ESS2222H

Tectonics and Planetary Dynamics
Lecture 3 – Stress, Strain

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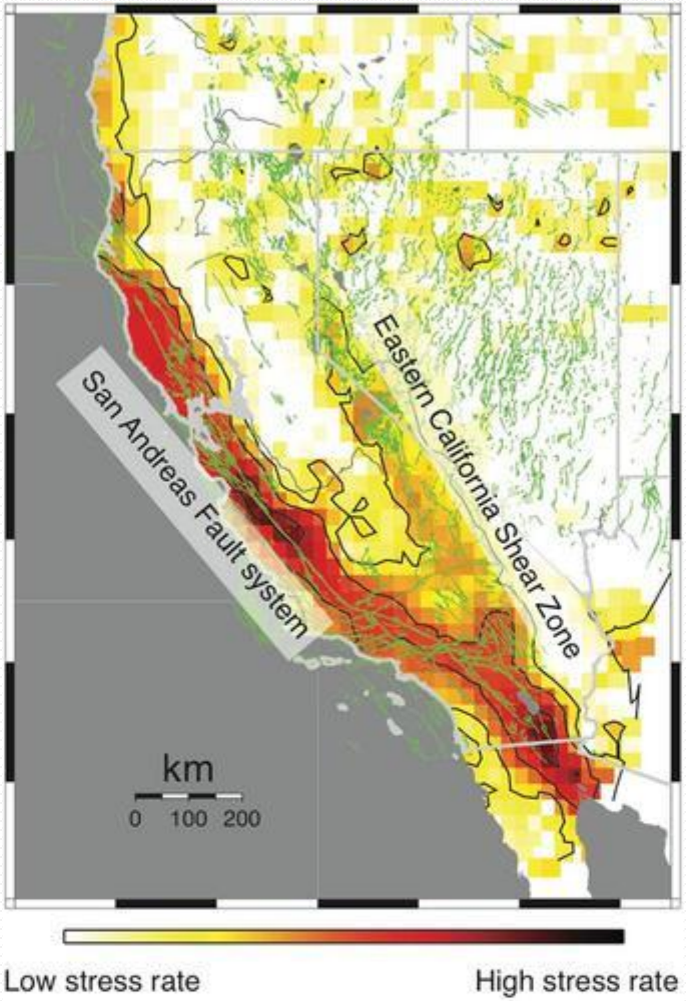
Stress, Deformation, and Strain

- ❑ **Stress & Strain Tensors**
- ❑ **Isobaric & Deviatoric Stress**
- ❑ **Principal Axes & Principal Stress**
- ❑ **Isotropic & Deviatoric Strain**
- ❑ **Mohr's Circle**

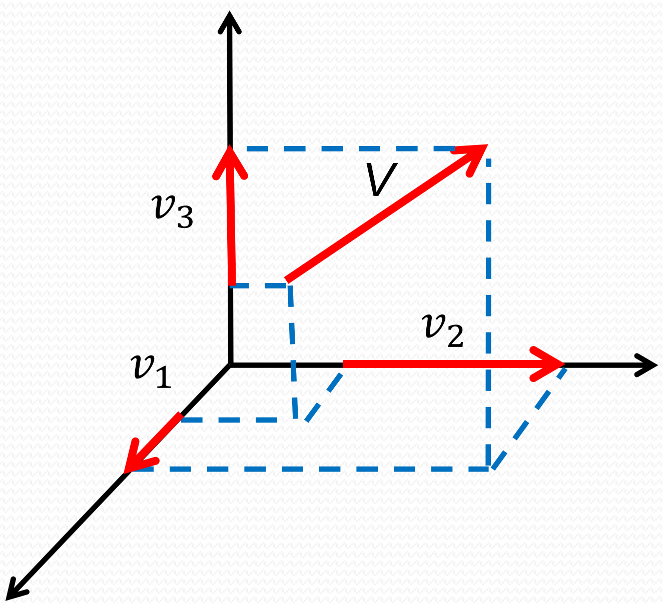


Stressing rate of the crust around California derived from two decades of geodetic measurements (USGS).

Tensors

Scalar (tensor of rank 0):
Scalar quantity is a tensor of rank zero, specified by a single component; like temperature (T), mass (m), density(ρ), etc.

Vector (tensor of rank 1):
Vector quantity is a tensor of rank 1, specified by three components; like velocity $V(v_1, v_2, v_3)$, gravitational acceleration $g(g_1, g_2, g_3)$, electric field $E(e_1, e_2, e_3)$, etc.



Tensors

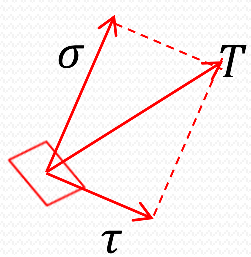
Tensor of rank 2:

Tensor of rank 2 is an algebraic object specified by nine components; like: stress tensor,

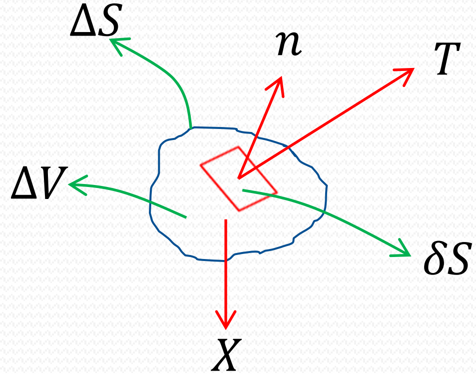
$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Traction vector

$T^n = lim \frac{\delta F}{\delta S} \quad \delta S \rightarrow 0$ ***force per unit area*** *acting on surface with orientation* ***n***



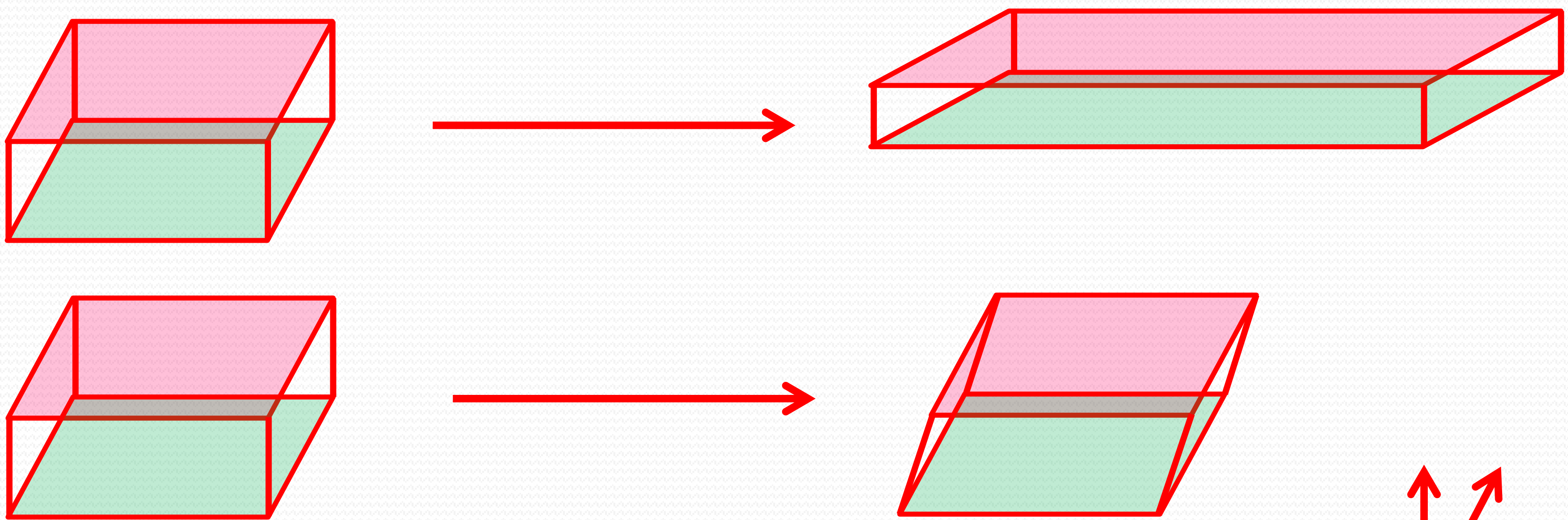
Convention*:*
Positive outward (tension)
Negative inward (compression)



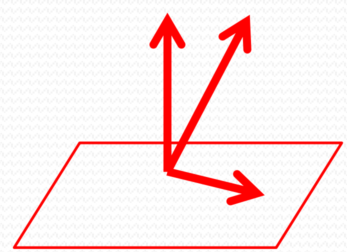
Stress Tensor

Why do we need a tensor of rank 2 for stress field?

Deformation of a volume element cannot be specified by a vector. Deformation of each volume face can in general be different (from the others).

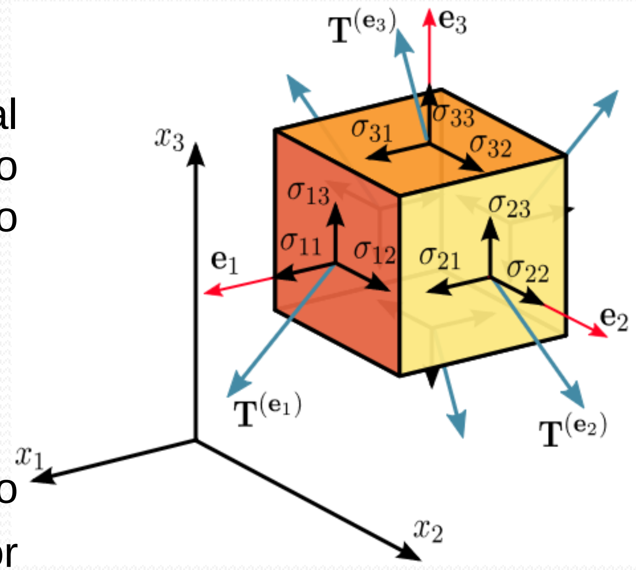


We need a vector for each face.



Stress Tensor

Stress tensor is a tensor of rank 2. Consider a cubic material element. The tractions on the three faces can be resolved into their Cartesian components, one normal and two tangential to the face on which the **traction** acts.



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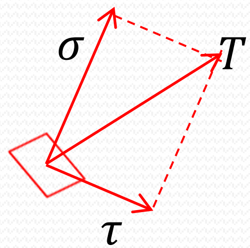
Consider the face with normal vector $\boldsymbol{e_1}$

T^{e1} can be decomposed into $\boldsymbol{\sigma}$ (in $\boldsymbol{e_1}$ dir.)and $\boldsymbol{\tau}$ (having two components in $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ dir.). We rename these vector components as σ_{11}, σ_{12}, σ_{11}, respectively, the first index representing the identity of surface and the second index representing the vector components.



For tree surfaces we have three vectors as:

$$T^{e1} = \sigma_{11}e_1 + \sigma_{12}e_2 + \sigma_{13}e_3$$
$$T^{e2} = \sigma_{21}e_1 + \sigma_{22}e_2 + \sigma_{23}e_3$$
$$T^{e3} = \sigma_{31}e_1 + \sigma_{32}e_2 + \sigma_{33}e_3$$

Stress Tensor

Einstein summation convention (summation notation)
Repeated indices are summed over.

In summation notation:

$$T^{ei} = \sigma_{ij} e_j \quad (\text{meaning: } T^{ei} = \sum_j \sigma_{ij} e_j) \qquad i, j = 1,2,3$$

$\boldsymbol{T^{ei}}$ is the **traction** force acting on a surface with **normal vector along $\boldsymbol{e_i}$**.

We can write these three vectors in a compact form, that is a tensor of rank 2:

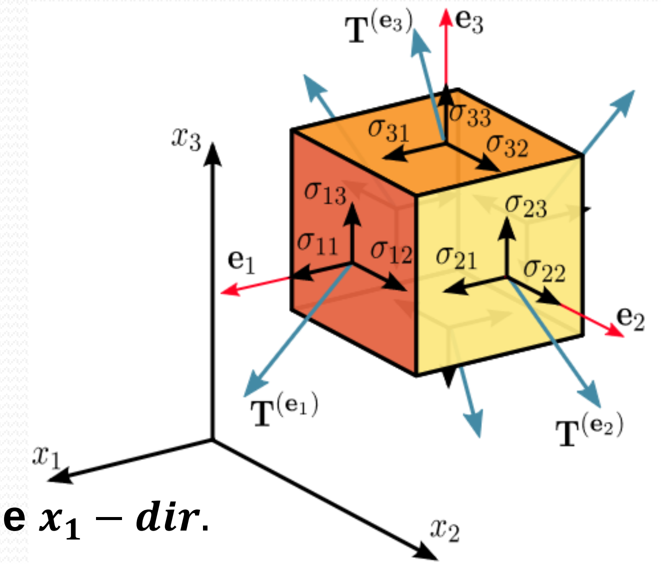
$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$x_1 \equiv x$
$x_2 \equiv y$
$x_3 \equiv z$

Equations of Equilibrium

The equations of equilibrium

$\sum F = 0$ (force balance)



$F_1 = 0 \quad (F_x)$
$F_2 = 0 \quad (F_y)$
$F_3 = 0 \quad (F_z)$

Consider the components of the surface force acting in the $x_1 - dir$.

$\sigma_{11} dx_2 dx_3$ and $\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3$

$\sigma_{21} dx_1 dx_3$ and $\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3$

$\sigma_{31} dx_1 dx_2$ and $\left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) dx_1 dx_2$

Note that $\frac{\partial \sigma_{11}}{\partial x_1}$, $\frac{\partial \sigma_{21}}{\partial x_2}$, and $\frac{\partial \sigma_{31}}{\partial x_3}$ are the rate of changes in σ_{11}, σ_{21}, and σ_{31}, respectively in x_1, x_2, and x_3 directions.

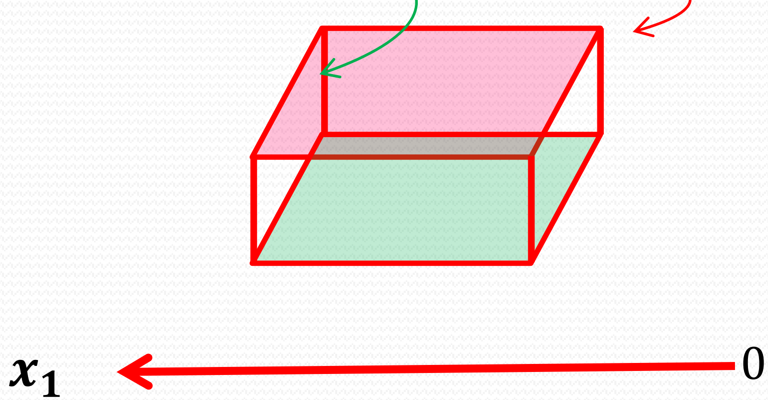
Equations of Equilibrium

And the body force components (like the gravity force acting the volume):

$$b_1 = \rho X_1 \, dx_1 dx_2 dx_3$$
$$b_2 = \rho X_2 \, dx_1 dx_2 dx_3 \qquad X_i: \quad the\ body\ force\ per\ unit\ mass\ \ in\ i - dir.$$
$$b_3 = \rho X_3 \, dx_1 dx_2 dx_3$$

The condition of equilibrium of forces in $x_1 - dir$.

$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 - \sigma_{11} dx_2 dx_3 +$$
$$\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 - \sigma_{21} dx_1 dx_3 +$$
$$\left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) dx_1 dx_2 - \sigma_{31} dx_1 dx_2 +$$
$$\rho X_1 \, dx_1 dx_2 dx_3 = 0$$



$$\rightarrow \quad \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho X_1 = 0$$

Equations of Equilibrium

Similarly if we repeat this in $x_2 - dir$ and $x_3 - dir$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho X_2 = 0,$$
$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho X_3 = 0$$

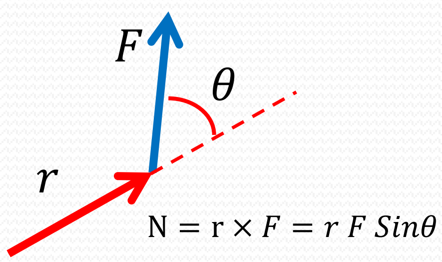
Or in compact form:

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho X_i = 0 \qquad \left(\frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} + \rho X_i = 0,\ for\ i = 1,2,3\right)$$

Equilibrium of moment of forces (Torque)

$\sum \boldsymbol{N} = \mathbf{r} \times \boldsymbol{F} = 0$ **Torque (torque balance)**

$$\boldsymbol{N_1} = 0 \qquad (\boldsymbol{N_x})$$
$$\boldsymbol{N_2} = 0 \qquad (\boldsymbol{N_y})$$
$$\boldsymbol{N_3} = 0 \qquad (\boldsymbol{N_z})$$



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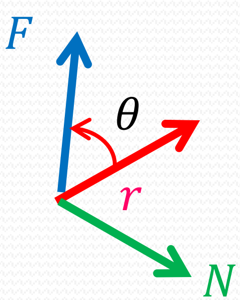

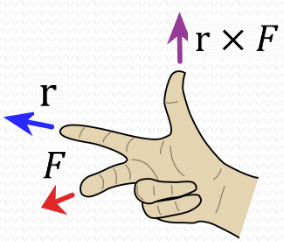

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Equations of Equilibrium

Equilibrium of moments about an axis parralel to x_1: $\boldsymbol{N_1 = 0}$

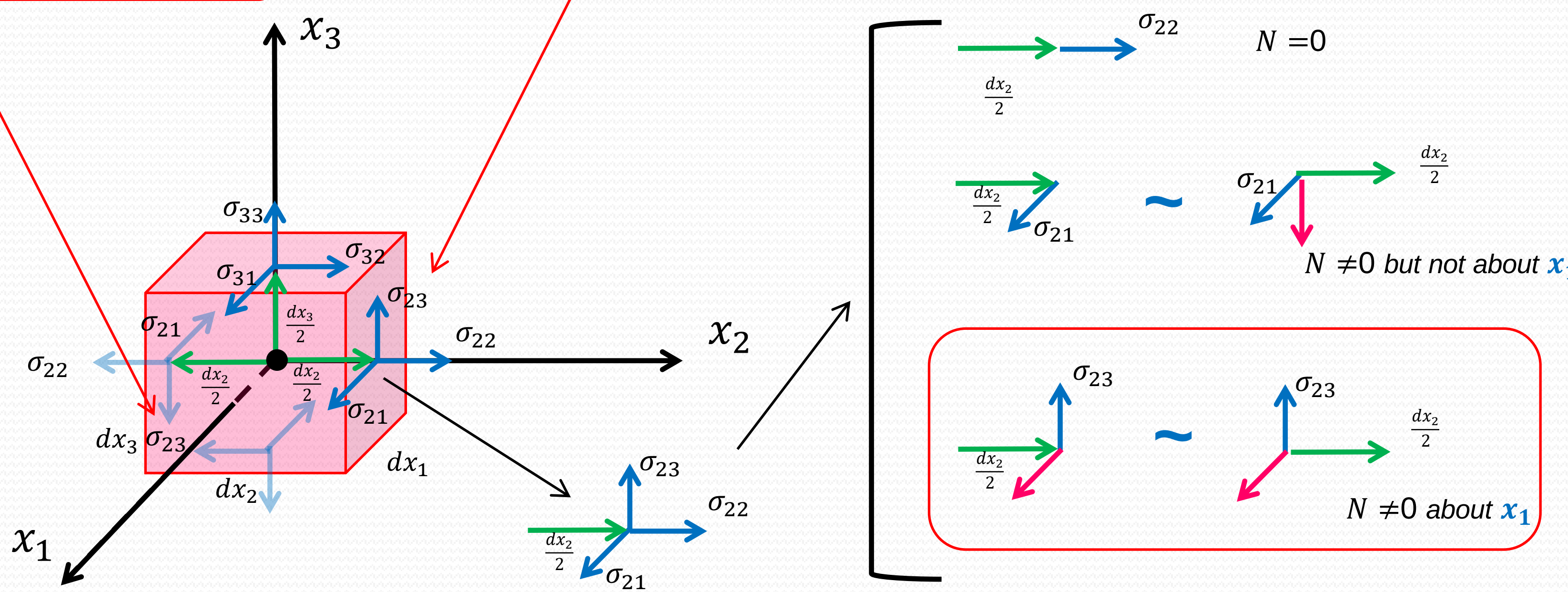
$$(\sigma_{23}dx_1dx_3)\frac{dx_2}{2} + \left(\sigma_{23} + \frac{\partial\sigma_{23}}{\partial x_2}dx_2\right)dx_1dx_3\frac{dx_2}{2} - (\sigma_{32}dx_1dx_2)\frac{dx_3}{2} - \left(\sigma_{32} + \frac{\partial\sigma_{32}}{\partial x_3}dx_3\right)dx_1dx_{12}\frac{dx_3}{2} = 0$$

$$\mathbf{r}\times\boldsymbol{F} = \left(\boldsymbol{\sigma_{23}} + \frac{\boldsymbol{\partial\sigma_{23}}}{\boldsymbol{\partial x_2}}\boldsymbol{dx_2}\right)\boldsymbol{dx_1dx_3}\frac{\boldsymbol{dx_2}}{\boldsymbol{2}}$$

F/A **A** **r**

$$\mathrm{N} = \mathrm{r}\times F = r\,F\,Sin\theta$$

$$\mathbf{r}\times\boldsymbol{F} = (\sigma_{23}dx_1dx_3)\frac{dx_2}{2}$$



Equations of Equilibrium

Dividing by $dx_1 dx_2 dx_3$

→ $\sigma_{23} = \sigma_{32}$

Similarly if we repeat this in $x_2 - dir$ and $x_3 - dir$ we obtain:

$\sigma_{13} = \sigma_{31}$, $\sigma_{12} = \sigma_{21}$

Or $\sigma_{ij} = \sigma_{ji}$

Using this symmetry

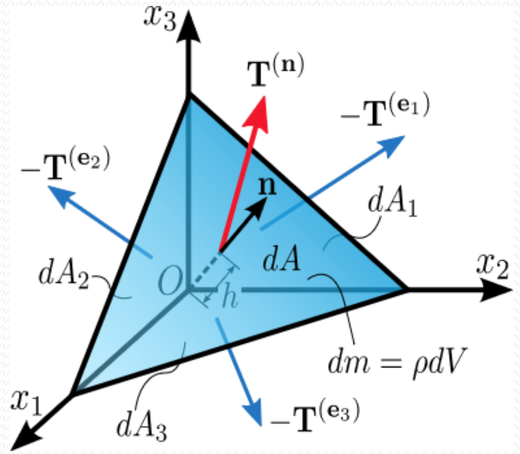
$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho X_i = 0 \quad \rightarrow \quad \frac{\partial \sigma_{ij}}{\partial x_j} + \rho X_i = 0$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \rightarrow \sigma_{ij} = \begin{pmatrix} \sigma_{11} & a & b \\ a & \sigma_{22} & c \\ b & c & \sigma_{33} \end{pmatrix}$$

six independent parameters

Stress on a surface – Cauchy's formula

Using Cauchy’s formula we can find traction on any surface with Normal vector n (n_1, n_2 n_3)



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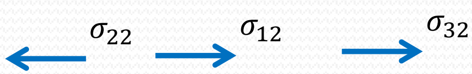
Consider a small tetrahedron

$\delta A_1 = \delta A \, n_1$ $\quad$ $n_1 = n\cos(\alpha)$ *component of n on* x_1

$\delta A_2 = \delta A \, n_2$ $\quad$ $n_2 = n\cos(\beta)$ *component of n on* x_2

$\delta A_3 = \delta A \, n_3$ $\quad$ $n_3 = n\cos(\gamma)$ *component of n on* x_3

$$\delta V = \frac{1}{3} h \ \delta A$$



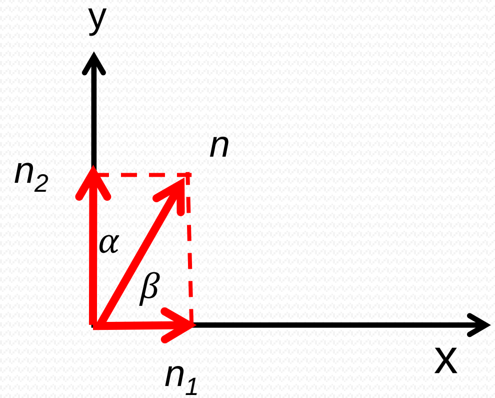
$$\sum \boldsymbol{F} = \mathbf{0}$$

Along $x_2 - dir$

$$\boldsymbol{F_x} = \mathbf{0}$$

$$T_2^n \delta A - \sigma_{12} \, n_1 \delta A - \sigma_{22} \, n_2 \delta A - \sigma_{32} \, n_3 \delta A + \frac{1}{3} \rho \, h \, \delta A \ \ X_2 = 0$$

δA_1 $\quad$ δA_2 $\quad$ δA_3



$n_1 = Cos \ \beta$

$n_2 = Cos \ \alpha$

For infinitesimally small volume $h \ \rightarrow 0$ ($h \, \delta A$ small relative to δA)

$$T_2^n = \sigma_{12} \, n_1 + \sigma_{22} \, n_2 + \sigma_{32} \, n_3 = \sigma_{i2} \, n_i$$

Cauchy's Formula

Similarly

$T_1^n = \sigma_{11}\, n_1 + \sigma_{21}\, n_2 + \sigma_{31}\, n_3$
$T_3^n = \sigma_{13}\, n_1 + \sigma_{23}\, n_2 + \sigma_{33}\, n_3$

In compact form

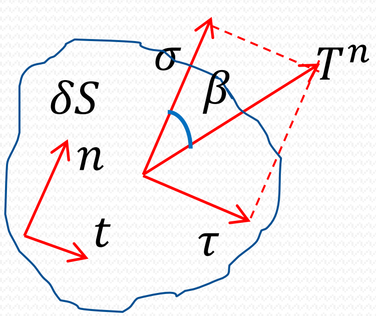
$T_i^n = \sigma_{ij}\, n_j$ **Cauchy's formula**

Or

$$\begin{pmatrix} T_1^n \\ T_2^n \\ T_3^n \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

Resolving the traction into two components

$\sigma = T^n \cdot n = |T^n|\, Cos\, \beta$
$\tau = T^n \cdot t = |T^n|\, Sin\, \beta$
$|n| = 1$
$|t| = 1$



Making use of Cauchy's formula

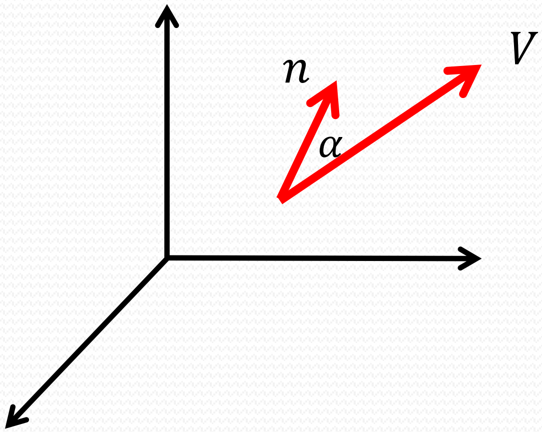
$\sigma = T_i^n n_i = \sigma_{ij}\, n_i\, n_j$ *normal stress*
$\tau = T_i^n t_i = \sigma_{ij}\, t_i\, n_j$ *shear stress*

$\tau = \sqrt{(T^n)^2 - \sigma^2}$

$(T^n)^2 = T^n \cdot T^n = T_i{}^n T_i{}^n =$
$(\sigma_{ij} n_j)(\sigma_{ik} n_k) = \sigma_{ij}\sigma_{ik} n_j n_k$

Note that

$V \cdot n = |V||n|\, cos\,(\alpha) = V_1 n_1 + V_2 n_2 + V_3 n_3$



Isotropic and Deviatoric Stress

Isotropic and deviatoric stress

In a **continuous** medium (like rocks, tectonic plates) complete specification of the state of system requires knowledge of the **stress tensor $\boldsymbol{\sigma_{ij}}$** at each point as **functions of the co-ordinates.** It is **useful** to **break** the stress **into two parts**, the isotropic and deviatoric parts:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

The isotropic stress

$\sigma_{ij}^0 = \frac{1}{3}\sigma_{kk}\delta_{ij} = \sigma_0\,\delta_{ij}$ *where* $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$ $\sigma_0 = \frac{1}{3}\sigma_{kk}$ *(mean normal stress)*

$$\sigma_{ij}^0 = \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}$$

$$\begin{cases} \delta_{ij} = 1 & if\ i = j \\ \delta_{ij} = 0 & if\ i \neq j \end{cases}$$

kronecker delta

The deviatoric stress

$$\sigma'_{ij} = \sigma_{ij} - \sigma_{ij}^0$$

$$\sigma'_{ij} = \begin{pmatrix} \sigma_{11} - \sigma_0 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_0 \end{pmatrix} \quad (\sigma_{kk} = 0)$$

Isotropic and Deviatoric Stress

→ $\boldsymbol{\sigma_{ij} = \sigma_0\ \delta_{ij} + \sigma'_{ij}}$

$$\sigma_{ij} = \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix} + \begin{pmatrix} \sigma_{11} - \sigma_0 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_0 \end{pmatrix}$$

Principal Axes and Principal Stress

Principal axes and principal stresses

In this coordinate system the only nonzero stress components are diagonal elements

$$\sigma_{ij}^{P} = \begin{pmatrix} \sigma_{11}^{P} & 0 & 0 \\ 0 & \sigma_{22}^{P} & 0 \\ 0 & 0 & \sigma_{33}^{P} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{1} & 0 & 0 \\ 0 & \sigma_{2} & 0 \\ 0 & 0 & \sigma_{3} \end{pmatrix}$$

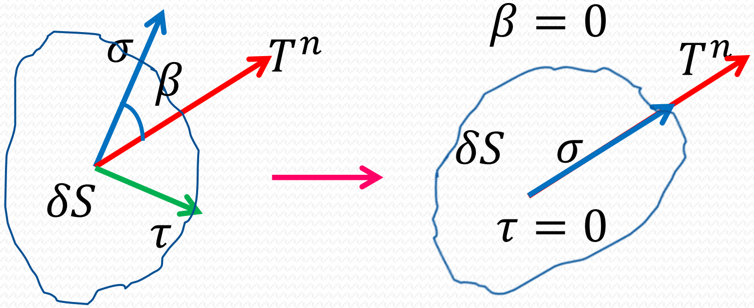
The usefulness of the reasoning in terms of principal axes and principal stresses lies in the fact that they give a **clear picture of the state of stress** at a point. We need to determine 6-componets; **three principal axes** and **three stress components**.

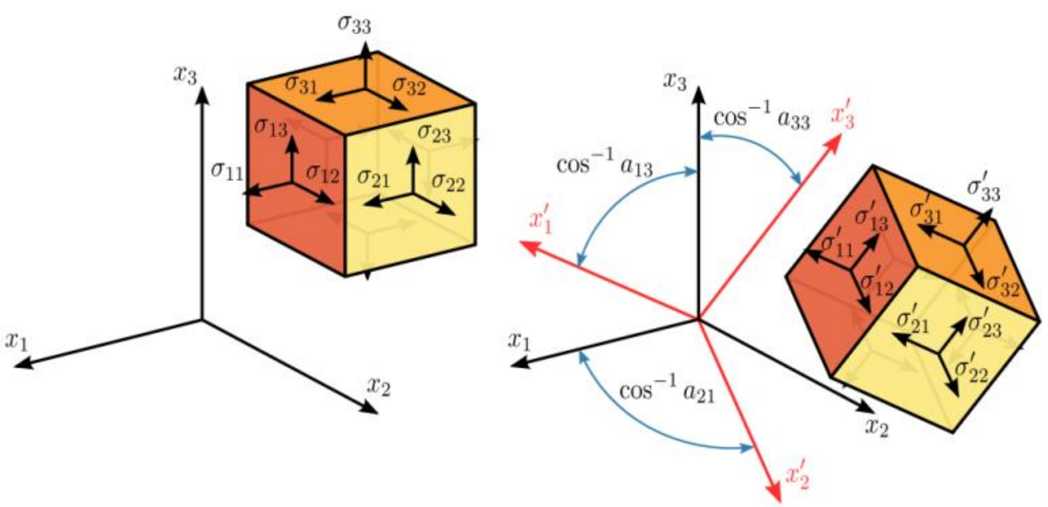
This is an eigenvalue problem.

Principal Axes and Principal Stress

Consider an **arbitrary-oriented surface** with **unit normal n**. In general the direction of the traction and that of the normal don not coincide unless the later is the principal axes.

In order to find principal axed (eigenvectors)
and the magnitude of normal stresses (eigenvalues)
With vanishing shear terms, we have to solve
solve an eigenvalue problem.





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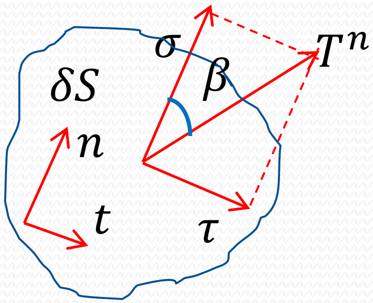
Principal Axes and Principal Stress

We had

$|\sigma| = T_i^n n_i = \sigma_{ij} n_i \, n_j$ *normal stress*

$|\tau| = T_i^n t_i = \sigma_{ij} t_i \, n_j$ *shear stress*

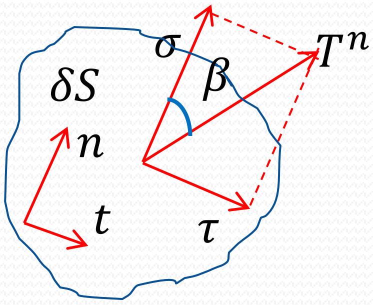


In principal coordinate system:

$$|\sigma| = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} =$$

$$\begin{bmatrix} \sigma_1 n_1 & \sigma_2 n_2 & \sigma_3 n_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

Principal Axes and Principal Stress



In principal coordinate system:

$$\sigma = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 + 0$$
$$\tau = \sigma_1 n_1 t_1 + \sigma_2 n_2 t_2 + \sigma_3 n_3 t_3 + 0$$

where $\sigma_1, \sigma_2, \sigma_3$ are **principal** stresses and n_i, t_i are unit normal and unit tangent to the surface element.

Principal Axes and Principal Stress

The **condition** for **n** to be a **principal direction** is on the basis of Cauchy's formula:

$$T_i^n = \sigma_{ij}\, n_j = \boldsymbol{\sigma}\, \boldsymbol{n_i} \qquad (\sigma_{11}\, n_1 + \cancel{\sigma_{12}}\, n_2 + \cancel{\sigma_{13}}\, n_3 = \sigma_{11}\, n_1 \equiv \sigma\, n_1, \quad for\ i = 1)$$

Using $\quad n_i = \delta_{ij} n_j \quad$ *(e.g.,* $n_1 = \delta_{11} n_1 + \cancel{\delta_{12}} n_2 + \cancel{\delta_{13}} n_3 = n_1$*)*

$$\sigma_{ij} n_j = \sigma\, \delta_{ij} n_j \quad \rightarrow \quad \left(\sigma_{ij} - \sigma\, \delta_{ij}\right) n_j = 0$$

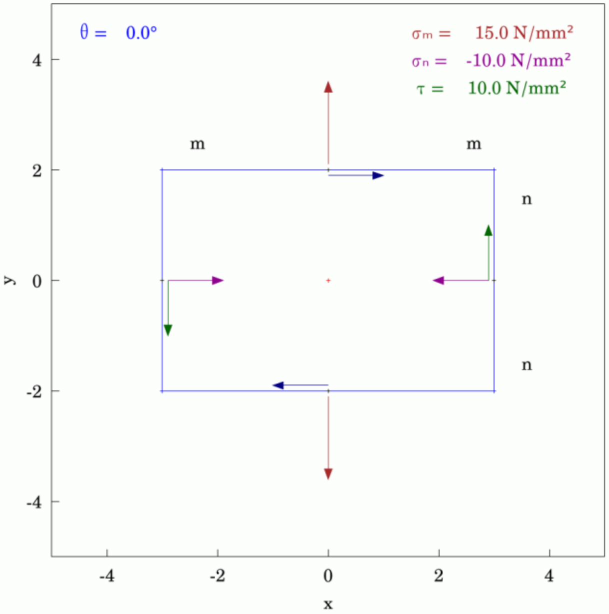
σ_{ij}*: stress tensor in the initial frame*

σ: any component of the stress in principal axes

$$(\sigma_{11} - \sigma)\, n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = 0$$
$$\sigma_{21} n_1 + (\sigma_{22} - \sigma)\, n_2 + \sigma_{23} n_3 = 0$$
$$\sigma_{31} n_1 + \sigma_{32} n_2 + (\sigma_{33} - \sigma)\, n_3 = 0$$

$$\begin{pmatrix} (\sigma_{11} - \sigma) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma) \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{\sigma}\ \boldsymbol{n} = \sigma\ \boldsymbol{n}$$



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Principal Axes and Principal Stress

Matrix-Vector multiplication

$$U = AV \quad \rightarrow u_i = (AV)_i = \sum_{j=1}^{3} \boldsymbol{A_{ij} v_j} \equiv \boldsymbol{A_{ij} v_j} = A_{i1} v_1 + A_{i2} v_2 + A_{i3}\, v_3$$

$$u_1 = A_{11} v_1 + A_{12} v_2 + A_{13}\, v_3$$
$$u_2 = A_{21} v_1 + A_{22} v_2 + A_{23}\, v_3$$
$$u_3 = A_{31} v_1 + A_{32} v_2 + A_{33}\, v_3$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Matrix multiplication

$$C = AB \quad \rightarrow C_{ik} = (AB)_{ik} = \sum_{j=1}^{3} \boldsymbol{A_{ij}B_{jk}} \equiv \boldsymbol{A_{ij}B_{jk}} = A_{i1}B_{1k} + A_{i2}B_{2k} + A_{i3}\,B_{3k}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

Principal Axes and Principal Stress

Determinant

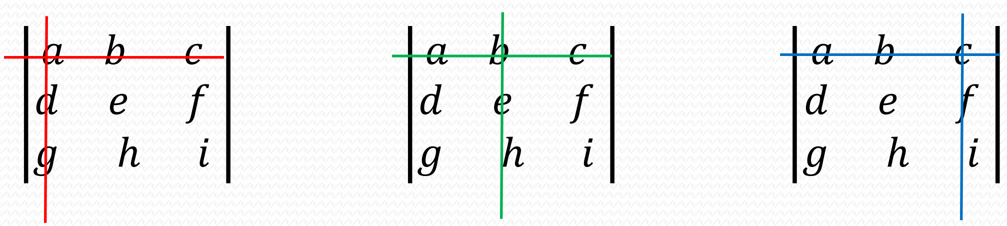
2-Dimension

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3-Dimension

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} =$$

$$(aei - afh) - (bdi - bfg) + (cdh - ceg)$$



Ex - A simple eigenvalue problem

$AV = \lambda V$

$A = \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}$ $\quad V = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ *eigenvector* $\quad \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\quad \lambda = ?$

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} (-6) * 1 + 3 * 4 \\ 4 * 1 + 5 * 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \end{pmatrix}$$

$\begin{pmatrix} 6 \\ 24 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ → $\lambda = 6$ eigenvalue of eigenvector $V = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$AV \quad = \quad \lambda V$

Principal Axes and Principal Stress

Formal solution for $\boldsymbol{V} = ?$ and $\boldsymbol{\lambda} = ?$

$$AV = \lambda V \quad \rightarrow AV = \lambda IV \qquad\qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow AV - \lambda IV = (A - \lambda I)V = 0$$

It can be proved that nontrivial solution $(V \neq 0)$ exits only if the **determinant of the coefficients vanishes.**

Therefore for non-zero V $(V \neq 0)$, λ can be obtained using the following determinant:

$$|A - \lambda I| = 0$$

Ex –

Find eigenvalues and eigenvectors of $\mathrm{A} = \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}$ (Note the $\sigma_{12} \neq \sigma_{21}$)

$$\left| \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \quad \rightarrow \quad \begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)(5-\lambda) - 3 * 4 = 0 \ \rightarrow \qquad \lambda = -7\,,\ \lambda = 6 \qquad eigenvalues$$

Principal Axes and Principal Stress

Eigenvectors ?

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

For $\boldsymbol{\lambda = 6}$

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} -6x + 3y = 6x \\ 4x + 5y = 6y \end{cases} \rightarrow x = 1, \quad y = 4 \rightarrow \mathrm{V} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{-6} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \end{pmatrix} = \mathbf{6} \begin{pmatrix} \mathbf{1} \\ \mathbf{4} \end{pmatrix}$$

For $\boldsymbol{\lambda = -7}$

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -7 \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} -6x + 3y = -7x \\ 4x + 5y = -7y \end{cases} \rightarrow x = -3, \quad y = 1 \rightarrow \mathrm{V} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{-6} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} \end{pmatrix}\begin{pmatrix} \mathbf{-3} \\ \mathbf{1} \end{pmatrix} = \mathbf{-7} \begin{pmatrix} \mathbf{-3} \\ \mathbf{1} \end{pmatrix}$$

3D – Case

Ex –

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix}$.

$(A - \lambda I)V = 0 \qquad |A - \lambda I| = 0$

$$\left| \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0 \quad \rightarrow \quad \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 5 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\lfloor(4-\lambda)(3-\lambda) - 5 * 4\rfloor = 0 \quad \rightarrow \quad \lambda = 2\,,\ \lambda = -1,\ \lambda = 8 \quad eigenvalues$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Principal Axes and Principal Stress

For $\boldsymbol{\lambda = -1}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 2x \\ 4y + 5z \\ 4y + 3z \end{pmatrix} = -1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{cases} 2x = -x \\ 47 + 5z = -y \\ 4y + 3z = -z \end{cases}$$

$$x = 0, \quad y = 1, \quad z = -1, \quad \rightarrow \mathrm{V} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} & \mathbf{5} \\ \mathbf{0} & \mathbf{4} & \mathbf{3} \end{pmatrix} . \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix} = \mathbf{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$$

Exercise – Find eigenvectors for $\lambda = 2$, $\lambda = 8$.

Normalization

In eigenvalue problems not all equations solved for eigenvectors are independent. We use normalization condition to constraint the solutions.

For $V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ we impose $\sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} \neq 1$ *in general*

We impose $|V'| = 1$

Normalized: $V' = \frac{1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $|V'| = 1$

Ex –

$$V = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$V' = \frac{1}{\sqrt{0+1+1}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Principal Axes and Principal Stress

Stress principal axes

$$\begin{pmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0, \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$\left|\sigma_{ij} - \sigma\,\delta_{ij}\right| = 0$$

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = -\sigma^3 + I_1\sigma^2 + I_2\sigma + I_3 = 0$$

where $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$ $\qquad -I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix}$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

Principal Axes and Principal Stress

It can be proved that a **real-valued symmetric matrix** always has always **three real-valued eigenvalues.**

→ $\sigma_1, \sigma_2, \sigma_3$ principal stresses

By convention: $\sigma_1 > \sigma_2 > \sigma_3$

Principal directions are obtained by solving $(\sigma_{ij} - \sigma\, \delta_{ij})\, n_j = 0$ for n_1, n_2, n_3 successively for the case $\sigma = \sigma_1,\ \sigma = \sigma_2,\ \sigma = \sigma_3$

$$\begin{pmatrix} \sigma_{11} - \sigma_i & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_i & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_i \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0 \qquad i = 1,2,3$$

Quantities I_1, I_2, and I_3 are the **invariants** of the stress tensor (**first, second and third invariant**).

Principal Axes and Principal Stress

Stress Tensor in Principal axes

$$[\sigma_{ij}] = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Invariants in principal coordinate:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_2 = \sigma_1\sigma_2\sigma_3$$

Maximum Shear Stress

Maximum Shear Stress

$\tau^2 = T^2 - \sigma^2$

Using **Cauchy's formula** $T_i^n = \sigma_{ij}\, n_j \rightarrow \left(T_i^n\right)^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$

$$\tau^2 = T^2 - \sigma^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - \left(\sigma_1\, n_1^2 + \sigma_2\, n_2^2 + \sigma_3\, n_3^2\ \right)^2 =$$
$$(\sigma_1 - \sigma_2)^2\,(n_1 n_2)^2 + (\sigma_2 - \sigma_3)^2\,(n_2 n_3)^2 + (\sigma_1 - \sigma_3)^2\,(n_1 n_3)^2$$

Note that $n_1^2\left(1 - n_1^2\right) = n_1^2\left(n_2^2 + n_3^2\right)$ and so on $(n_1^2 + n_2^2 + n_3^2 = 1)$

The planes on which the shear stress is **maximum** are obtained from the condition

$$\frac{\partial \tau}{\partial n_1} = \frac{\partial \tau}{\partial n_2} = 0$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$, there is no need for the third derivative

Maximum Shear Stress

Applying this to τ^2-equation, after some algebra, the extreme values of τ are

$$\begin{cases} \tau = \frac{1}{2}(\sigma_1 - \sigma_2) & \text{for } n_1 = n_2 = \sqrt{\frac{1}{2}},\ n_3 = 0 \\ \tau = \frac{1}{2}(\sigma_1 - \sigma_3) & \text{for } n_1 = n_3 = \sqrt{\frac{1}{2}},\ n_2 = 0 \\ \tau = \frac{1}{2}(\sigma_2 - \sigma_3) & \text{for } n_2 = n_3 = \sqrt{\frac{1}{2}},\ n_1 = 0 \end{cases}$$

Since $\sigma_1 > \sigma_2 > \sigma_3$ → $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$

Examples

Ex. – The principal values (eigenvalues) for the following stress tensor

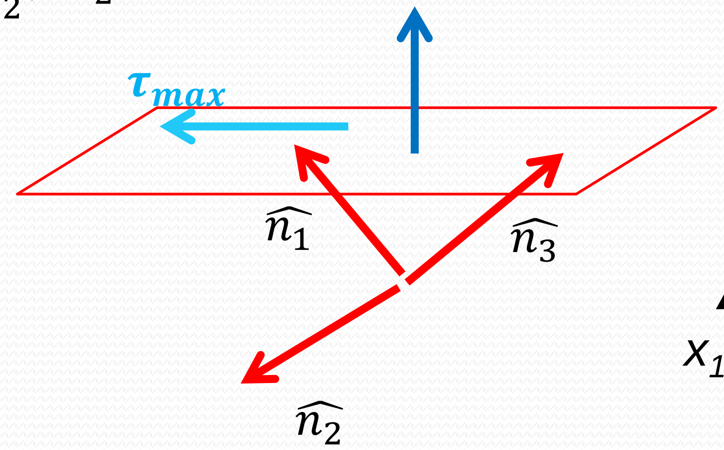
$$[\sigma_{ij}] = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{pmatrix}$$

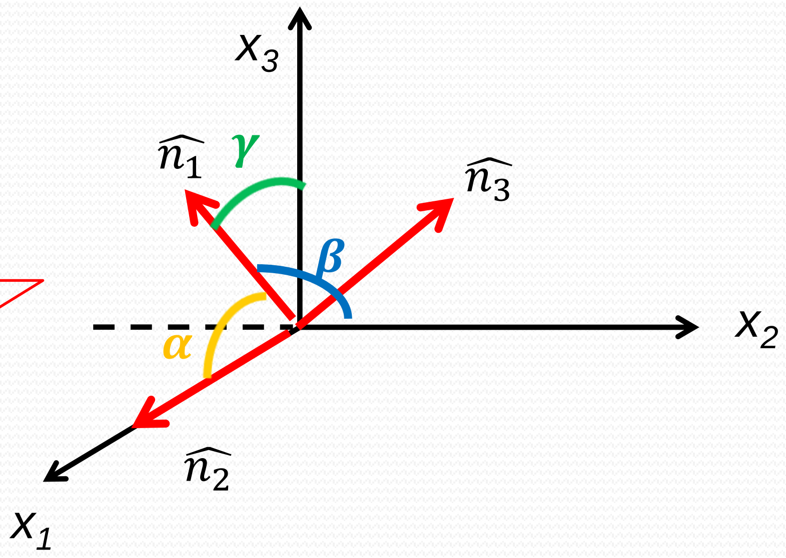
are $\sigma_1 = 10, \quad \sigma_2 = 5, \quad \sigma_3 = -15 \quad (\sigma_1 > \sigma_2 > \sigma_3)$

So the maximum shear stress is given by:

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(10 + 15) = 12.5$$

for $n_1 = n_3 = \sqrt{\frac{1}{2}}, \; n_2 = 0$





Minimum Shear Stress

From

$$\tau^2 = T^2 - \sigma^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 =$$
$$(\sigma_1 - \sigma_2)^2 (n_1 n_2)^2 + (\sigma_2 - \sigma_3)^2 (n_2 n_3)^2 + (\sigma_1 - \sigma_3)^2 (n_1 n_3)^2$$

we see that τ has a **minimum** ($\tau = 0$) for the following choices of n:

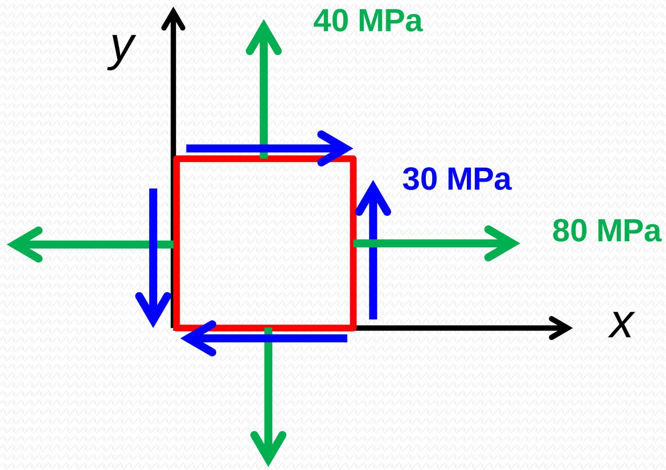
$$n_1 = \pm 1, \quad n_2 = n_3 = 0$$
$$n_2 = \pm 1, \quad n_1 = n_3 = 0$$
$$n_3 = \pm 1, \quad n_1 = n_2 = 0$$

Which shows **no shear stress act on three planes with normal in the co-ordinate directions** (principal planes) as one would expect from the choice of the principal axes as co-ordinate system.

Examples

Ex. – Find the principal axes and eigenvalues for the following stress tensor:

$$[\sigma_{ij}] = \begin{pmatrix} 80 & 30 \\ 30 & 40 \end{pmatrix}$$



Sol.

$$(\sigma_{ij} - \sigma\,\delta_{ij})\,n_j = 0$$

$$|\sigma_{ij} - \sigma\,\delta_{ij}| = 0 \qquad \begin{vmatrix} 80-\sigma & 30 \\ 30 & 40-\sigma \end{vmatrix} = 0$$

$$(80-\sigma)(40-\sigma) - 30^2 = 0$$

$$\sigma^2 - 120\sigma + 2300 = 0 \qquad \sigma_1 = 96.05, \quad \sigma_2 = 23.95 \ \textit{eigenvalues}$$

For $\sigma_1 = 96.05\ MPa$:

$$\begin{pmatrix} \sigma_{11}-\sigma & \sigma_{12} \\ \sigma_{21} & \sigma_{22}-\sigma \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = 0 \quad \rightarrow \quad \begin{bmatrix} 80-96.05 & 30 \\ 30 & 40-96.05 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (80-96.05)\,n_1 + 30\,n_2 = 0 \\ n_1^2 + n_2^2 = 1 \end{cases} \longrightarrow \quad \text{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0.88 \\ 0.47 \end{bmatrix}$$

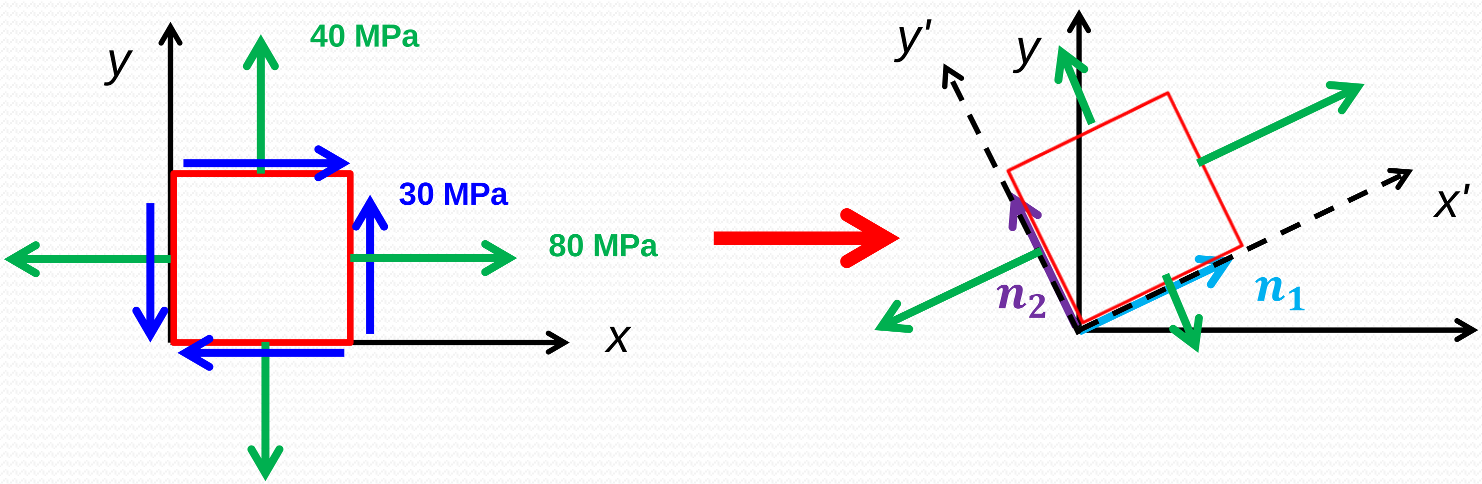
Alternatively, if we assume $n_2 = 1$

$$(80 - 96.05)\, n_1 + 30 = 0 \quad \rightarrow n_1 = \frac{30}{16.95} \quad \rightarrow \quad n = \frac{1}{\sqrt{\left(\frac{30}{16.95}\right)^2 + 1}} \begin{bmatrix} \frac{30}{16.95} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.88 \\ 0.47 \end{bmatrix}$$

Similarly for $\sigma_2 = 23.95$: $\quad n = \begin{bmatrix} -0.47 \\ 0.88 \end{bmatrix}$

Rename $n = \begin{bmatrix} 0.88 \\ 0.47 \end{bmatrix}$ as $n_1 = \begin{bmatrix} 0.88 \\ 0.47 \end{bmatrix}$ corresponding to the eigenvalue $\sigma_1 = 96.05$

and $n = \begin{bmatrix} -0.47 \\ 0.88 \end{bmatrix}$ as $n_2 = \begin{bmatrix} -0.47 \\ 0.88 \end{bmatrix}$ corresponding to the eigenvalue $\sigma_2 = 23.95$



$$\sigma' = \begin{pmatrix} 96.05 & 0 \\ 0 & 23.95 \end{pmatrix}$$

Ex. – The state of stress at a point is given by:

$$[\sigma_{ij}] = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

Find:

a) the traction vector acting on a plane with unit normal of:

$n = \frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{2}{3}e_3$

b) the shear and normal components of this traction vector.

Sol.

a) $T_i^n = \sigma_{ij}\, n_j$ *Cauchy's formula*

$$\begin{pmatrix} T_1^n \\ T_2^n \\ T_3^n \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ 9 \\ -3 \end{pmatrix}$$

$$T^n = -\frac{2}{3}e_1 + 3e_2 - e_3$$

Examples

b) $\sigma = T^n \cdot n = \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + (3)\left(\frac{2}{3}\right) + (-1)\left(-\frac{2}{3}\right) \approx 2.4$

$$\tau = \sqrt{(T^n)^2 - (\sigma)^2} = \sqrt{\left[\left(-\frac{2}{3}\right)^2 + (3)^2 + (-1)^2\right] - [(2.4)^2]} \approx 2.1$$

Ex. – The state of stress at a point is given by:

$$[\sigma_{ij}] = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{pmatrix}$$

Determine principal stress components and principal directions.

Sol.

$$(\sigma_{ij} - \sigma\, \delta_{ij})\, n_j = 0$$
$$|\sigma_{ij} - \sigma\, \delta_{ij}| = 0$$

Examples

$$\begin{vmatrix} 5-\sigma & 0 & 0 \\ 0 & 6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{vmatrix} = (-10+\sigma)(5-\sigma)(15+\sigma) = 0$$

$\sigma_1 = 10, \qquad \sigma_2 = 5, \qquad \sigma_3 = -15$

For $\sigma_1 = 10\ MPa$:

$$\begin{bmatrix} 5-\sigma & 0 & 0 \\ 0 & 6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\sigma = \sigma_1)$$

$$\left\{\begin{array}{l} -5n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 - 16n_2 - 12n_3 = 0 \\ 0n_1 + 12n_2 + 9n_3 = 0 \end{array}\right.$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$n_1 = 0, \quad n_2 = -\frac{3}{5}, \quad n_3 = \frac{4}{5}$

$$n = -\frac{3}{5}\, e_2 + \frac{4}{5}\, e_3$$

Examples

Similarly for $\sigma_2 = 5\ MPa$:

$$\begin{bmatrix} 5-\sigma & 0 & 0 \\ 0 & 6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\sigma = \sigma_2)$$

$$\begin{cases} 0n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 - 11n_2 - 12n_3 = 0 \\ 0n_1 - 12n_2 - 4n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{cases}$$

$n_1 = 1, \quad n_2 = 0, \quad n_3 = 0$

$n = e_1$

And for $\sigma_3 = -15\ MPa$:

$$\begin{bmatrix} 5-\sigma & 0 & 0 \\ 0 & 6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\sigma = \sigma_3)$$

$$\begin{cases} 20n_1 + 0n_2 + 0n_3 = 0 \\ 0n_1 + 9n_2 - 12n_3 = 0 \\ 0n_1 - 12n_2 + 16n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{cases}$$

$n_1 = 0, \quad n_2 = \frac{4}{5}, \quad n_3 = \frac{3}{5}$

$n = \frac{4}{5}\ e_2 + \frac{3}{5}\ e_3$

Examples

Directions of the principal axes with respect to the original frame

$n = -\frac{3}{5}\ e_1 + \frac{4}{5}\ e_3$ rename as $n_1 = -\frac{3}{5}\ e_2 + \frac{4}{5}\ e_3$

$n = e_1$ rename as $n_2 = e_1$

$n = \frac{4}{5}\ e_2 + \frac{3}{5}\ e_3$ rename as $n_3 = \frac{4}{5}\ e_2 + \frac{3}{5}\ e_3$

Note that the components of each principal direction, are the direction cosines.

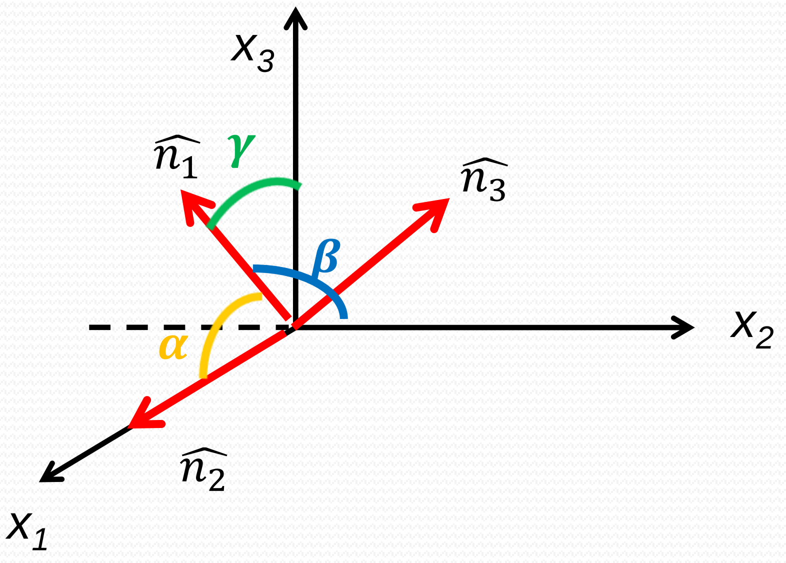
e.g.,

$n_1 = -\frac{3}{5}\ e_2 + \frac{4}{5}\ e_3$

$Cos\alpha = 0$

$Cos\ \beta = -\frac{3}{5}$

$Cos\ \gamma = \frac{4}{5}$



Examples

Ex. – The stress tensor at a point in two different co-ordinate systems are given by:

$$[\sigma_{ij}] = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{pmatrix} \qquad [\sigma'_{ij}] = \begin{pmatrix} \frac{3}{2} & \frac{3}{\sqrt{2}} & \frac{1}{2} \\ \frac{3}{\sqrt{2}} & 3 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{3}{2} \end{pmatrix}$$

The first invariant:

$$I_1 = 2 + 3 + 1 = \frac{3}{2} + 3 + \frac{3}{2} = 6$$

Verify that the second and third invariants are:

$I_2 = 6$, and $I_3 = -3$

for the matrix in the original coordinate and rotated frame..