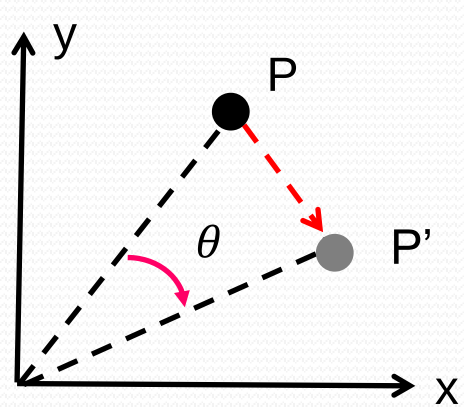
a) Active transformation

Point moves from position P to P' by rotating clockwise by an angle θ about the origin.

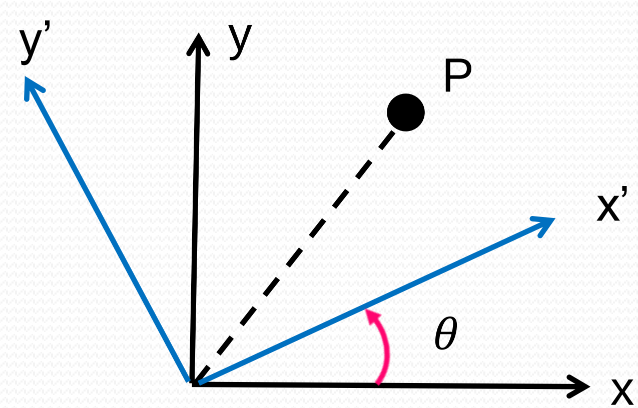
b) Passive transformation

Point does not move, instead, the coordinate system rotates counterclockwise by an angle θ about its origin.

The coordinates of P' in the active case (that is, relative to the original coordinate system) are the same as the coordinates of P relative to the rotated coordinate system



Active



Passive

Rotation matrix in 3D

$$R_P = \begin{pmatrix} \cos(x',x) & Cos(x',y) & Cos(x',z) \\ Cos(y',x) & Cos(y'y) & Cos(y',z) \\ Cos(z',x) & Cos(z',y) & Cos(z',z) \end{pmatrix}$$

Transpose of a matrix (A^T)

$$\boldsymbol{A^T}_{\boldsymbol{ij}} = \boldsymbol{A}_{\boldsymbol{ji}}$$

Vector and Tensor Rotation

Rotation matrix

A rotation matrix is a transformation matrix used to perform a rotation in Euclidean space.

Two dimensional rotation

$$R_{2D}(\theta) = \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$$ *(Active rotation)* $\left\{ R_{2D}(\theta) = \begin{pmatrix} \cos\theta & Sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} \right.$ *(Passive rotation)* $\left.\right\}$

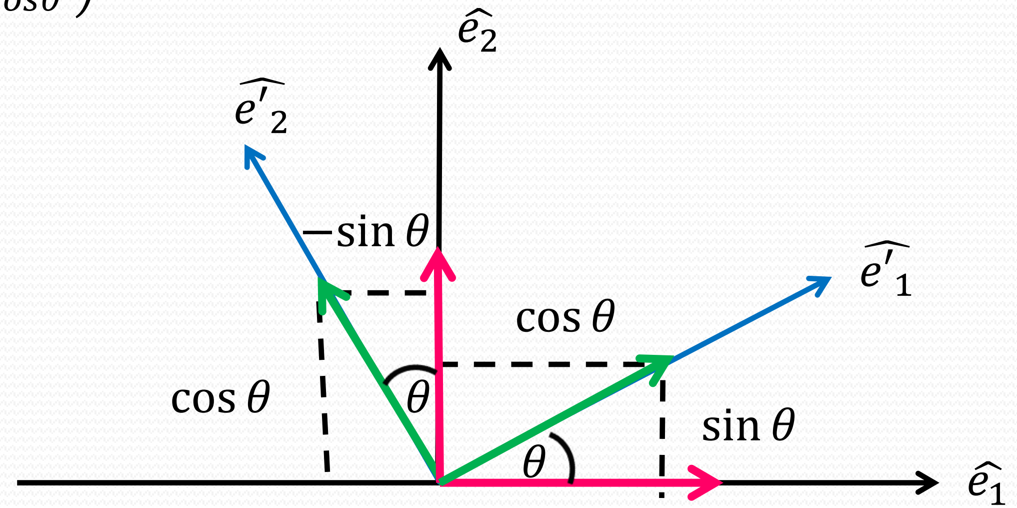
$$\widehat{e'}_1 = R_{2D}\,\widehat{e_1} = \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$$

$$\widehat{e'}_2 = R_{2D}\,\widehat{e_2} = \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix}$$

$$V' = R_{2D}\,V = \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} =$$

$$\begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} v_x\, cos\theta - v_y\, sin\theta \\ v_x \sin\theta + v_y \cos\theta \end{pmatrix}$$

$$R_{2D}(-\theta) = \begin{pmatrix} \cos\theta & sin\theta \\ -\sin\theta & cos\theta \end{pmatrix}$$



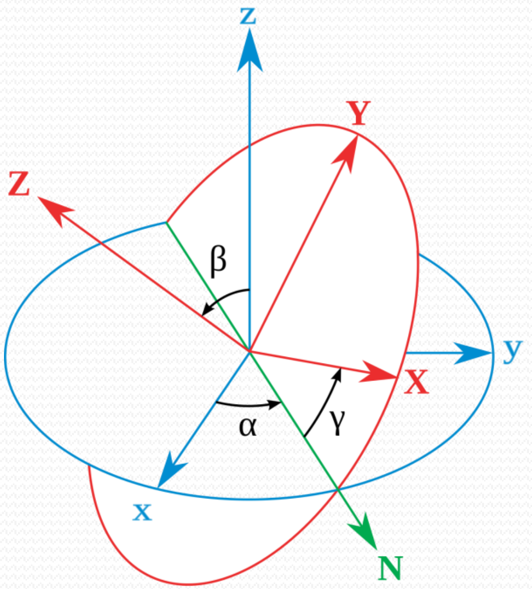
Vector and Tensor Rotation

Three dimensional rotation

$$R_{3D-x}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & \sin\theta & cos\theta \end{pmatrix}$$ *(Active rotation)*

$$R_{3D-y}(\theta) = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}$$

$$R_{3D-z}(\theta) = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



https://commons.wikimedia.org/w/index.php?curid=3362239

$$R_{3D} = R_x(\alpha)R_y(\beta)R_z(\gamma) = \begin{pmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

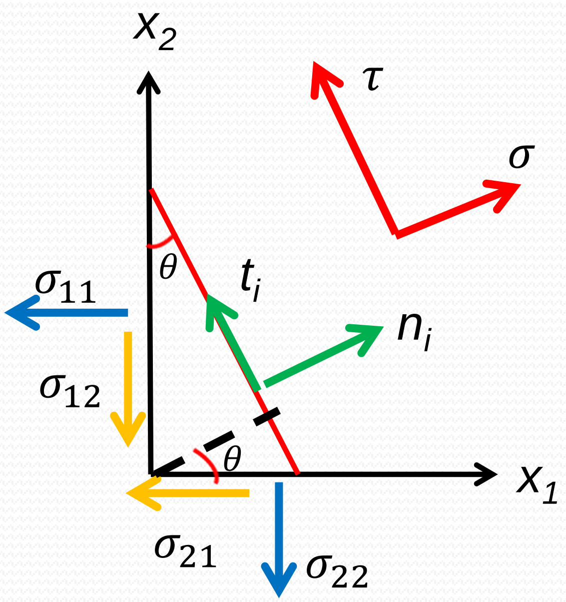
$$\begin{pmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & -sin\gamma \\ 0 & \sin\gamma & cos\gamma \end{pmatrix}$$

$$V' = R_{3D}\, V \qquad \begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = R_{3D} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Vector and Tensor Rotation

Rotation of stress tensor

$$\begin{cases} n_1 = Cos\,\theta \\ n_2 = Sin\,\theta \\ t_1 = -Sin\,\theta \\ t_2 = Cos\,\theta \end{cases} \rightarrow \begin{cases} \widehat{n_A} = \begin{pmatrix} Cos\,\theta \\ Sin\,\theta \end{pmatrix} \\ \widehat{n_B} = \begin{pmatrix} -Sin\,\theta \\ Cos\,\theta \end{pmatrix} \end{cases}$$

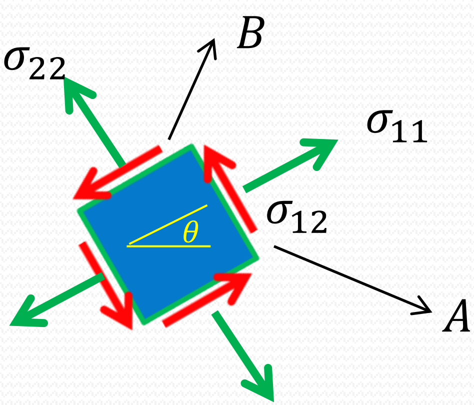


We obtained:

$$\sigma = \sigma'_{11} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22})\ \ Cos\,2\theta + \sigma_{12}\ Sin\,2\theta$$

$$\sigma = \sigma'_{22} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \frac{1}{2}(\sigma_{11} - \sigma_{22})\ \ Cos\,2\theta - \sigma_{12}\ Sin\,2\theta$$

$$\tau = \sigma'_{12} = -\frac{1}{2}(\sigma_{11} - \sigma_{22})\ Sin\,2\theta + \sigma_{12}\ Cos\,2\theta$$



Vector and Tensor Rotation

Using

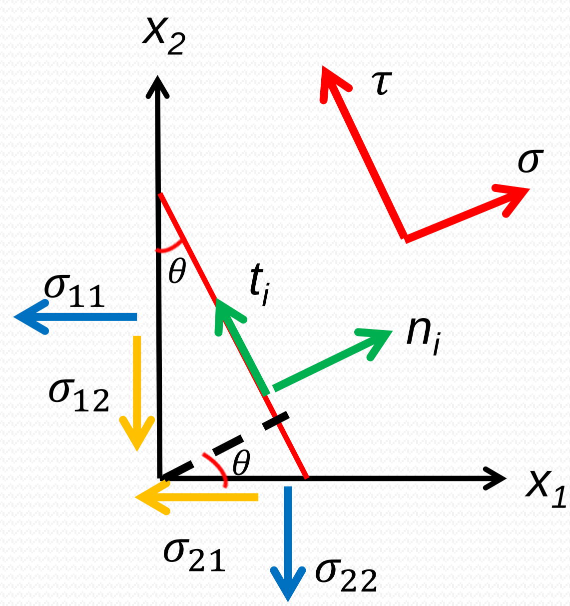
$Cos^2\theta \equiv \frac{1}{2}(1 + Cos\ 2\theta)$

$Sin^2\theta \equiv \frac{1}{2}(1 - Cos\ 2\theta)$

$Cos\ \theta\ \ Sin\ \theta \equiv \frac{1}{2}Sin\ 2\theta$

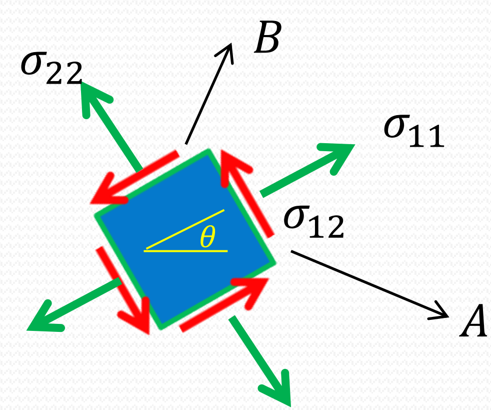
These equations can be cast into:

$$\sigma'_{11} = \sigma_{11}Cos^2\theta + \sigma_{22}Sin^2\theta + 2\sigma_{12}Sin\ \theta\ Cos\ \theta$$
$$\sigma'_{22} = \sigma_{11}Sin^2\theta + \sigma_{22}Cos^2\theta - 2\sigma_{12}Sin\ \theta\ Cos\ \theta$$
$$\sigma'_{12} = (\sigma_{22} - \sigma_{11})\ Sin\ \theta\ Cos\ \theta + \sigma_{12}(Cos^2\theta - Sin^2\theta)$$



In matrix notation

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$$



Or

$$\boldsymbol{\sigma}' = \boldsymbol{R\sigma R^T}$$

Where

$R = \begin{pmatrix} \cos(x',x) & Cos(x',y) \\ Cos(y',x) & Cos(y'y) \end{pmatrix}$ and $R^T{}_{ij} = R_{ji}$

Vector and Tensor Rotation

$R = \begin{pmatrix} \cos(x',x) & Cos(x',y) \\ Cos(y',x) & Cos(y'y) \end{pmatrix}$ and $R^T{}_{ij} = R_{ji}$

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix}$$

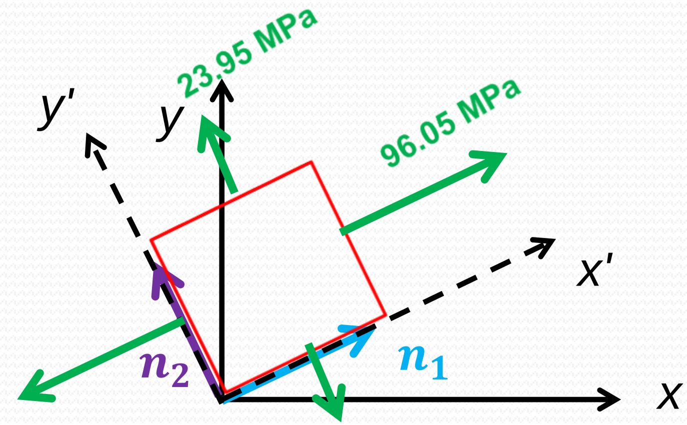
$$\boldsymbol{\sigma' = R\sigma R^T}$$

$\rightarrow \quad \sigma'_{ij} = a_{im}a_{jn}\sigma_{mn}$

Examples

Ex. – Find the stress tensor in principal axes by rotation for the following stress tensor.

$$[\sigma_{ij}] = \begin{pmatrix} 80 & 30 \\ 30 & 40 \end{pmatrix}$$



Sol.

Solving an eigenvalue problem we obtained:

Rename $n = \begin{bmatrix} 0.88 \\ 0.47 \end{bmatrix}$ as $n_1 = \begin{bmatrix} 0.88 \\ 0.47 \end{bmatrix}$ corresponding to the eigenvalue $\sigma_1 = 96.05$

and $n = \begin{bmatrix} -0.47 \\ 0.88 \end{bmatrix}$ as $n_2 = \begin{bmatrix} -0.47 \\ 0.88 \end{bmatrix}$ corresponding to the eigenvalue $\sigma_2 = 23.95$

and

$$\sigma' = \begin{pmatrix} 96.05 & 0 \\ 0 & 23.95 \end{pmatrix}$$

$$\theta = \frac{1}{2} \arctan[2\sigma_{12}/(\sigma_{11} - \sigma_{22})]$$

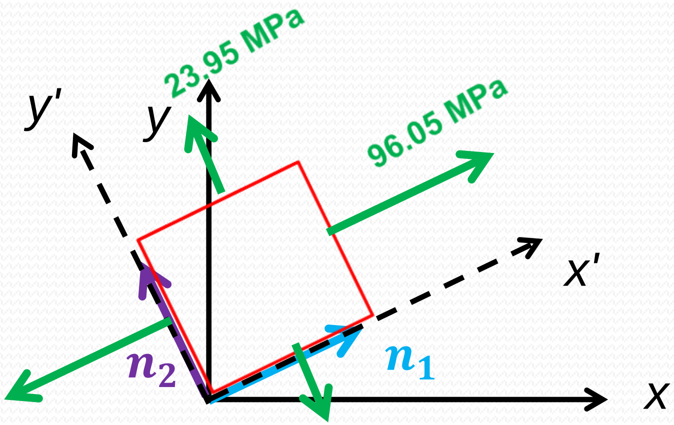
Examples

We had obtained the angle of rotation to the principal coordinate:

$$\theta = \frac{1}{2}\arctan[2\sigma_{12}/(\sigma_{11} - \sigma_{22})]$$

$$\theta = \frac{1}{2}\arctan[2 \times 30/(80 - 40)]$$

$$\theta = \frac{1}{2}\arctan[2\sigma_{12}/(\sigma_{11} - \sigma_{22})] = 28.15^o$$



$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{pmatrix} = \begin{pmatrix} 0.882 & 0.472 \\ -0.472 & 0.882 \end{pmatrix} \begin{pmatrix} 80 & 30 \\ 30 & 40 \end{pmatrix} \begin{pmatrix} 0.882 & -0.472 \\ 0.472 & 0.882 \end{pmatrix} =$$

$$\begin{pmatrix} 0.882 & 0.472 \\ -0.472 & 0.882 \end{pmatrix} \begin{pmatrix} 80 & 30 \\ 30 & 40 \end{pmatrix} \begin{pmatrix} 0.882*80+0.472*30 & (-0.472)*80+0.882*30 \\ 0.882*30+0.472*40 & (-0.472)*30+0.882*40 \end{pmatrix} =$$

$$\begin{pmatrix} 0.882 & 0.472 \\ -0.472 & 0.882 \end{pmatrix} \begin{pmatrix} 84.72 & -11.3 \\ 45.34 & 21.12 \end{pmatrix} =$$

$$\begin{pmatrix} 96.12 & 0.00 \\ 0.00 & 23.96 \end{pmatrix}$$

Examples

Ex. – The stress tensor in a coordinate frame is given by:

$$[\sigma_{ij}] = \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \quad \text{(Mpa)}$$

Fid the stress tensor in a rotated coordinate system described by the following rotation matrix:

$$[a_{ij}] = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

Using

$$\boldsymbol{\sigma' = R\sigma R^T} \qquad (\sigma'_{ij} = a_{im}a_{jn}\sigma_{mn})$$

$$\boldsymbol{R^T{}_{ij} = R_{ji}}$$

Examples

$$\boldsymbol{\sigma}' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}^T$$

$$\boldsymbol{\sigma}' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$\boldsymbol{\sigma}' = \begin{pmatrix} \frac{120}{\sqrt{3}} & \frac{40}{\sqrt{3}} & \frac{120}{\sqrt{3}} \\ \frac{80}{\sqrt{2}} & \frac{20}{\sqrt{2}} & \frac{-20}{\sqrt{2}} \\ \frac{120}{\sqrt{6}} & \frac{-20}{\sqrt{6}} & \frac{-180}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$$

$$\boldsymbol{\sigma}' = 20 \begin{pmatrix} \frac{14}{\sqrt{9}} & \frac{4}{\sqrt{6}} & \frac{-4}{\sqrt{18}} \\ \frac{4}{\sqrt{6}} & \frac{3}{\sqrt{4}} & \frac{7}{\sqrt{12}} \\ \frac{-4}{\sqrt{18}} & \frac{7}{\sqrt{12}} & \frac{23}{\sqrt{36}} \end{pmatrix}$$